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BILATERAL SEARCH AND VERTICAL HETEROGENEITY*

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Just like perfect (frictionless) matching models, a search model is proposed that is characterized by bilateral search and vertical heterogeneity. It allows for a generally specified utility function. The equilibrium allocation is unique and exists in iterated strict dominance. The model is robust with the perfect matching model as frictions disappear. Nonetheless, the equilibrium allocations are surprisingly odd. For multiplicatively separable preferences, the distributions are partitioned endogenously. And for a wide range of preferences, matching sets are naturally disconnected.

1. INTRODUCTION

Two-sided matching models without frictions solve for the equilibrium allocation of ex ante heterogeneous agents. Here, this perfect matching model is extended to a search model, i.e., matching with frictions. Contrary to other work, general preferences are considered and a general algorithm is developed that allows me to show that the equilibrium allocation exists and is unique. This algorithm enables the derivation and characterization of the equilibrium allocation for *any* utility function. Equilibrium allocations have features not encountered in the perfect matching literature. These features are derived even though the model is shown to be robust: With search frictions disappearing, the allocation coincides with the perfect matching allocation.

The central characteristics of the perfect matching model with nontransferable utility (marriage, allocating physicians to hospitals, etc.) are the presence of heterogeneity of types and the bilateral decision: Only when there is a double coincidence of wants between a male and a female with heterogeneous preferences will they engage in marriage. Without frictions, the two-sided perfect matching model where agents are vertically heterogeneous (i.e., there exists a ranking of the types) has an allocation that exhibits positive assortative mating: The highest-ranked female is matched with the highest-ranked male, etc. This is due to the bilateral nature of acceptance. The second-highest-ranked male would like to be matched with the highest-type female, but the female will not accept marriage. She can do better being matched with the highest-type male.

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Introducing a search friction certainly does not decrease the true representation of some markets. After all, in a marriage market, for example, it may not be too difficult to find a spouse, but in order to do as well as possible, some longer (costly) search may be optimal. The main difference is that with search frictions, the top female will accept males over a certain range, since waiting too long is costly.

The main contribution of this article is to show that provided that the distribution of singles is stationary, a (Nash) equilibrium allocation exists and is unique. This is true for any specification of the utility function. This result is surprising in two respects. First, uniqueness. The strategies of one sex are monotonic in the strategies of the other sex, so a continuum of strategies would be expected. However, an argument of iterated elimination of dominated strategies only leaves one strategy to survive. Second, existence. Whatever preferences are assumed, the allocation can always be found using the recursive elimination method. Given existence and uniqueness, equilibrium is characterized. First, the distribution of types is endogenously partitioned for preferences that are multiplicatively separable. This is nontrivial, since preferences are type-dependent, while, by definition of endogenous partitioning, strategies are not. Second, for some preferences, matching sets are disconnected. This implies that you are rejected when you propose a match with some types, even though you are accepted by both higher and lower types. Finally, the model is shown to be robust. With frictions disappearing, the equilibrium allocation coincides with the equivalent allocation in perfect matching.

There are substantial differences with parallel results both in the existing literature and in simultaneously conducted research. A considerable number of authors have looked at this problem for a specific utility function where the utility derived is equal to the type matched with. The papers by McNamara and Collins (1990), Burdett and Coles (1997), and Bloch and Ryder (1995) all have these specific utility functions. All of them derive the partitioning result, since their preferences are a limit case of multiplicative separability. The result derived here applies to a more general class of utility functions. The general partitioning result has been discovered independently by Smith (1996). Smith also looks at general preferences but provides a different proof and solution. The main novelty of the approach here is the intuitive appeal of the proof and its wide applicability to any utility specification. The fact that the equilibrium is shown to exist in a strong concept like iterated dominance provides significant behavioral foundations for both the resulting equilibrium allocation and the method or algorithm of obtaining it.

This appealing and intuitive method and solution are derived under the assumption of a stationary distribution of singles. A similar approach is adopted in McNamara and Collins (1990) and Bloch and Ryder (1995). Endogenizing the distribution in itself does not pose any problem (this is done in the Appendix). The problem is to find a fixed point for the equilibrium distribution, and this goes at the expense of the intuitive derivation of the equilibrium allocation.[2] Burdett and Coles (1997) show that due to thick market externalities, for some parameter values, multiple steady-state distributions can be supported in equilibrium. My contribution is to show that given a distribution of singles, the *allocation* is unique for any preferences.

[2] Smith (1996) provides a proof for a fixed point of the endogenous distribution of singles.

The generalization of the perfect matching model to the search model is very much modeled in the tradition of the literature. The main aspect, however, is that both sides of the market search (i.e., there is bilateral search) and that there exists an ex ante heterogeneity of the types. In this marriage model, individuals of one sex will meet potential partners at random, and there are only a limited number of meetings per unit of time. Given perfect information, the type of the potential partner is observed on meeting and can be accepted or rejected. If the type is too low, it may pay to wait until a higher type is met. Accepting, however, only implies that a match materializes provided there is a double coincidence of wants (i.e., the other party decides to accept as well). In the presence of bilateral search, the decision to form a match cannot be enforced unilaterally. The utility derived from a match is represented by any cardinal utility function that satisfies vertical heterogeneity. The model considered features nontransferable utility.

In this framework, agents will choose strategies to accept or reject potential partners that come along in order to maximize the value function of searching. Entirely counterintuitive, the uniqueness and existence result derives from the fact that the equilibrium solution can be solved for using an iterated strict dominance argument. The intuition is that with vertical heterogeneity, the top types of both sexes are most desired by all, so they can be assured to be accepted by all types. Hence they have an iterated strict dominant strategy. Given these strategies, this argument equally applies to the next but top types, etc. In the presence of search frictions, they have to accept a range of types with positive mass so that a finite number of iterations will suffice.

The basic model is presented in Section 2. Even though the marriage vocabulary prevails most dominantly in this article, the model is easily generalizable to any market where pairs of agents trade nondivisible, heterogeneous goods. In Section 3, the model is solved, and it is shown that a unique iterated strict dominance equilibrium exists and that for multiplicatively separable utility functions both distributions of singles are endogenously partitioned. Section 4 provides the intuition behind the elimination procedure and discusses several possible equilibrium outcomes in function of the payoff specification. The equivalence of the model with the Gale-Shapley-Becker model is rigorously proved in Section 5. Some concluding remarks are made in Section 6. The Appendix provides a proof for the main proposition and derives the endogenous distribution of singles.

2. THE BASIC MODEL

Consider two disjoint sets of infinitely lived individuals: females and males. They are intrinsically heterogeneous in type. Only one dimension of heterogeneity will be considered so that their type can be represented by one variable θ. This type can be interpreted as a measure of either beauty, wealth, sexual attraction, etc., or as a composite measure of all these characteristics. Females and males are distinguished by θ_f and θ_m, respectively. Both populations of singles are cumulatively distributed according to $F_i(\theta)$ over $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$, $i \in \{f, m\}$ [with $f_i(\theta)$ the density function] and have equal measure one.

Individuals can be in two possible states. They can either be matched to a partner or they can be single. When single, they are looking for a partner to be matched to. Partners of a different sex meet randomly, and on meeting, they can perfectly observe the type of the other sex. At that moment, they will decide whether to accept or reject a match with the partner met. A match is materialized only when both partners decide to accept each other. The decision is bilateral and cannot be enforced unilaterally.

Being single is a dire state. It does not yield any intrinsic utility. The state of being matched, on the other hand, brings all potential pleasure that exists in this world. It is modeled as an instantaneous utility derived at the moment the match is formed, i.e., when both individuals decide to accept the match, the marriage is instantaneously "consum(mat)ed." The nontransferable utility to an individual of sex i characterized by type θ_i from being matched to a type θ_j is $u_i(\theta_j, \theta_i)$, with u continuous. In general, no symmetry is required: $u_i \neq u_j$. Preferences exhibit vertical heterogeneity, $\partial u_i/\partial \theta_j > 0$. This implies that there is a ranking of the types of the other sex on which all individuals agree. All men agree that Juliette Binoche is the most beautiful woman, and all women have no doubts about who is the least endowed man. Note that utility is type-dependent without any restrictions. Showing that an equilibrium exists in the presence of a general utility specification is exactly the objective of this article. Clearly, utility is cardinal, since a search model intrinsically puts a cardinal value on the time of search. The general utility specification allows for any cardinal value of the vertically heterogeneous preference orderings as long as the values are bounded: $u(\underline{\theta}_i) > 0$, $u(\bar{\theta}_i) < \infty$.[3]

Frictions are modeled in the standard way: Instantaneous utility and search costs are collapsed into value functions using a stochastic friction. Here, a constant returns to matching[4] search technology is specified as follows: When single, an individual bumps into someone of the other sex with probability β. This arrival rate β is distributed according to a Poisson process. Infinitely lived agents are not matched for life. With probability α, a match is dissolved.[5] For the purpose of this article, on-the-job search, endogenous separation, and polygamy are ruled out.

Crucially, not all potential partners met will yield a match. In the first place, an individual may not be entirely satisfied with the type of the other sex and will prefer to search further until a more preferred type is met. Second, an individual may be very willing to enter a match, but the potential partner may wish to postpone the match. An individual's strategy will be determined subject to being accepted, so in the first instance, a strategy of an individual will be determined taking the strategies of all other players as given. An equilibrium will then be a rule such that an agent maximizes the value function taking into account that all other agents adopt such a maximizing strategy: a Nash equilibrium.

An individual's optimizing strategy will be derived from maximizing the value functions V_0 and V_1 in both possible states, the value for being single and matched, re-

[3] In what follows, the notation $u(\cdot)$ may be used to signify $u_i(\cdot, \theta_i)$ when there is no confusion possible.

[4] Constant returns to matching implies that the number of meetings in the market is proportional to the number of individuals searching. As a result, the number of meetings per person is constant.

[5] Modeling finitely lived agents with an exogenous inflow of new births yields the same results.

spectively. They will, in general, be different depending on the type θ_i. The value functions of both states are written in the form of Bellman equations that give the current option value, given a positive interest rate r.

(1) $rV_0(\theta_i) = \beta \max E_{\theta_j}\big[0, u(\theta_j, \theta_i) + V_1(\theta_i) - V_0(\theta_i) \mid \text{given acceptance by } \theta_j\big]$

(2) $rV_1(\theta_i) = \alpha\big[V_0(\theta_i) - V_1(\theta_i)\big]$

When single, a potential partner is met with probability β. The type of the partner is randomly drawn from the pool of singles. Provided the type θ_j accepts the match, the instantaneous utility derived is $u(\theta_j, \theta_i)$. Marriage will be proposed if being matched to θ_j yields a higher utility than the value of looking further until a more preferred type is met. This is the case when $u(\theta_j, \theta_i) + V_1(\theta_i)$ is higher than the expected value of remaining single $V_0(\theta_i)$. Since separation occurs with fixed probability α, the option value of being matched is given by α times the residual value of switching from being matched to being single. Note that being single has an option value associated with it that is positive. Being single does not yield any intrinsic instantaneous utility, but there exists the probability of being matched at some future point in time. Not only is V_0 positive, it is even bigger than V_1. This is the result of the modeling strategy. The utility of marriage is assumed to be an instantaneous flow of utility at the time of meeting.

The decision of an individual of type θ_i is either to accept or reject a type θ_j that is met. I will represent this by the binary variable $\pi_i(\theta_j, \theta_i)$, which is defined as $\pi_i(\theta_j, \theta_i) = 1$ if a match with θ_j is accepted by θ_i and $\pi_i(\theta_j, \theta_i) = 0$ if it is rejected. Clearly, acceptance does not necessarily imply that a match materializes, given the bilateral nature of the decision to form a match. A type of the other sex θ_j accepts a type θ_i if $\pi_j(\theta_i, \theta_j) = 1$. Whether a type θ_i is accepted is given by the inverse function of $\pi_j(\theta_i, \theta_j)$. Hence, once any potential trading partner is met, the match is materialized with probability $\psi_i(\theta_i)$:

$$\psi_i(\theta_i) = \int_{\Theta_j} \pi_i(x, \theta_i)\pi_j(\theta_i, x)\, dF_j(x) \tag{3}$$

where $F_j(\theta_j)$ is the cumulative density function of singles in the market.

REMARK 1. Since the whole population is not single at the same time, the measure of people searching is not equal to the measure of the population. More important, since in general not all types have the same strategy (i.e., the strategy is type-dependent), the distribution of singles $F_i(\theta_i)$ is not equal to the distribution of the entire population, say, $H_i(\theta_i)$. In the Appendix, the relation between the H and F is derived. All the results go through with an endogenously derived distribution of singles, both in the steady state and out of the steady state.

There are, however, two reasons why the results are derived under an exogenously given distribution of singles F. First, I do not have a proof for a fixed point of the endogenous distribution. Second, there is a source of multiplicity of steady-state equilibria that is independent of the potential multiplicity this article shows not to exist. Burdett and Coles (1997) provide an example where different beliefs about the steady-state distribution can be supported in equilibrium. This multiplicity is due to thick market externalities in the search technology, very much as in Diamond (1982). The main contribution of Proposition 1 below is to show that given a distribution of singles (of which more than one may exist), there exists a unique equilibrium allocation. Below it will become apparent that that in itself is a most nontrivial result.

Equation (1) can now be rewritten in terms of the binary variables π_i and π_j and the distribution of single males and females $F(\theta_i)$ and $F(\theta_j)$.

$$(1') \qquad rV_0(\theta_i) = \beta \int_{\Theta_j} \pi_i(x, \theta_i)\pi_j(\theta_i, x)[u(x, \theta_i) + V_1(\theta_i) - V_0(\theta_i)]\, dF_j(x)$$

An individually optimal solution to Equations (1′) and (2) for a type θ_i is a strategy of acceptance π_i such that he or she is indifferent between remaining single and being matched. An equilibrium requires that individuals use a strategy such that they accept all matches for which the value of being matched is higher than the value of remaining single. In what follows, this will be referred to as a *reservation strategy*: A type θ_j is offered marriage if $u(\theta_j, \theta_i) + V_1(\theta_i) - V_0(\theta_i) \geq 0$. The reservation strategy implies that for each θ_i there must then be a critical type $\theta_j = \phi_j(\theta_i)$ that solves the equation

$$(4) \qquad u(\phi_j) \geq V_0(\theta_i) - V_1(\theta_i)$$

REMARK 2. The reservation strategy restricts the strategy space because it rules out strategies where lower-type males choose to reject a high-type female because they know they will be rejected anyway. This would yield a degenerate equilibrium where everyone rejects everyone. Because I impose the strategy "accept all types for which the expected value of a match is higher than the marginal type," high types cannot be rejected strategically.

An optimal strategy $\pi_i(\theta_j, \theta_i)$ will be determined in function of the critical type ϕ_j associated with the strategy (4). An imperfect matching equilibrium can now be defined using the notion of Nash equilibrium. It is a list of optimizing strategies taking into account that all other agents use their optimizing strategy.

DEFINITION 1. For given distributions of singles F_i and F_j, an imperfect matching equilibrium is a list $\{\pi_i(\theta_j, \theta_i), \pi_j(\theta_i, \theta_j)\}$, $\forall \theta_i \in \Theta_i$, $\forall \theta_j \in \Theta_j$ satisfying:

(a) Equations (1′) and (2)
(b) The reservation strategy (4)

3. THE RESULTS: EXISTENCE AND UNIQUENESS

In this section, the main result of existence and uniqueness of the imperfect matching equilibrium is derived. For this purpose, two lemmas are shown. Lemma 1 claims that a reservation strategy implies that all types θ_j above the critical type ϕ_j are accepted and all types below are rejected. Lemma 2 proceeds to prove that there is a unique reservation strategy holding the strategy of all other players constant. With these lemmas, the main result in Proposition 1 can be shown.

Lemma 1 provides the relation between the strategy π_i and the reservation type ϕ_j.

LEMMA 1. *A reservation strategy* $\pi_i(\theta_j, \theta_i)$ *satisfies*:

$$\pi_i(\theta_j, \theta_i) = 1 \qquad \text{if } \theta_j \geq \phi_j(\theta_i)$$

$$\pi_i(\theta_j, \theta_i) = 0 \qquad \text{if } \theta_j < \phi_j(\theta_i).$$

PROOF. $\phi_j(\theta_i)$ has to satisfy the reservation strategy $u(\phi_j, \theta_j) \geq V_0(\theta_i)$. Since $u_{\theta_j} > 0$ and $\partial[V_0(\theta_i) - V_1(\theta_i)]/(\partial\theta_j) = 0$, the lemma is always (never) satisfied for $\theta_j \geq (<)\phi_j(\theta_i)$. It follows that any $\theta_j \geq (<)\phi_j(\theta_i)$ will be accepted (rejected), so $\pi_i(\theta_j, \theta_i) = 1$ $(= 0)$. ■

This lemma shows that reservation strategies generate sets of acceptance that are connected. This follows from the assumption of vertical heterogeneity and is not true in general. Just consider a utility function that is not monotonic in θ_j. It may generate multiple ϕ_j, and hence Lemma 1 is not satisfied.

As a result of Lemma 1, the use of the decision variables π_i may at this stage appear cumbersome notation, since strategies are monotonic in θ_j: $\pi_i = 1$ always constitutes a connected set in θ_j. However, not only do we need $\pi_i(x, \theta_i)$ but also its inverse $\pi_i(\theta_j, x)$. In general, $\pi_i = 1$ is not a connected set in θ_i. As a result, with the decision variables, the calculation of integrals can be made without knowing the internal boundaries of the disconnected sets.

Lemma 2 shows that, given the strategies of all other players, the reservation strategy is unique. Using the reservation strategy (4), Equations (1′) and (2) can be rewritten as

$$T_i(\phi_j) = (r + \alpha)u(\phi_j) - \beta\gamma_i(\phi_j) = 0 \tag{5}$$

with $\gamma_i(\phi_j) = \int_{\Theta_j} \pi_i(x, \theta_i)\mu_i(x, \theta_i)[u(x, \theta_i) - u(\phi_j)]\, dF_j(x)$. The first-order condition (5) embodies the tradeoff made by every individual agent. From Lemma 1, all types above the reservation type $\phi_j(\theta_i)$ are accepted, and a match is materialized if you are accepted by these types. Given acceptance, vertical heterogeneity implies that the higher the reservation value, the higher is the expected value of the match. The cost of increasing the reservation value, though, is that the probability of leaving the pool of singles decreases: Being more choosy means that (utilityless) waiting times increase. In the limit, the prince(ss) of your dreams arrives with probability zero; hence the expected time of being single is infinite and utility is zero. Moreover, without a direct search cost, the opportunity cost of waiting is utility foregone while you could be matched to a partner. Solving Equation (5) yields a critical type $\phi_j(\theta_i)$, $\forall\theta_i$, and hence a reservation strategy $\pi_i(\theta_j, \theta_i)$, $\forall\theta_i$. Lemma 2 shows that it is unique.

LEMMA 2. *Given* π_j, *and for* $\phi_j' = \max\{\theta_j \in \Theta_j \mid \pi_j(\theta_j) = 1\}$:

(*a*) ϕ_j *is the unique solution to* $T_i(\phi_j) = 0$.

(*b*) $\phi_j \leq \phi_j'$.

PROOF. First, it follows from the definitions of $\psi_i(\theta_i)$ and ϕ'_j that for $\phi_j > \phi'_j$, $\psi_i = 0$, and as a result, $\gamma_i = 0$. Since $u(\theta_j) > 0$, $\forall \theta_j$ [from $\partial u/\partial \theta_j > 0$ and $u(\underline{\theta}_j) > 0$], it follows that $T_i(\phi_j) > 0$ for $\phi_j > \phi'_j$ and that there is no solution to $T_i(\phi_j) = 0$ in $(\phi'_j, \bar{\theta}_j)$. Next, $\partial T_i/\partial \phi = (r+\alpha)\partial u/\partial \phi - \beta(\partial \gamma/\partial \phi) > 0, \forall \phi_j \in \Theta_j$ since $\partial \gamma/\partial \phi_j = -\partial u(\phi_j, \theta_i)/\partial \theta_j \int_{\Theta_j} \pi_i(x, \theta_i)\pi_i(\theta_i, x)\, dF_j(x) < 0$ and $\partial u/\partial \theta_j > 0$. Given that $T_i(\phi_j) > 0$ for $\phi_j > \phi'_j$ and that $\partial T_i/\partial \phi > 0$, a solution to $T_i(\phi_j) = 0$ is in $[\underline{\theta}_j, \phi'_j]$. This establishes (b) $\phi_j \leq \phi'_j$.

Since T_i is strictly increasing, the solution is unique. If $T_i(\underline{\theta}_j) \leq 0$, there is an interior solution. If $u(\phi_j) \geq V_0(\theta_i) - V_1(\theta_i)$ holds with strict inequality for some θ_i, there is no interior solution $T_i(\phi_j) = 0$. An optimizing agent will then choose the unique ϕ_j as the minimum $\theta_j \in \Theta_j$, satisfying the reservation strategy. This maximizes the expected value $V_0(\theta_i) - V_1(\theta_i)$, and the solution is a corner solution. This establishes that (a) ϕ_j is the unique solution to $T_i(\phi_j) = 0$. ■

The proof of uniqueness of the reservation value is made using the fact that the value function is monotonic in the reservation value. Part (b) of the lemma shows that the reservation value cannot be above the highest type that is willing to accept you. On the other hand, if there is no interior solution below that, the solution is the corner solution $\underline{\theta}_j$. Together with Lemma 1, it then follows that any type above the reservation type is accepted. Just as a unique optimal response in a normal-form game does not imply a unique Nash equilibrium, uniqueness of the reservation strategy, given the strategies of all other players, does not imply that the equilibrium is unique. This is illustrated in Figure 1.

The lower graph is the reservation strategy of all types θ_i: Above the reservation type, all θ_j are accepted; below, they are rejected. The upper graph is the reservation

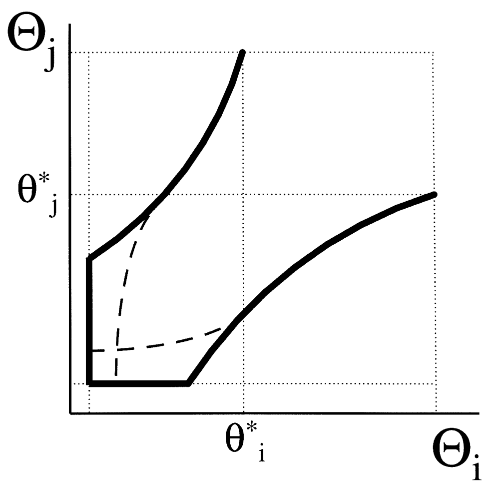

FIGURE 1

RESERVATION STRATEGIES

value for all types θ_j: To the right of the graph, all θ_i are accepted; to the left, all are rejected. Clearly, not all θ_j are willing to accept a match with types left of θ_i^*. Only the types θ_j below the upper graph (i.e., inverse of the reservation strategy of the types θ_j) will accept. The vertical distance between the two graphs is then the range over which matches are materialized. Measured over the distribution F_j, it determines the probability of acceptance ψ_j.

It is also clear from the figure that a unique optimal response does not necessarily imply a unique Nash equilibrium. Lowering the higher of the two graphs will imply that the unique best response will result in the lower graph shifting down as well. As a result, many equilibria may be envisioned: one unique response for each strategy of the other players. This makes the following proposition all the more surprising.

PROPOSITION 1. *An imperfect matching equilibrium exists in iterated elimination of strictly dominated strategies and is unique.*

PROOF. See Appendix.

In the next section the intuition behind this result will be discussed for different possible cases. There it will emerge what the algorithm behind the elimination process involves. The intuition initially will be helped by discussing the characteristics of one special case of the result.

It specifies a wide class of utility functions for which the properties of the equilibrium schedules $\phi_j(\theta_i)$ and $\phi_i(\theta_j)$ partition the distributions. In Proposition 2, it is shown that this is the case for multiplicatively separable utility functions of both sexes $u(\theta_j, \theta_i) = v(\theta_j)w(\theta_i)$, $\forall i, j$ (but not necessarily identical $u_i \neq u_j$). That is, females of a certain range of types only match with males of a certain range. Outside the partitions, there is no matching. This follows from the fact that individuals within one partition have identical reservation strategies. This is unexpected because for multiplicatively separable utility functions, the utility is type-dependent, whereas the strategy within the partition is type-independent!

PROPOSITION 2. *For multiplicatively separable utility functions, the distributions of types are endogenously partitioned.*

PROOF. First, the necessary condition is derived for which a strategy is independent of the type. From Lemma 2, there exists a unique reservation strategy, given the strategy of all other players. Type independence of the reservation strategy will occur when, taking π_j as given, $\partial\phi_j(\theta_i)/\partial\theta_i = 0$, i.e., when a lower type has the same reservation value. With $\partial T/\partial\phi > 0$, the implicit function theorem implies that $\partial T/\partial\theta_i = 0$, which implies that

$$(6) \qquad (r+\alpha)\frac{\partial u(\phi_j)}{\partial\theta_i} - \beta\int_{\Theta_j}\pi_i(x,\theta_i)\pi_j(\theta_j,x)\left[\frac{\partial u(x)}{\partial\theta_i} - \frac{\partial u(\phi_j)}{\partial\theta_i}\right]dF_j(x) = 0$$

Under condition (6), the equilibrium mapping $\phi_j(\theta_i)$ is type-independent for a given π_j and F_j.

Second, multiplicatively separable utility functions. Consider the general formulation of such a function: $u(\theta_j, \theta_i) = v(\theta_j)w(\theta_i)$. For this case, $T(\phi_j) = 0$ can be written as

$$(r+\alpha)v(\theta_j) - \beta \int_{\Theta_j} \pi_i(x, \theta_i)\pi_j(\theta_i, x)[v(x) - v(\phi_j)]\, dF_j(x) = 0 \tag{7}$$

It is easily verifiable that $\partial T/\partial\theta_i = 0$, provided $\pi_j(\theta_i, \theta_j)$ is independent of θ_i, i.e., $\pi_j(\theta_i^1, \cdot) = \pi_j(\theta_i^2, \cdot)$, $\forall \theta_i^1 \neq \theta_i^2$. By requiring that u_j too is multiplicative, this is automatically satisfied $\forall \theta_i$ in the same partition. ■

4. DISCUSSION OF THE RESULTS

Proposition 2 provides a good example in order to get some insight into the algorithm of the iterated elimination of dominated strategies that establishes the existence and uniqueness of equilibrium. Consider Figure 2.

Start with the most optimistic guess: All types θ_i are accepted by all θ_j. Then, all θ_i choose the same type-independent reservation value θ_j^*, given a multiplicative utility function. In the proof of Proposition 1, it is shown that this reservation strategy θ_j^* is an upper bound. Being accepted by less than all the types θ_j (i.e., a less than the most positive guess) would certainly lower the reservation value. By the same argument, θ_i^* is an upper bound for all θ_j. An upper bound implies that all strategies above it are dominated. Dominated strategies can be eliminated.

The implication is that at least for $\forall \theta_i \geq \theta_i^*$, the optimistic guess is true: Any θ_j will accept them. From Lemma 2, the strategy $\phi_j = \theta_j^*$ is the unique best response for those types. It is an iterated strict dominance strategy. Likewise, the iterated strict

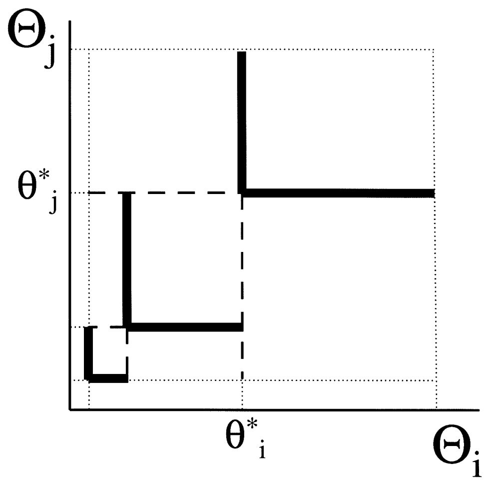

FIGURE 2

PARTITIONING

dominance strategy for all $\theta_j \geq \theta_j^*$ is θ_i^*. This gives rise to the first subset of the partition.

Now, conditional on the iterated strict dominance strategies of the types in the highest subset, all lower types can make a next-to-most optimistic guess (all $\theta_i < \theta_i^*$ are now rejected by all $\theta_j \geq \theta_j^*$) about being accepted. Provided that they are accepted by all types below the highest partition, they will choose a revised (type-independent) reservation strategy (the second horizontal line). Again, the iterated dominance argument implies that the second partition is formed by all types that are accepted above the revised upper bound. This goes on until all types belong to precisely one subset, establishing the partition.

The partitioning result is quite surprising. Although utility functions are type-dependent, the strategies are not. For a special case of multiplicatively separable utility functions with $\partial u/\partial\theta_i = 0$, the result is fairly intuitive because the utility function is type-independent.[6] It follows that the first-order condition (5) is type-independent. Utility derived and hence the opportunity cost are identical ex ante for types within one partition. Hence they will solve for the same solution. With type-dependent but separable utility functions, this is equally the case but for different reasons. Consider, for example, the case where higher types derive more utility from being matched with a high type of the other sex (i.e., the utility exhibits strategic complementarities). It follows that the expected value of being matched is increasing in type. On the one hand, higher types will be more choosy and have higher reservation values. On the other hand, without direct search costs, the cost of search is the opportunity cost of not being matched. As a result, the search cost is increasing in type. The higher types are more impatient and choose lower reservation values. For multiplicative utility functions, these two effects cancel out against each other, and the first-order condition (7) is homogeneous of degree zero in the own type.

Note further that in case $u(\underline{\theta}_i) = u(\underline{\theta}_j) = 0$, the number of partitions goes to infinity. At the bottom it is always more lucrative to wait a bit more and not accept the lower types because they yield utility going to zero. The proof, however, is beyond the purpose of this article.

The uniqueness result is independent of both the specification of the payoff function and whether both sexes have the same payoff functions. The proof for existence and uniqueness uses a generalized iterated strict dominance argument, as discussed for the case of multiplicatively separable utility functions. Imagine, for example, that the reservation value is increasing in type when accepted by all other types, as is the case in Figure 1. All strategies above this schedule are dominated. It follows that there exists a pair (θ_i^*, θ_j^*) above which all types are accepted with certainty. Hence all types above (θ_i^*, θ_j^*) have a unique iterated strict dominant strategy. Taking these dominant strategies as given, all types below will revise their upper bound above which all strategies are dominated, so they choose a new reservation value below the

[6] The partitioning result, obtained in different frameworks, has always been derived for a special case of the type-independent utility function: $u_i = \theta_j$. See McNamara and Collins (1990), Bloch and Ryder (1995), and Burdett and Coles (1997). The exception is Smith (1996), who looks at multiplicative payoffs and derives a similar result to the one in Proposition 2.

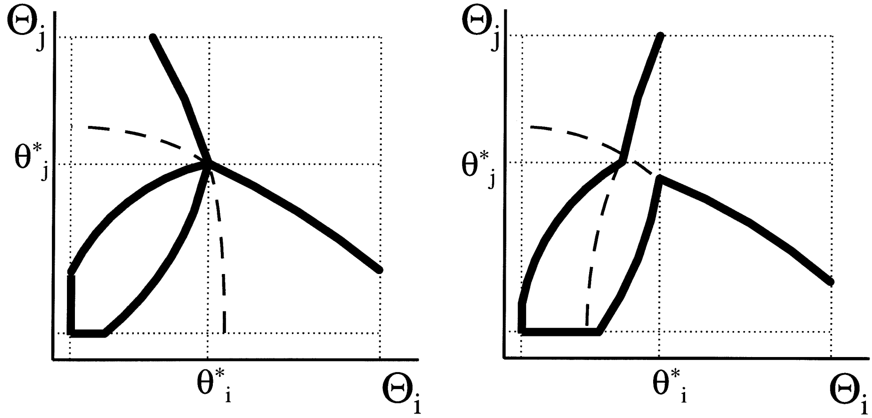

FIGURE 3

DOWNWARD-SLOPING RESERVATION STRATEGIES

dashed line. A new pair (θ_i^*, θ_j^*) exists for which there is now an iterated dominant strategy. This can then be repeated a finite number of times.

The proof for the case where the reservation schedule is decreasing in type needs an additional feature. Consider Figure 3.

In the panel on the left, the case is depicted for reservation schedules decreasing in type, e.g., for utility functions like $u_i = \theta_j + \theta_i$. The dashed line is the (decreasing) reservation value conditional on acceptance by all types of the other sex. All reservation values above this schedule are strictly dominated. At the intersection of the two dashed schedules, the pair (θ_i^*, θ_j^*) is defined. All types above have a dominant strategy, given by the fat line and equal to the dashed line. The types immediately below (θ_i^*, θ_j^*) are now accepted by some types above their first reservation schedule but not by all. Hence they will revise their upper bound downward. However, given acceptance by some, it is shown that now they also will have a lower bound. This holds for both sexes. Given the lower bound of the other sex, they will revise their upper bound, and given the upper bound of the other sex, they will revise the lower bound. As in the Cournot *tâtonnement* process, this goes on infinitely long until the unique reservation schedule is determined.

The panel on the right in Figure 3 is merely a variation on the same theme. One schedule is upward-sloping; the other, downward. Again, by eliminating dominated strategies starting from the top [i.e., above (θ_i^*, θ_j^*)], the whole schedule can be constructed uniquely.

REMARK 3. Figure 3 clearly illustrates that the slope of the schedule ϕ_j is not only a function of the utility function. It can be shown that for utility functions exhibiting log supermodularity (i.e., $u_{12}u > u_1u_2$), the reservation value, given acceptance by all, is increasing in type and decreasing if it is log submodular (see Smith, 1996). However, the equilibrium schedule is not necessarily downward-sloping over the whole range even if there is log submodularity but π_j is type-independent. This is, for example, the case at the upper part of the distribution. In Figure 3, even though in both cases at least one of the utility functions is log submodular, at the lower end the schedule is upward-sloping. The reason is that in that range, π_j is type-dependent.

5. PERFECT MATCHING EQUIVALENCE

In this section it is shown that the equilibrium is indeed the generalization of the perfect matching model. First, the perfect matching model is defined in more detail. Second, it is shown that the bilateral search model with vertical heterogeneity yields the same outcome as the perfect matching model when the search friction disappears in the limit. The search friction disappears when waiting time goes to zero, i.e., when the arrival rate β goes to infinity.

The perfect matching model used as the benchmark here is the model first discussed by Gale and Shapley (1962) and rigorously explained in Roth and Sotomayor (1990). Originally, it was formulated for a finite number of agents and for any set of preferences. Here, it will be extended to a continuum of agents, and the preferences will be such that they exhibit vertical heterogeneity, the Beckerian aspect (Becker, 1973–74). In what follows it will be referred to as the *Gale-Shapley-Becker model*. Consider two disjoint sets of agents Θ_i and Θ_j, both with mass one. Individuals are characterized by a type θ_i, cumulatively distributed over $F_i(\theta_i)$. Vertical heterogeneity of preferences can be represented by any utility function $u(\theta_j, \theta_i)$ as long as $\partial u/\partial \theta_j > 0$. A matching μ is defined as a one-to-one correspondence from $\Theta_i \cup \Theta_j$ onto itself of order two [i.e., $\mu^2(\theta_i) = \theta_i$] such that $\mu(\theta_i) \in \Theta_j$ and $\mu(\theta_j) \in \Theta_i$. A matching μ is individually rational if it is not blocked by any individual agent. It is stable if it is individually rational and if it is not blocked by any pair of agents, one female and one male. This establishes that a stable matching is a core concept and thus a cooperative equilibrium. It can be shown that there exists a unique stable matching $\mu(\theta_i) = \theta_j \Leftrightarrow F_i(\theta_i) = F_j(\theta_j)$: In equilibrium, only individuals of the same rank match.

The equivalence between perfect matching and search can now be established. Note, however, that there is an entirely different use of equilibrium concept: cooperative versus noncooperative equilibrium. What will be shown is that the noncooperative search equilibrium yields the same outcome as the cooperative stable matching without friction when the search friction is infinitely small (i.e., $\lim \beta \to \infty$). It actually can be shown that the stable matching is equivalent to the trembling hand equilibrium, which rules out degenerate noncooperative equilibria. Note also that the restriction to reservation strategies has a similar impact.

PROPOSITION 3 EQUIVALENCE: *The Gale-Shapley-Becker perfect matching model is the limit case of the search model when trading opportunities arrive instantaneously* (*i.e.*, $\lim \beta \to \infty$).

PROOF. For $\lim \beta \to \infty$, the system of Equations (1) and (2) collapses. The value of being single now coincides with the value of being matched, since a match is instantaneously realized. It follows that the value of being single has to equal the expected value of being matched: $V_0(\theta_i) = E[V_1(\theta_i) \mid \pi_j = 1]$. An individual θ_i will choose a reservation value ϕ_j such as to maximize the expected value of being matched subject to being accepted. This implies that

$$\max_{\phi_j} EV_1(\theta_i) = \frac{\int_{\Theta_j} \pi_i(x, \theta_i)\pi_j(\theta_j, x)u(x)\, dF_j(x)}{\int_{\Theta_j} \pi_i(x, \theta_i)\pi_j(\theta_j, x)\, dF_j(x)} \tag{8}$$

For a given π_j, EV_1 is monotonically increasing in ϕ_j, as long as $\pi_j = 1$. This follows immediately from Equation (8) and the fact that $\partial u/\partial\theta_j > 0$. The solution to the maximization problem is a corner solution: EV_1 is maximized when ϕ_j is maximized, such that $\pi_j = 1$. With $\phi'_j = \max\{\theta_j \in \Theta_j \mid \mu_i(\theta_j) = 1\}$, the optimal choice of ϕ_j is $\phi_j(\theta_i) = \phi'_j(\theta_i)$, $\forall\theta_i$. Likewise, $\phi_i(\theta_j) = \phi'_i(\theta_j)$, $\forall\theta_j$. Applying the algorithm in the proof of Proposition 1 then gives the following allocation: A type θ_i will match with θ_j if and only if $F_i(\theta_i) = F_j(\theta_j)$. This is equivalent to the concept of stability in perfect matching μ: $\mu(\theta_i) = \theta_j \Leftrightarrow F_i(\theta_i) = F_j(\theta_j)$. ■

6. CONCLUDING REMARKS

In the tradition of the long-standing line of research in perfect matching models (Gale and Shapley, 1962), this article has introduced frictions by considering a search model with nontransferable utility. As is the case in this literature, the main objective has been to solve for the equilibrium allocation. In this article, a general algorithm is provided to find this allocation. Existence and uniqueness are shown. The equilibrium concept is iterated elimination of dominated strategies, for which there are strong behavioral foundations. In addition, the search model is robust with the perfect matching model as soon as frictions disappear.

Nonetheless, the characterization of the allocation gives rise to surprising insights. For multiplicatively separable utility functions, the distribution of types is partitioned, and for a wide class of preferences, disconnected matching sets arise naturally.

As is the case with the perfect matching models, applying the results to a labor market environment should be done with extreme care. First, utility is nontransferable. There is no room for the endogenous determination of wages. Introducing some kind of wage bargaining once a match is materialized would be a necessary feature. Second, the way the model is set up implies that the value of being single is higher than the value of being matched. This arises because the utility of marriage is instantaneous and it is realized on engaging in the match. In a labor market, wages need to be modeled as continuous flow when matched. This would automatically imply that the value of being matched exceeds the value of being in the market.

Nonetheless, still a wide range of possible applications remain wherever there is endogenous partner formation, i.e., the allocation of pairs of heterogeneous agents. After all, the perfect matching literature started off with the example of allocating physicians to hospitals. The approach here provides new insights when matching is costly.

APPENDIX: ENDOGENOUS DISTRIBUTION OF SINGLES

Let $H_i(\theta_i)$ be the distribution of the entire population of sex i and $H_i^s(\theta_i)$ be the distribution of singles. All these distributions can be time-variant. If n_i is the fraction of singles of type θ_i at a particular moment in time, the density function $h^s(\theta_i)$ of singles associated with the population density $h(\theta_i)$ is given by

$$h^s(\theta_i) = \frac{n_i}{\int_{\Theta_i} n_i dh(\theta_i)} h(\theta_i) \tag{A.1}$$

At any moment in time, the law of motion is given by $\dot{n}_i = \beta\psi_i n_i + \alpha(1 - n_i)$. Out of the steady state, the distribution of singles h^s changes over time, but in a steady state, $F = H_t^s$. This is also true out of steady state if players do not hold rational expectations and believe that the observed distribution will not change over time.[7] If agents hold full rational expectations, they will take into account the change in the distribution over the expected duration $(\beta\psi_i)^{-1}$ of being single. The belief about the distribution of singles then satisfies $F(\theta_i) = \int_0^{(\beta\psi_i)^{-1}} H_t^s(\theta_i)\,dt$.

Endogenizing the distribution of singles leaves the existence and uniqueness of the allocation intact (although there is a new source of multiplicity). In addition, the characterization of equilibria for given preferences and the perfect matching equivalence still hold.

REMARK A.1. For the remainder of the proofs, the following notation is used.

- $\pi_i^1(\theta_k) \geq \pi_i^2(\theta_k)$, $k \in \{i, j\}$ means that for a given θ_{-k}, $\pi_i^1(\theta_j, \theta_i) \geq \pi_i^2(\theta_j, \theta_i)$, $\forall\theta_k$ and with strict inequality for some k with positive mass.
- $\pi_i^1(\theta_k) = \pi_i^2(\theta_k)$ if for a given θ_{-k}, $\pi_i^1(\theta_j, \theta_i) = \pi_i^2(\theta_j, \theta_i)$, $\forall\theta_k$.
- $(\theta_i^1, \theta_j^1) < (\theta_i^2, \theta_j^2)$ if at least one of the following two equations holds with strict inequality: $\theta_i^1 < \theta_i^2$ or $\theta_j^1 < \theta_j^2$.
- $(\theta_i^1, \theta_j^1) = (\theta_i^2, \theta_j^2)$ if both $\theta_i^1 = \theta_i^2$ and $\theta_j^1 = \theta_j^2$.

REMARK A.2. With every value of $\phi_j(\theta_i)$, there is associated a value $\pi_i(\theta_j, \theta_i)$. It follows that the whole schedule $\phi_j(\theta_i)$, $\forall\theta_i$ is defined by $\pi_i(\theta_j, \theta_i)$. In terms of notation, $\pi_i = \tau_i(\pi_j)$ is the reaction function τ_i yielding the unique solution π_i for a given π_j. That is, for a given π_j, T_i is solved $\forall\theta_j$.

PROOF OF PROPOSITION 1. First, we proceed by proving Lemma A.1, which will be used in the argument of iterated elimination of dominated strategies. The main implication of this lemma is that the reservation strategy always has an upper bound.

LEMMA A.1. (*a*) *An optimizing agent will always accept matches within a range of agents with strictly positive mass*; (*b*) $\pi_j^1(\theta_j) \geq \pi_j^2(\theta_j)$ *implies that* $\phi_j^1 > \phi_j^2$; *and* (*c*) $\pi_j(\theta_j) \geq 0$ *implies that there is an upper bound on the reservation value* ϕ_j.

[7] This corresponds to what is called a "partial rational expectations belief" in Burdett and Coles (1997).

PROOF. (a) A population with zero mass implies that $\gamma_i = 0$. Since $u(\theta_j) > 0$, $T_i(\phi_j) > 0$. $T_i(\phi_j) = 0$ can only be satisfied for some $\gamma_i = 0$. This implies accepting a population with strictly positive mass. This applies to all types of both sexes, since $u(\theta_j) > 0$ and $u(\theta_i) > 0$, $\forall \theta_i, \theta_j$.

(b) $\pi_j^1(\theta_j) \geq \pi_j^2(\theta_j)$ *ceteris paribus* implies that $\gamma_i^1 \geq \gamma_i^2$, by definition of γ_i. If ϕ_j^1 is the unique solution to $T_i(\phi_j^1 \mid \pi_i^1) = 0$, i.e., $T_i(\phi_j^1) = 0$ given π_j^1, then it follows that $T_i(\phi_j^1 \mid \pi_j^2) > 0$, since $T_\phi > 0$. The unique solution to $T_i(\phi_j^2 \mid \pi_j^2) = 0$ then satisfies $\phi_j^1 > \phi_j^2$.

(c) For $\pi_i(\theta_j) \geq 0$, $\gamma_i > 0$. Since $T_\phi > 0$, a decreasing ϕ_j implies a decreasing $T(\phi_j)$. As a result, there will exist a value X satisfying $u(X) > 0$ such that $T(X) < 0$. No agent will choose such a reservation strategy. Hence there is a lower bound X^* where $T(X^*) = 0$. If $X^* \notin \Theta_j$, $X^* = \{\min \theta_j \in \Theta_j \mid \theta_j > X, T(X) = 0\}$. ■

Iterated elimination implies n iterations. Therefore, the following notation is introduced. First, because of the argument of iterated elimination, the variable $\mu_i(\theta_j, \theta_i) = \pi_j(\theta_i, \theta_j)$ is introduced in order to distinguish the acceptance rule by others from the strategy by other players. Clearly, in equilibrium they are the same. $\Pi_i(\theta_j, \theta_i; n) = \Pi_i(n)$, $\forall i, j$ is the schedule $\pi_i(\theta_j, \theta_i)$ calculated in iteration n, provided $\mu_i = 1$. $\phi_j(\theta_i; n)$ is the reservation value associated with $\Pi_i(\theta_j, \theta_i; n)$, provided $\mu_i = 1$. Likewise, $\mu_i(n)$ is $\mu_i(\theta_j, \theta_i)$, $\forall i, j$ in iteration n.

In each iteration n, the algorithm below will allow one to determine the unique strategies for a connected set with positive mass. Given the outcome of the anterior iterations that all types $(\theta_i, \theta_j) \geq [\theta_i^*(n-1), \theta_j^*(n-1)]$ have determined their unique strategy, the nth iteration starts. It consists of five steps (below). It can be established that there exists a set of dominated strategies (i.e., there is a maximum reservation value) for all remaining types of both sexes. These are determined in steps 1 through 3. These dominated strategies imply that all types of the other sex higher than the reservation value will never be rejected. In step 4, the connected set of all types is determined that will never be rejected irrespective of other players' strategies, which is the result of the other sex's dominated strategies. It follows that all these types have a unique strategy (step 5) that is the result of iterated elimination of dominated strategies within this iteration. If the connected set is empty, the unique strategy for a strictly positive connected set is determined according to Lemma A.2. As a result, after the nth iteration, all $(\theta_i, \theta_j) \geq [\theta_i^*(n), \theta_j^*(n)]$ have a unique iterated strict dominant strategy.

1. After $n-1$ iterations, the schedules $\Pi_i(n-1)$ and $\Pi_j(n-1)$ are uniquely determined for all $(\theta_i, \theta_j) \geq [\theta_i^*(n-1), \theta_j^*(n-1)]$. It follows that in the next iteration the schedules $\mu_i(n)$ and $\mu_j(n)$ are uniquely defined in that range. For the other types, maximal acceptance [i.e., $\mu(n) = 1$] allows one to determine the dominated strategies. Hence, determine $\mu_i(n) = \Pi_j(n-1)$ if $\theta_j > \theta_j^*(n-1)$; $\mu_i(n) = 1$ otherwise. Likewise, $\mu_j(n) = \Pi_i(n-1)$ if $\theta_i > \theta_i^*(n-1)$; $\mu_j(n) = 1$ otherwise. At the start of the procedure $(n = 1)$, $\mu_i(n) = 1$, $\forall \theta_j$ and $\mu_j(n) = 1$, $\forall \theta_i$.
2. $\Pi_i(n) = \tau_i[\mu_i(n)]$ and $\Pi_j(n) = \tau_j[\mu_j(n)]$

3. Consider all types $(\theta_i, \theta_j) < [\theta_i^*(n-1), \theta_j^*(n-1)]$. Taking into account the unique strategies of all higher types and by determining $\mu_i(n)$ and $\mu_j(n)$ in terms of maximal acceptance [i.e., from step 1, there exists no $\mu_i \geq \mu_i(n)$], it follows from Lemma 2 that all reservation strategies $\theta_j > \phi_j(\theta_i; n)$ are strictly dominated for all θ_i. From Lemma 1, $\Pi_i(n) = 1$ for all $\theta_j > \phi_j(\theta_i)$. Likewise, all reservation strategies $\theta_i > \phi_i(\theta_j; n)$ are strictly dominated for all θ_j and $\Pi_j(n) = 1$ for all $\theta_i > \phi_i(\theta_j; n)$.
4. Define

$$
\text{(A.2)}\qquad [\theta_i^*(n), \theta_j^*(n)] = \begin{cases} \min \theta_i, \min \theta_j) \mid \Pi_i(n)\Pi_j(n) = 1 \\ \forall \theta_i \leq \theta_i^*(n-1)\text{: } \Pi_j(n) = 1, \forall \theta_j \leq \theta_j^*(n-1) \\ \forall \theta_j \leq \theta_j^*(n-1)\text{: } \Pi_i(n) = 1, \forall \theta_i \leq \theta_i^*(n-1) \end{cases}
$$

In the first round ($n = 1$), define $\theta_i^*(0) = \bar{\theta}_i$ and $\theta_j^*(0) = \bar{\theta}_j$. Note that it follows from Equation (A.2) that $[\theta_i^*(n), \theta_i^*(n-1)]$ and $[\theta_j^*(n), \theta_j^*(n-1)]$ are connected sets. If $[\theta_i^*(n), \theta_j^*(n)] < [\theta_i^*(n-1), \theta_j^*(n-1)]$, at least one of these sets is nonempty. The unique iterated strict dominant strategies are determined in step 5. Alternatively, if $[\theta_i^*(n), \theta_j^*(n)] = [\theta_i^*(n-1), \theta_j^*(n-1)]$, the sets are empty, and the pair $[\theta_i^*(n), \theta_j^*(n)]$ and the unique iterated strict dominant strategies are determined according to Lemma A.2 below.
5. From step 3 and from Equation (A.2), all types $\theta_i \in [\theta_i^*(n), \theta_i^*(n-1)]$ and $\theta_j \in [\theta_j^*(n), \theta_j^*(n-1)]$ have $\mu_i(n) = \mu_j(n) = 1$ independently of any other player's strategy (because the strategies are dominated), since $\mu_i = \Pi_j$ and $\mu_j = \Pi_i$. The reservation strategy of all these types is thus independent of the strategy of any other player. By eliminating the dominated strategies, all these types have a unique iterated strict dominant strategy $\Pi_i(n)$ and $\Pi_j(n)$, respectively (from Lemma 2a).

   This iterative procedure is repeated until $\Pi_i(N) = \Pi_i(N+1)$ and $\Pi_j(N) = \Pi_j(N+1)$. Because $[\theta_i^*(n), \theta_j^*(n)] < [\theta_i^*(n-1), \theta_j^*(n-1)]$ and from Lemma A.1a, every agent chooses to accept matches from a population with strictly positive mass. As a result, the populations eliminating strictly dominated strategies in every iteration have strictly positive mass. It follows that the equilibrium list $[\Pi_i(N), \Pi_j(N)]$ is obtained after a finite number of N iterations. ■

LEMMA A.2. *If according to Equation* (A.1) $[\theta_i^*(n), \theta_j^*(n)] = [\theta_i^*(n-1), \theta_j^*(n-1)]$, *a new pair can be defined such that* $[\theta_i^*(n), \theta_j^*(n)] < [\theta_i^*(n-1), \theta_j^*(n-1)]$ *and such that there exists a unique iterated strict dominant strategy for all types in the interval* $\{[\theta_i^*(n), \theta_i^*(n-1)], [\theta_j^*(n), \theta_j^*(n-1)]\}$.

PROOF. First, $[\theta_i^*(n), \theta_j^*(n)] = [\theta_i^*(n-1), \theta_j^*(n-1)]$ implies that both $\mu_i(n) \geq 0$ for $\theta_j > \phi_j(n-1)$ and $\mu_j(n) \geq 0$ for $\theta_i > \phi_i(n-1)$. This follows from Lemma A.1a and A.1b. If it is not satisfied, say, for sex i, there would exist a range of dominated strategies with strictly positive mass below θ_j^* for the types of sex i. Hence the connected set would be nonempty, and the equality no longer holds. Therefore, the pair can be redefined such that the set is nonempty:

$$\text{(A.3)} \qquad \left[\theta_i^*(n), \theta_j^*(n)\right] = \begin{cases} \min\left\{\theta_i \mid \mu_i(n) \geq 0, \forall \theta_j > \phi_j(n-1)\right\} \\ \min\left\{\theta_j \mid \mu_j(n) \geq 0, \forall \theta_i > \phi_i(n-1)\right\} \end{cases}$$

The proof now is similar in spirit to the proof of dominance solvability of the Cournot model in Gabay and Moulin (1980) and Moulin (1984) and involves a Cournot *tâtonnement* process. First, additional notation is introduced for this stage of the elimination process only. $\Pi_i(s \mid n)$ is the subiteration s that determines the schedule $\Pi_i(n)$. Elimination of strictly dominated strategies will occur by defining an upper bound and a lower bound in every subiteration s: $\Pi_i^u(s \mid n)$ and $\Pi_i^l(s \mid n)$. $\mu_i^u(s \mid n)$ and $\mu_i^l(s \mid n)$ are analogously defined. Likewise for individuals of type j. From Lemma A.1c and given $\mu_i(n) \geq 0$, a lower bound on the reservation strategy exists. Given $\mu_i^l(1 \mid n) = \mu_i(n-1)$ if $\theta_j \geq \theta_j^*(n-1)$ and $\mu_i^l(1 \mid n) = 0$ otherwise, $\Pi_i^l(1 \mid n) = \tau_i[\mu_i^l(1 \mid n)]$. All strategies $\pi_i \geq \Pi_i^l(1 \mid n)$ are strictly dominated. Likewise for $\Pi_j^l(1 \mid n)$. On the other hand, from Lemma 2, it can be established that for upper bounds $\Pi_i^u(1 \mid n) = \tau_i[\Pi_j^l(1 \mid n)]$ and $\Pi_j^u(1 \mid n) = \tau_j[\Pi_i^l(1 \mid n)]$, all strategies $\pi_i \leq \Pi_i^u(1 \mid n)$ and $\pi_j \leq \Pi_j^u(1 \mid n)$ are strictly dominated. In every following iteration, $\Pi_i^l(s \mid n) = \tau_i[\Pi_j^u(s-1 \mid n)]$ and $\Pi_j^l(s \mid n) = \tau_j[\Pi_i^u(s-1 \mid n)]$ are determined. From Lemma A.1c, all strategies $\pi_i \geq \Pi_i^l(s \mid n)$ and $\pi_j \geq \Pi_j^l(s \mid n)$ are strictly dominated. Likewise, all strategies $\pi_i \leq \Pi_i^u(s \mid n) = \tau_i[\Pi_j^l(s \mid n)]$ and $\pi_j \leq \Pi_j^u(s \mid n) = \tau_j[\Pi_i^l(s \mid n)]$ are strictly dominated. If this procedure is repeated ad infinitum, $\Pi_i^l(\infty \mid n)$ and $\Pi_i^u(\infty \mid n)$ will converge to $\Pi_i(n)$ and $\Pi_j^l(\infty \mid n)$ and $\Pi_j^u(\infty \mid n)$ to $\Pi_j(n)$: (1) $\Pi_i^l(\infty \mid n) \geq \Pi_i(n)$ and $\Pi_i^u(\infty \mid n) \leq \Pi_i(n)$; (2) $\Pi_i^l(\infty \mid n) = \tau_i[\Pi_j^u(\infty \mid n)] = \tau_i[\tau_j(\Pi_i^l(\infty \mid n)]$, which is possible only if $\Pi_i^l(\infty \mid n) = \Pi_i(n)$; (3) similarly for $\Pi_j^l(\infty \mid n) = \Pi_j(n)$. The same reasoning holds for $\Pi_i^u(\infty \mid n) = \Pi_i(n)$ and $\Pi_j^u(\infty \mid n) = \Pi_j(n)$.

There is a unique strategy $\Pi_i(n)$ and $\Pi_j(n)$ for all types $\theta_i \in [\theta_i^*(n), \theta_i^*(n-1)]$ and $\theta_j \in [\theta_j^*(n), \theta_j^*(n-1)]$. Hence the pair $[\theta_i^*(n), \theta_j^*(n)]$ is defined as in Equation (A2). ■

REFERENCES

BECKER, G., "A Theory of Marriage: Part I," *Journal of Political Economy* 81 (1973), 813–46.

———, "A Theory of Marriage: Part II," *Journal of Political Economy* 82(2 pt. 2) (1974), S11–26.

BLOCH, F., AND H. RYDER, "Two-Sided Search, Marriages and Matchmakers," mimeo, HEC, September 1995.

BURDETT, K., AND M. COLES, "Marriage and Class," *Quarterly Journal of Economics* 112(1) (1997), 141–68.

———, "Aggregate Demand Management in Search Equilibrium," *Journal of Political Economy* 90 (1982), 881–94.

GABAY, D., AND H. MOULIN, "On the Uniqueness of Nash Equilibrium in Noncooperative Games," in Bensoussan, Kleindorfer, and Tapien, eds., *Applied Stochastic Control in Econometric and Management Science* (Amsterdam: North Holland, 1980).

GALE, D. AND L. SHAPLEY, "College Admission and the Stability of Marriage," *American Mathematical Monthly* 69 (1962), 9–15.

MCNAMARA, J., AND E. COLLINS, "The Job Search Problem as an Employer-Candidate Game," *Journal of Applied Probability* 28 (1990), 815–27.

MOULIN, H., "Dominance Solvability and Cournot Stability," *Mathematical Social Sciences* 7 (1984), 83–102.

ROTH, A. AND M. SOTOMAYOR, *Two-Sided Matching. A Study in Game-Theoretic Modelling and Analysis* (Cambridge UK: Cambridge University Press, 1990).

SMITH, L., "The Marriage Model with Search Frictions," Massachusetts Institute of Technology mimeo, February 1996.