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On the uniqueness of stable marriage matchings

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Abstract

A sufficient condition for uniqueness is identified on the preferences in the marriage problem, i.e. two-sided one-to-one matching with non transferable utility. For small economies this condition is also necessary. This class of preferences is broad and they are of particular relevance in economic applications.

Keywords: Marriage problem; Matching; Uniqueness

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1. Introduction

The marriage problem introduced by Gale and Shapley (1962) is a decision problem that studies the allocation of agents from two disjoint sets, males and females. Gale and Shapley solved the problem showing that for any type of preferences of males over females and vice versa, a stable matching equilibrium exists. The basic existence result has been extended to a whole set of new findings, an extensive review of which is found in Roth and Sotomayor (1990). While existence has been widely studied, less attention has been devoted to the uniqueness of the stable marriage matching. This short paper derives a condition on the preference ordering that yields uniqueness of equilibrium. This result is useful for two reasons. First, understanding of uniqueness is desirable when models are used as a tool for representing economic environments[1]. Second, it turns out that the preferences for which the equilibrium is unique are of appealing economic relevance: the set is broad and it contains those preference orderings that are commonly assumed.

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[1]A substantial part of the non-cooperative game theory has been devoted precisely to deriving uniqueness results. Milgrom and Roberts (1994) provide a method to extend uniqueness results from one model to another one with a suitable transformation of the strategy spaces.

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In recent research, the marriage problem has been used to explain matching of differentiated agents, in the context of savings (see Cole et al., 1992) and in matching models with friction (Smith, 1997a,b; Burdett and Coles, 1997; Eeckhout, 1999; Burdett and Wright, 1998; Kiyotaki and Wright, 1989). All these models have in common that, for the preferences they define, the stable marriage matching is unique.[2]

In this paper, the class of preferences is identified for which the set of stable matchings is the singleton. By representing each of the sets of males and females as an ordered set, the preferences require that no male or female prefers the mate of the opposite sex with the same rank order below his or her own order. Given such a preference ordering, by a recursive argument starting at the highest ranked mates, any other stable matching would be blocked by the identity matching, i.e. the matching where mates of the same rank order match.

Preference orderings that satisfy this condition are vertical heterogeneity and horizontal heterogeneity. *Vertical heterogeneity* implies that all agents of the same sex have identical preferences over the mates of the opposite sex, i.e. there is a ranking over the mates. This is the standard assumption of identical preferences with different endowments (as in Cole et al., 1992; Smith, 1997a; Burdett and Coles, 1997; Eeckhout, 1999). The endowment in the marriage model is the desirability by the opposite sex. *Horizontal heterogeneity* implies that each agent has a different most preferred mate (i.e. there is no common rank order over the mates). This type of preference can be thought of as two concentric circles (one for each sex) along which agents are distributed. The preferences are such that each prefers a mate that is closer by (this is the case in Smith, 1997b; Burdett and Wright, 1998; Kiyotaki and Wright, 1989).

2. The marriage problem

Consider two finite and disjoint sets of agents: the set of females $\mathscr{F}$ and the set of males $\mathscr{M}$. The typical agent in $\mathscr{F}$ is indicated by A, B, C, X, Y, Z and likewise in $\mathscr{M}$ by a, b, c, x, y, z. Each set is of equal size and the total (even) number of agents is $\#(\mathscr{F} \cup \mathscr{M}) = N$. When referring to ordered sets, we denote $\mathscr{F} = (X_i)$ and $\mathscr{M} = (x_i)$, $i \in I$, where I is the partially ordered set $(1, 2, \ldots, N/2)$ with the order relation $\geq$ a linear order[3].

All males have preferences over the females and females have preferences over the males. Preferences are, as usual, assumed to be complete, reflexive and transitive. We will consider strict preferences only. If female X prefers being married to x than to y, we denote $x \succ_X y$. Likewise for a male x who prefers X to Y: $X \succ_x Y$. All agents of the other sex are *acceptable*, i.e. all agents prefer being matched to any agent of the opposite sex rather than being single (i.e. $x_i \succ_X X$, $\forall x_i$). The preference ordering $\succ_x$ of male x over all females will also be denoted by the ordered set $(A, B, C, \ldots, Z)_x$. Similar for the females. The preference profile $(\succ_k)_{k \in \mathscr{F} \cup \mathscr{M}}$ is the list of preferences $\succ_k$ of all agents. All agents have perfect information about the preferences and there are no frictions in the market (i.e. any pair can be formed at no cost).

[2] That does not imply that the equilibria of those models are unique. These models solve for games in which the marriage problem is merely one stage in a larger extensive form game, or a particular case when frictions disappear for example.

[3] A linear order is a binary relation $\geq$ that is transitive, reflexive and total. An order relation is transitive if $x \geq y$ and $y \geq z$ implies $x \geq z$; reflexive if $x \geq x$; total if for any $x \neq y$, either $x \geq y$ or $y \geq x$ but not both.

The one-to-one matching problem is the three-tuple $(\mathcal{F}, \mathcal{M}, \succ)$ that consists of a set of females and males, and a preference profile. An allocation is a matching of females and males such that each agent is paired with at most one agent of the other set.

Definition 1. A matching μ is a one-to-one correspondence from $\mathcal{F} \cup \mathcal{M}$ onto itself of order two $(\mu^2(x)=x)$ such that $\mu(x) \in \mathcal{F}$ and $\mu(X) \in \mathcal{M}$.

All matchings μ are *individually rational* as all matchings are acceptable. A matching μ is blocked by two agents (X, x) if they are not assigned to each other and each of them prefers being matched to each other than to the partner assigned under μ: $\{X, x\} \in \mathcal{F} \times \mathcal{M}$ block μ if $x \succ_X \mu(X)$ and $X \succ_x \mu(x)$. We are interested in stable matchings.

Definition 2. A matching μ is *stable* if it is individually rational and if it is not blocked by any pair of agents $(X, x) \in \mathcal{F} \times \mathcal{M}$. We denote the set of stable matchings μ for a given marriage problem $(\mathcal{F}, \mathcal{M}, \succ)$ by $\mathcal{S}$.

Gale and Shapley (1962) show that for any problem $(\mathcal{F}, \mathcal{M}, \succ)$, a stable matching exists. In general however, the stable matching is not unique. It is shown in Roth and Sotomayor (1990) that for any problem $(\mathcal{F}, \mathcal{M}, \succ)$, the set of stable matchings $\mathcal{S}$ is a lattice under the partial orders for males and females: i.e. there exists a *male-optimal* stable matching μ^M that, out of all $\mu \in \mathcal{S}$, is preferred by all males $x \in \mathcal{M}$. Likewise, there exists a *female-optimal* stable matching μ^F.

Given a stable matching always exists, here is an example to illustrate that in general $\mathcal{S}$ is not the singleton.

Example 1. Consider $\mathcal{F} \cup \mathcal{M} = \{A, B, C, a, b, c\}$ and the preference profile

$$(a, b, c)_A, (b, a, c)_B, (a, b, c)_C$$

$$(B, A, C)_a, (A, B, C)_b, (A, B, C)_c$$

This marriage problem has two stable matchings $\mathcal{S} = \{\mu^F, \mu^M\}$, $\mu^F(A, B, C) = (a, b, c)$ and $\mu^F(A, B, C) = (b, a, c)$ (where $\mu(A, B, C) = (a, b, c)$ denotes: $\mu(A) = a$ and $\mu(B) = b$ and $\mu(C) = c$).

In the next section, we derive a sufficient condition on the preferences in order for $\mathcal{S}$ to be the singleton. We also investigate for which class of marriage problems this condition is necessary. When denoting the unique stable matching by μ^*, it immediately follows from $\mathcal{S} = \{\mu^*\}$ the singleton, that $\mu^M = \mu^F = \mu^*$.

3. Uniqueness

We can now state the main theorem.

Theorem 1. *Consider two ordered sets $\mathcal{F} = (X_i)$ and $\mathcal{M} = (x_i)$. If the preference profile satisfies*

$$\begin{aligned} &\forall X_i \in \mathscr{F}: x_i \succ_{X_i} x_j, \quad \forall j > i \\ &\forall x_i \in \mathscr{M}: X_i \succ_{x_i} X_j, \quad \forall j > i \end{aligned} \tag{1}$$

then there is a unique stable matching $\mu^*(X_i) = x_i$, $\forall i \in (1, 2, \ldots, N/2)$.

Proof. Consider a preference profile satisfying (1). Suppose now there exists a stable matching μ' different from μ^* with for some i $\mu'(X_i) = x_k$, $k \neq i$. Given the definition of a stable matching and given acceptability there must also exist some $j \neq k$ such that $\mu'(x_j) = X_l$, $l \neq j$. Let

$$\lambda = \min\{i\colon \mu'(X_i) = x_k, k \neq i\}$$

and

$$\gamma = \min\{j\colon \mu'(x_j) = X_l, l \neq j\}$$

Since $\mu^*(X_\lambda) = x_\gamma$, it follows that $\lambda = \gamma$.

Then $\mu'(X_\lambda) = x_k$ implies $\lambda < k$. Likewise, $\mu'(x_\lambda) = X_l$ implies $\lambda < l$. Given preferences (1), it follows that

$$x_\lambda \succ_{X_\lambda} x_k$$

$$X_\lambda \succ_{x_\lambda} X_l$$

As a consequence, X_λ and x_λ form a blocking pair and hence μ' is not a stable match. A contradiction. Since a stable matching always exists (Gale and Shapley, 1962) $\mathscr{S} = \{\mu^*\}$ is the singleton where $\mu^*(X_i) = x_i$, $\forall i \in (1, 2, \ldots, N/2)$. □

The intuition behind the proof is straightforward. Suppose each individual can be given a rank i. The proof can be seen as some recursive elimination process. The preferences are defined such that for $i = 1 = \min I$, both the male and the female must prefer each other above anyone else. Hence, they rank each other as highest. Clearly, that pair will always be a blocking pair unless they are matched with each other. Now consider both sexes with type $i = 2$. Given the preferences (1), they must prefer type 2 above all other types above all types $i > 2$. It may be that they most prefer the partner of type 1, but they can never be matched with type 1 since then both types 1 (i.e. each from the opposite sex) would form a blocking pair. Hence, both sexes of type 2 will be matched to each other. If each of them is matched to a type $j > 2$, together they will form a blocking pair. In general, any two partners with the same rank order i must prefer each other above any partner of a lower rank, given preferences (1). Since all types of a higher order i already form a blocking pair (from our recursive argument above), they also form a blocking pair if matched with a lower type. As long as there is no type of the other sex with a lower order which is preferred to the type with equal order, equilibrium is unique.

The preference ordering satisfying (1) can also be expressed in terms of a ring.

Definition 3. A (strict) ring is an ordered subset of males and females $(x_1, x_2, \ldots, x_j) \in \mathscr{F} \cup \mathscr{M}$ $(j \geq 3)$ such that

$$x_{i+1} \succ_{x_i} x_{i-1}, \quad \forall i = 1 \ldots j \text{ (modulo } j)$$

In what follows we refer to a strict ring when we mention ring. Note also that because agents do not have preferences over agents of the same sex, (*i*) the elements of a ring in the marriage problem necessarily alternate between males and females; and (*ii*) the number of elements in the ring is always even and no smaller than 4. For example, the ring (A, c, B, b) implies $B \succ_c A$, $b \succ_B c$, $A \succ_b B$, $c \succ_A b$.

Rings have been shown to play an important role in the existence of matching problems. Recently, Chung (1998) has shown that the absence of 'odd rings', i.e. rings such that the number of elements is odd, is a sufficient condition for showing the existence of a stable matching in the Roommate problem. The roommate problem is a many-to-one matching problem for which, contrary to the marriage problem, a stable matching does not always exist.

Now given the ordered sets $\mathscr{F} = (X_i)$ and $\mathscr{M} = (x_i)$, consider any ring of the following type: $(x_k, X_k, x_l, X_l, \ldots, x_r, X_r)$ or $(x_k, X_l, x_l, X_m, \ldots, x_r, X_k)$ where $k < l < m < \cdots < r$. We can now show that the preference profile does not contain any of the rings of that type.

Lemma 2. *Consider the ordered sets $\mathscr{F} = (X_i)$ and $\mathscr{M} = (x_i)$. The preference profile that satisfies* (1) *contains no ring of the type $(x_k, X_k, x_l, X_l, \ldots, x_r, X_r)$ or $(x_k, X_l, x_l, X_m, \ldots, x_r, X_k)$ where $k < l < m < \cdots < r$.*

Proof. In Appendix A. □

4. Vertical and horizontal heterogeneity

As Corollaries of Theorem 1, we now consider two particular preference profiles that generate a unique stable matching and that are commonly assumed in economic applications. The first is referred to as vertical heterogeneity. Let all females have identical preferences over the males, and all males have identical preferences over the females. There is a common (*objective*) ranking over the other sex. Even though preferences are identical, endowments (i.e. own rank in the other sex's preference order) are not.

Corollary 3. (Vertical heterogeneity) *Consider two ordered sets $\mathscr{F} = (X_i)$ and $\mathscr{M} = (x_i)$. If the preference profile satisfies*

$$\begin{aligned} &\forall X_i \in \mathscr{F}: x_k \succ_{X_i} x_j, \quad \forall k < j \\ &\forall x_i \in \mathscr{M}: X_k \succ_{x_i} X_j, \quad \forall k < j \end{aligned} \tag{2}$$

then there is a unique stable matching $\mu^*(X_i) = x_i$.

Proof. Immediate from Theorem 3 and (1). □

An interesting relation between vertical heterogeneity and uniqueness has been exploited for different purposes by Gusfield and Irving (1989). They study algorithms and the asymptotic time they need run before obtaining the stable matchings. In order to derive the lower bound on the running time, they show conditions for the women canonical preferences (i.e. vertical heterogeneity) that yield unique stable matchings.

When all agents have different preferences over the other sex, but each agent has a different most preferred mate and in addition is the most preferred by that mate, then the preference profile satisfies horizontal heterogeneity. There is a *subjective* ranking over the other sex. This is the case for example when agents can be represented in a metric space that constitutes two concentric circles, each sex on one circle. All agents prefer the mate on the other circle that is nearer to them to one who is at a further distance.

Corollary 4. (Horizontal heterogeneity) *Consider two ordered sets* $\mathscr{F}=(X_i)$ *and* $\mathscr{M}=(x_i)$. *If the preference profile satisfies*

$$\begin{aligned}&\forall X_i \in \mathscr{F}: x_i \succ_{X_i} x_j, \forall j\\&\forall x_i \in \mathscr{M}: X_i \succ_{x_i} X_j, \forall j\end{aligned} \tag{3}$$

then there is a unique stable matching $\mu^*(X_i)=x_i$.

Proof. Immediate from Theorem 1 and (1). □

These are two extreme cases of the preference profiles satisfying (1). In fact, one way is to consider (1) as a convex combination of (2) and (3).

5. Necessary condition

The preference ordering (1) is a sufficient condition for uniqueness. For small marriage problems, it is also a necessary condition: first this is shown for $N \le 4$ and then for $N \le 6$. It is not a necessary condition for large marriage problems, which is shown by counterexample for $N=8$.

Lemma 5. *For* $N=4$ *and given the ordered sets* $\mathscr{F}=\{X_i\}$ *and* $\mathscr{M}=\{x_i\}$. *If the set of stable matchings* $\mathscr{S}$ *is the singleton* $\mu^*(X_i)=x_i$, *then the preference profile satisfies* (1).

Proof. In Appendix A. □

Lemma 6. *For* $N=6$ *and given the ordered sets* $\mathscr{F}=\{X_i\}$ *and* $\mathscr{M}=\{x_i\}$. *If the set of stable matchings* $\mathscr{S}$ *is the singleton* $\mu^*(X_i)=x_i$, *then the preference profile satisfies* (1).

Proof. In Appendix A. □

One final result as a corollary to Lemma 6.

Corollary 7. *Let* $N=6$. *If the set of stable matchings* $\mathscr{S}$ *is the singleton, then at least one female and one male each get the most preferred partner.*

Proof. From Lemma 6, the preference profile satisfies (1). That implies that female X_1 and male x_1

respectively have the preference $x_1 \succ_{X_1} x_j$, $\forall j > 1$ and $X_1 \succ_{x_1} X_j$, $\forall j > 1$. Now the unique stable matching $x_i = \mu^*(X_i)$ implies $x_1 = \mu^*(X_1)$. $\square$

We now provide an example for $N = 8$ where condition (1) is not necessary[4].

Example 2. Consider the following preferences

$$(b, a, c, d)_A, (a, b, c, d)_B, (b, c, d, a)_C, (c, d, a, b)_D$$

$$(C, A, B, D)_a, (D, B, C, A)_b, (A, C, B, D)_c, (C, D, B, A)_d$$

The male optimal matching μ^M is equal to the female optimal matching μ^F where $\mu^M(A, B, C, D) = \mu^F(A, B, C, D) = (a, b, c, d)$. It is clear that these preferences do not satisfy (1). Whatever the ordered sets $\mathscr{F}$ and $\mathscr{M}$ (for example (A, B, C, D) and (a, b, c, d)) the condition is always violated ($C \succ_a A$ and $b \succ_A a$). Equivalently, there also exist rings (A, b, B, a) and (a, C, c, A).

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Appendix A

Proof of Lemma 2. Take any pair (i, j), and let $i < j$. (1) implies both $x_i \succ_{X_i} x_j$ and $X_i \succ_{x_i} X_j$. Then there may exist a Y such that (x_j, X_i, x_i, Y) is a ring or z such that (X_j, x_i, X_i, z) is a ring. The first ring implies $Y \succ_{x_i} X_i$. However, under (1) this requires $Y \neq X_j$. Likewise, (1) requires $z \neq x_j$. Since this applies to any (i, j), this implies there are no rings with 4 elements of the type (x_j, X_i, x_i, X_j) or (X_j, x_i, X_i, x_j).

Now take any n-tuple $(i, j, \ldots, l, k)$ and let $i < j < l < k$. Then (1) implies $x_i \succ_{X_i} x_j$ and $x_l \succ_{X_l} x_k$ and $X_i \succ_{x_i} X_j$ and $X_l \succ_{x_l} X_k$. Again there may exist a Y and a z such that $(x_k, X_i, x_i, \ldots, X_l, x_l, Y)$ and $(X_k, x_i, X_i, \ldots x_l, X_l, z)$ are rings. Under (1) this requires $Y \neq X_k$ and $z \neq x_k$. Hence there is no ring with $2n$ elements of the type $(x_k, X_i, x_i, \ldots, X_l, x_l, X_k)$ or $(X_k, x_i, X_i, \ldots, x_l, X_l, x_k)$. $\square$

Proof of Lemma 5. Let $\mu^*(A, B) = (a, b)$. There are two possible rings of the type (a, B, b, A) and (A, b, B, a). Now if $\mathscr{S}$ is the singleton, then $\mu'(A, B) = (b, a) \neq \mu^*$ will be blocked. This implies either:

1. that the pair (a, A) block μ', i.e. $a \succ_A b$ and $A \succ_a B$, so that the ordered sets (a, B, b, A) and (A, b, B, a) are not rings; or

[4] I am grateful to Ahmet Alkan for suggesting this example.

2. that the pair (b, B) block μ', i.e. $b \succ_B a$ and $B \succ_b A$, so that the ordered sets (a, B, b, A) and (A, b, B, a) are not rings. □

Proof of Lemma 6. We prove the Lemma by contradiction. Then, if (1) is not satisfied, there is at least one other equilibrium μ', either in addition to μ^* (hence no uniqueness) or μ^* is not an equilibrium (hence no uniqueness with $\mathcal{S} = \{\mu^*\}$).

For $N = 6$, (1) implies

$$a \succ_A b \text{ and } a \succ_A c \text{ and } A \succ_a B \text{ and } A \succ_a C$$

$$b \succ_B c \text{ and } B \succ_b C$$

If (1) does not hold, at least one of the conditions is violated.

If $b \succ_A a$ and/or $B \succ_a A$ and the remaining conditions of (1) then $\mu'(A, B, C) = (b, a, c) \in \mathcal{S}$.

If $c \succ_A a$ and/or $C \succ_a A$ and the remaining conditions of (1) then $\mu'(A, B, C) = (c, b, a) \in \mathcal{S}$.

If $c \succ_B b$ and/or $C \succ_b B$ and the remaining conditions of (1) then $\mu'(A, B, C) = (a, c, b) \in \mathcal{S}$.

If $b \succ_A a$ and $c \succ_B b$ and/or $B \succ_a A$ and $C \succ_b B$ and the remaining conditions of (1) then $\mu'(A, B, C) = (b, c, a) \in \mathcal{S}$.

If $c \succ_A a$ and $c \succ_B b$ and/or $C \succ_a A$ and $C \succ_b B$ and the remaining conditions of (1) then $\mu'(A, B, C) = (c, a, b) \in \mathcal{S}$.

If $b \succ_A a$ and $B \succ_a A$ and $c \succ_A a$ and $C \succ_a A$ and $c \succ_B b$ and $C \succ_b B$ then $\mu^* \not\in \mathcal{S}$.

This implies that $\mathcal{S} = \{\mu^*\}$ is not the unique equilibrium. □

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