Sorting and Decentralized Price Competition

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Motivation

- Role of prices in the classic assignment problem? Complementarities are common in:
  - labor market
  - business partnerships
  - product markets (car quality, driver's milage) ; (size of house, size of family)

- Becker (1973): competitive matching market
  - full information about prices and types, perfect trade
  - Concern: important trade imperfections (unemployment, waiting times)

- Shimer and Smith (2000): random search
  - no information about prices and types, imperfect trade
  - Concern: No information is a strong assumption

- Our approach: decentralized price competition
  - full information about prices and types, imperfect trade
    (e.g. due to mis-coodination)
    (competitive search / directed search)

Motivation

- We uncover a natural economic explanation for the forces that govern the matching patterns (when good types match with other good types?)
- Insights:
  - New condition for positive sorting (between Becker and Shimer-Smith)
  - New condition for negative sorting
  - Clear economic interpretation of the driving forces

MOTIVATION

- Two key aspects to matching:
  (1) The quality of the match ("match value motive"):

  (2) The probability (speed) of trade ("trading-security"):

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    (generalized: $1/(1-a)$ - *root-supermodularity*, where a is el. of subst. in matching)

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  –AM even with some supermodularity: nowhere *root*-sm

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Related Literature

Decentralized Price Competition

Peters (1984,1991,1997a,2000), Moen (1997), Acemoglu, Shimer (1999a,b), Burdett, Shi, Wright (2001), Shi (2001), Mortensen, Wright (2002), Rocheteau, Wright (2005), Galenianos, Kircher ('06), Kircher ('07), Delacroix, Shi ('06)

General Matching Function

Comp. Search (Moen (1997),...), Dir. Search (Menzio 2007)

Assortative Matching

Becker (1973) , Burdett, Coles (1997), Shimer, Smith (2000)

Competing Auctions - Ex post Screening

McAfee (1993), Peters (1997b), Shi (2002), Shimer (2005), Eeckhout and Kircher (2008)

THE MODEL

- *Players*
  - Measure $S(1)$ sellers: observable types $y \in [\underline{y}, \bar{y}]$ dist $S(y)$
  - Measure 1 buyers: private type $x \in [\underline{x}, \bar{x}]$ i.i.d. from $B(x)$
  - Unit demands and supplies

- *Payoffs* of trade between (x, y) at price p:
  - Buyer: utility $f(x, y) - p$
  - Seller: profit p
  - No trade: payoffs normalized to zero

THE MODEL

THE EXTENSIVE FORM

2 stage extensive form:

1. Sellers post prices: $G(y, p)$ seller distribution of (y, p)
2. Buyers observe G and choose y, p
   - $H(y, p)$ buyer distribution over (y, p).
   - If buyer meets such a seller, he gets the good and pays p

Matching Technology:

- Let λ be buyer-seller ratio (depends on (y, p))
- Matching prob.: Seller $m(\lambda)$; Buyer: $q(\lambda) = m(\lambda)/\lambda$
- $m, q \in [0, 1]$, $m' > 0$, $q' < 0$, $m'' < 0$

The Model

Matching Function

Interpretation of different $\lambda(y, p)$

1. anonymous strategies (buyer miscoordination)
2. spacial separation (Acemoglu 1997)
3. market makers providing trading platforms (Moen 1997)

Examples of Matching Function

1. anonymous strategies [urn-ball]: $m_1(\lambda) = 1 - e^{-\lambda}$
2. fraction $1 - \beta$ buyers get lost: $m_2(\lambda) = 1 - e^{-\beta\lambda}$
3. random on island [telegraph-line]: $m_3(\lambda) = \lambda/(1 + \lambda)$
4. CES: $m_4(\lambda) = (1 + k\lambda^{-r})^{-1/r}$

Number of matches: $M(b, s) = sM(\frac{b}{s}, 1) = sm(\lambda)$

PAYOFFS AND OPTIMAL DECISIONS GIVEN G AND H

- Queue length $\lambda(y,p)$ on equilibrium path (given G and H):

$$\int_{\mathcal{A}} \lambda(\cdot,\cdot) dG = \int_{\mathcal{A}} dH \quad \forall\, \mathcal{A} \subset \mathcal{Y} \times \mathcal{P},$$

- Stage 2: Buyer x obtains utility $U(x)$ according to

$$\max_{(y,p)\in \operatorname{supp} G \cup z} q(\lambda(y,p))(f(x,y)-p). \tag{1}$$

- Stage 1: Seller y optimizes according to

$$\max_{p\in\mathcal{P}} m(\lambda(y,p))p. \tag{2}$$

- Subgame Perfection "off-equilibrium-path"
  Acemoglu and Shimer (1999b): $\lambda(y,p)$ s.t.

$$\begin{aligned} U(x) &= q(\lambda(y,p))\,(f(x,y)-p) \text{ for some } x \\ U(x) &\geq q(\lambda(y,p))\,(f(x,y)-p) \text{ for all } x \end{aligned}$$

EQUILIBRIUM

Recall:

(1) Buyer's Problem: $\max_{(y,p)\in \mathrm{supp} G \cup z} \; q(\lambda(y,p))(f(x,y)-p)$

(2) Seller's Problem: $\max_{p\in\mathcal{P}} m(\lambda(y,p))p$

DEFINITION

An equilibrium is a pair $(G^\star, H^\star)$ that have full measure and for all measurable subsets $\mathcal{A}$ of the quality-price space $\mathcal{Y}\times\mathcal{P}\cup z$:

Sellers: $G^\star(\mathcal{A}) \leq S(y\in\mathcal{Y} \mid \exists\, p$ that solves (2) and $(y,p)\in\mathcal{A})$

Buyers: $H^\star(\mathcal{A}) \leq B(x\in\mathcal{X} \mid \exists\, (y,p)$ that solves (1) and $(y,p)\in\mathcal{A})$.

Definition (Assignment Function)

$\mu(y) \in \mathcal{X}$: buyer type that wants to trade with seller y

Assortative Matching

- $\mu'(y) > 0$: Positive Assortative Matching (+AM) (for matched types)
- $\mu'(y) < 0$: Negative Assortative Matching (–AM) (for matched types)

Assortative Matching

Root-Supermodularity

Definition

A function $f(x, y)$ is:

Supermodular $\quad \frac{\partial^2 f(x,y)}{\partial x \partial y} > 0 \quad \Leftrightarrow f_{xy}(x, y) > 0$

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Log-supermodular	$\frac{\partial^2 \log f(x,y)}{\partial x \partial y} > 0$	$\Leftrightarrow f_{xy}(x, y) > \frac{f_x(x,y) f_y(x,y)}{f(x,y)}$

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Root-supermodular	$\frac{\partial^2 \sqrt{f(x,y)}}{\partial x \partial y} > 0$	$\Leftrightarrow f_{xy}(x, y) > \frac{1}{2} \frac{f_x(x,y) f_y(x,y)}{f(x,y)}$

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n-root-supermodular	$\frac{\partial^2 \sqrt[n]{f(x,y)}}{\partial x \partial y} > 0$	$\Leftrightarrow f_{xy}(x, y) > \frac{n-1}{n} \frac{f_x(x,y) f_y(x,y)}{f(x,y)}$

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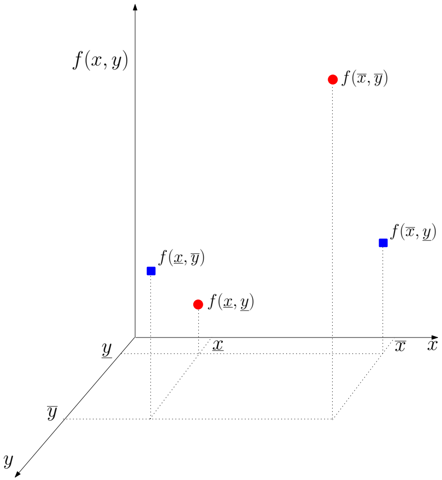
Extreme cases of n-root-supermodular:
$n = 1$: Supermodular; $n \to \infty$ log-supermodular

Assortative Matching

Log – Root – Supermodularity



Supermodular: $f_{xy} > 0$

Example $\quad f = (x + y)^{\alpha}$, $\alpha > 1$

ASSORTATIVE MATCHING

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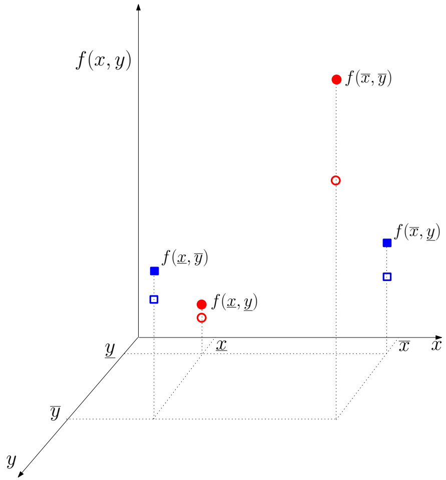
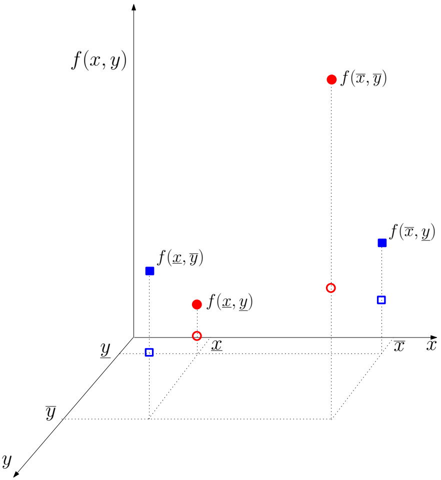


Supermodular: $f_{xy} > 0$ $\sqrt{f}$-sup.: $f_{xy} > \frac{1}{2} f_x f_y / f$

Example $f = (x + y)^{\alpha}, \alpha > 1$ $f = (x + y)^{\alpha}, \alpha > 2$

Assortative Matching

Log – Root – Supermodularity



Supermodular: $f_{xy} > 0$		$\sqrt{f}$-sup.: $f_{xy} > \frac{1}{2} f_x f_y / f$	$\log f$-sup.: $f_{xy} > 1 f_x f_y / f$
Example	$f = (x+y)^\alpha,\ \alpha > 1$	$f = (x+y)^\alpha,\ \alpha > 2$	$f = \beta^{x+y}$

ASSORTATIVE MATCHING

MAIN INSIGHTS

- *n*-root-supermod needed to overcome NAM ($n \in [0, 1]$)
- *n* equals elasticity of substitution in matching
- *n* results simple (efficiency) trade-off
  - complementarities in production
  - complementarities in search technology

Illustration of -AM

Private Values

Illustration of -AM

Private Values

1 . The quality of the match.
2 . The probability (speed) of trade.

ILLUSTRATION OF -AM

PRIVATE VALUES

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(e.g. opportunity cost to seller: $y = -c$)

- Frictionless: optimal assignment is indeterminate
(no "match value motive")

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  (no "match value motive")
- Frictions: Equilibrium is –AM
- High value buyer pays high p to avoid no-sale
  ("trading-security motive")
- Low type seller is more interested in price than prob.
  (so low seller types provide trading security for buyers)

Illustration of -AM

Private Values

- With private values: single crossing
- Buyers' indifference curves in 2-dimensional plane

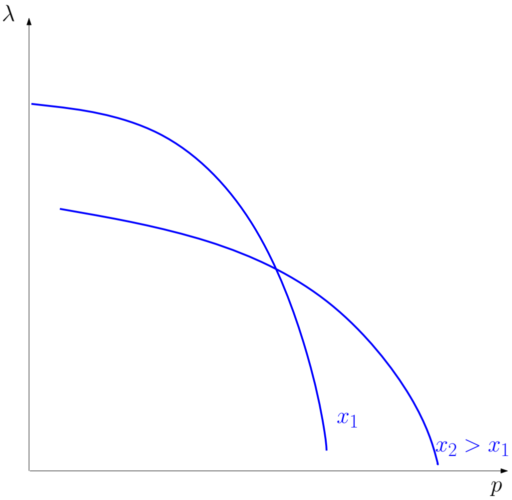


Illustration of -AM

Private Values

- With private values: single crossing
- Sellers' isoprofit curves in 2-dimensional plane

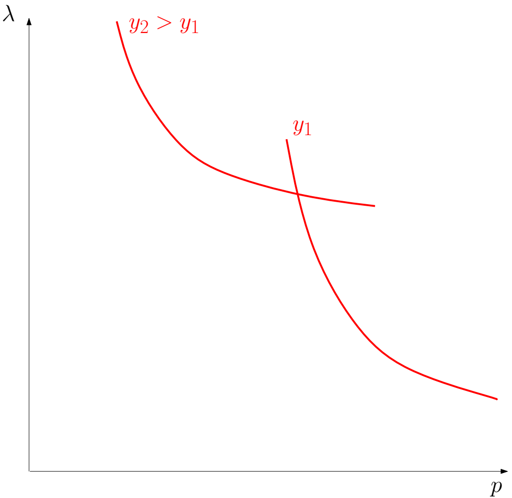
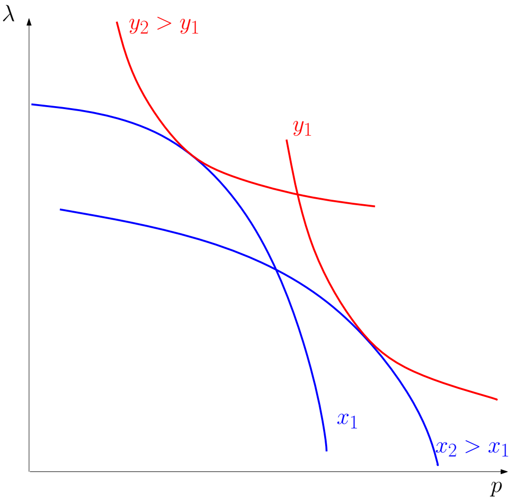


Illustration of -AM

Private Values

- With private values: single crossing
- –AM: High y_2 matches with low x_1



Assortative Matching

Main Theorems

There exist $\bar{n}$ and $\underline{n}$ in $[0, 1]$ such that

Theorem (+AM under $\bar{n}$-Root-Supermodularity)

+AM for all type distr. iff $f(x, y)$ is $\bar{n}$- root-supermodular.
-AM for all type distr. iff $f(x, y)$ is nowhere $\underline{n}$-root-supermod.

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The assortative assignment is constrained efficient.

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Theorem (Efficiency)

The assortative assignment is constrained efficient.

Proposition: q^{-1} *convex and derivatives bounded:*
+AM for all distr. iff $f(x, y)$ is square-root-supermodular.

Corollary: *-AM for all distr. if $f(x, y)$ is weakly submod.*

Proposition: If matching function is not CES
+AM for some distr. even if $f(x, y)$ not $\bar{n}$-root-supermod.

Proposition: If matching function is not CES
-AM for some distr. even if $f(x, y)$ is $\underline{n}$-root-supermod.

Positive Assortative Matching

Proof: +AM iff $f(x, y)$ $\bar{n}$-root-supermodular

Seller y:

$$max_{p \in \mathcal{P}} m(\lambda(p, y))p$$

where $\lambda(y, p)$, satisfies buyer optimization

$$\begin{aligned} U(x) &= q(\lambda(p, y))[f(x, y) - p(y)], \text{ for } x = \mu^\star(y) \\ U(x') &\geq q(\lambda(p, y))[f(x', y) - p(y)], \text{ for all } x' \end{aligned}$$

Seller y's problem is equivalent to (for any p attract x that gives highest possible λ; cf. Competing Mechanisms):

$$\max_{x,p,\lambda} \pi = m(\lambda)p$$

$$\text{s.t.} \qquad \frac{m(\lambda)}{\lambda}[f(x, y) - p] = U(x).$$

POSITIVE ASSORTATIVE MATCHING

PROOF: +AM IFF $f(x,y)$ $\bar{n}$-ROOT-SUPERMODULAR

After substituting the constraint:

$$\max_{x\in\mathcal{X},\lambda\geq 0} m(\lambda)f(x,y)-\lambda U(x).$$

First Order Conditions:

$$\begin{aligned} m'(\lambda)f(x,y)-U(x) &= 0 \\ m(\lambda)f_x(x,y)-\lambda U'(x) &= 0 \end{aligned}$$

Hessian for SOC:

$$\begin{pmatrix} m''(\lambda)f(x,\mu) & m'(\lambda)f_x(x,\mu)-U'(x) \\ m'(\lambda)f_x(x,\mu)-U'(x) & m(\lambda)f_{xx}(x,\mu)-\lambda U''(x) \end{pmatrix}.$$

Along Equilibrium Allocation:

$$\mu'\left[f_{xy}-\underbrace{\frac{m'(\lambda)q'(\lambda)}{q(\lambda)m''(\lambda)}}_{a(\lambda)}\frac{f_x(x,\mu)f_y(x,\mu)}{f(x,\mu)}\right]>0,$$

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After substituting the constraint:

$$\max_{x \in \mathcal{X}, \lambda \geq 0} m(\lambda) f(x, y) - \lambda U(x).$$

First Order Conditions:

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Hessian for SOC:

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$$\mu' \left[f_{xy} - \underbrace{\frac{m'(\lambda) q'(\lambda)}{q(\lambda) m''(\lambda)}}_{a(\lambda)} \frac{f_x(x, \mu) f_y(x, \mu)}{f(x, \mu)} \right] > 0,$$

Intuition and Explanation

What is $a(\lambda)$?

- It is the *elasticity of substitution* σ_M between buyers and sellers in the matching function $M(b, s) = sm(b/s)$.

$$a(\lambda) = \frac{M_b(\lambda, 1)M_s(\lambda, 1)}{M_{bs}(\lambda, 1)M(\lambda, 1)}$$

Why is it important?

- The Hosios' condition: entry of sellers into one (x, y) based on *derivative of matches with respect to sellers (M_s)*.
- Our condition connects different (x, y) combinations via the *elasticity of substitution between buyers and sellers (σ_M)*.

Interpretation in terms of "match value" and "trading security":

$$\underbrace{f_{xy}}_{\text{match value improvement}} - \underbrace{a(\cdot)f_x f_y/f}_{\text{loss due to no trade}} > 0$$

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$$\text{If } f(x,y) \text{ CRTS}: \quad \sigma_f^{-1} > \sigma_M \iff \sigma_f \cdot \sigma_M < 1$$

Positive Assortative Matching

Under Square-Root-Supermodularity

Assume q^{-1} convex, first and second derivatives bounded.

Proposition: PAM $\forall\ B, S \;\Leftrightarrow\; f$ is square-root-sm.

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Proposition: PAM $\forall\, B, S \;\Leftrightarrow\; f$ is square-root-sm.

$$f_{xy}(x,y) > a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)}, \quad a(\lambda) = \frac{m'(\lambda)q'(\lambda)}{q(\lambda)m''(\lambda)}$$

Necessary: +AM $\forall$ distr. $\Rightarrow$ Root-supermodularity

Reason: $a(0) = 1/2$, binding when some sellers cannot trade

$$\begin{aligned}
& q(\lambda) = m(\lambda)/\lambda \\
\Rightarrow\; & q'(\lambda) = (m'(\lambda) - q(\lambda))/\lambda \quad \text{bounded} \quad \Rightarrow m'(0) = q(0) \\
\Rightarrow\; & q''(\lambda) = (m'' - 2q')/\lambda \quad \text{bounded} \quad \Rightarrow q'(0) = m''(0)/2 \\
\Rightarrow\; & a(0) = m'(0)q'(0)/[m''(0)q(0)] = 1/2
\end{aligned}$$

Sufficient: Root-supermodularity $\Rightarrow$ +AM $\forall$ distr.

Reason: $a(\lambda) \leq 1/2$ if and only if $1/q(\lambda)$ is convex in λ.

NEGATIVE ASSORTATIVE MATCHING

OBTAINS ALWAYS UNDER WEAK SUBMODULARITY

$$f_{xy}(x,y) < a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)}, \quad a(\lambda) = \frac{m'(\lambda)q'(\lambda)}{q(\lambda)m''(\lambda)}$$

Sufficient: $f(x,y)$ weakly Sub-Mod $\Rightarrow$ -AM $\forall$ distr.

Reason: inequality always holds

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Necessary?

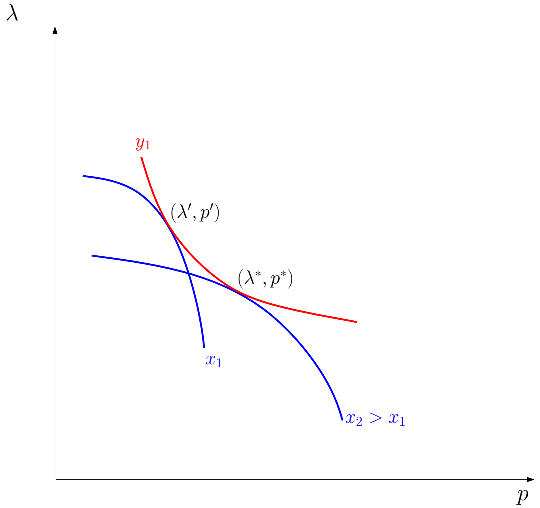
Yes for Urn-Ball (m_1): -AM $\forall$ distr. $\Rightarrow$ $f(x, y)$ weakly Sub-Mod

No for Telegraph-Line (m_5): nowhere Root-Sup-Mod $\Rightarrow$ -AM $\forall$ distr.

Assortative Matching

Graphical Interpretation

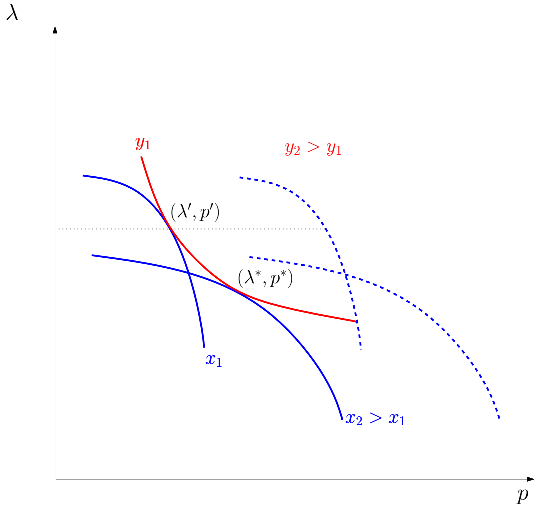
- IC in (λ, p, y), project in (λ, p) and vary y



Assortative Matching

Graphical Interpretation

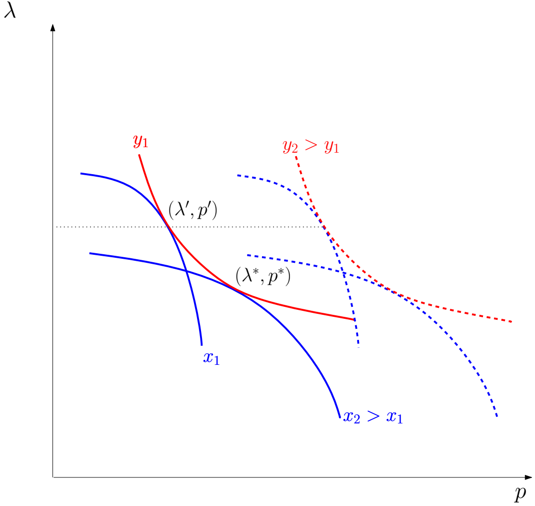
- Parallel shifts, identical distance when $f = x + y$



Assortative Matching

Graphical Interpretation

- Slope of iso-profit curve is flatter



ASSORTATIVE MATCHING

GRAPHICAL INTERPRETATION

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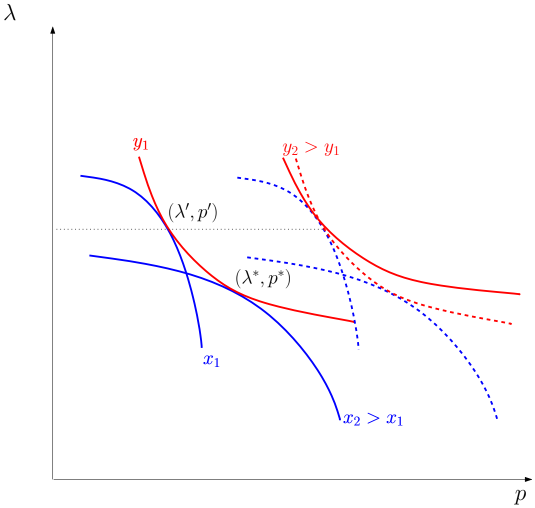
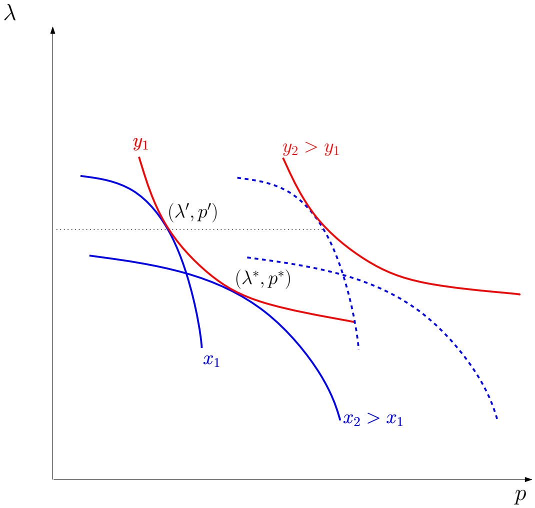


Assortative Matching

Graphical Interpretation

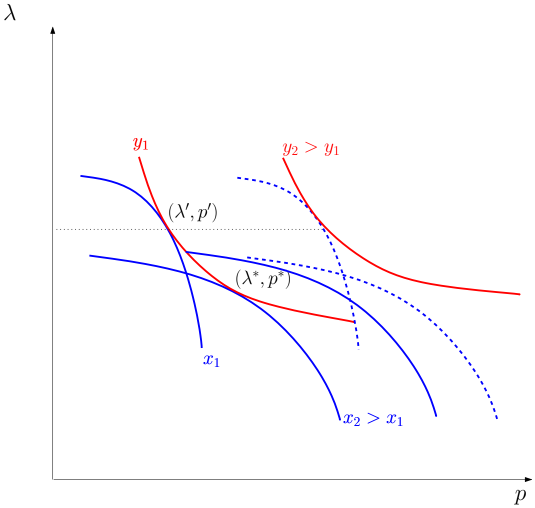
- High y_2 will match with low x_1



Assortative Matching

Graphical Interpretation

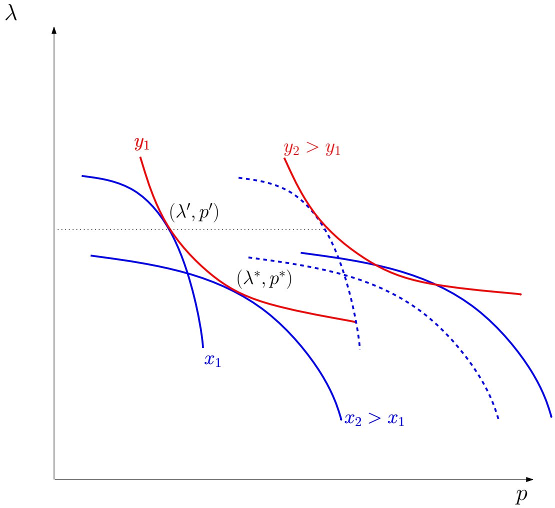
- High *x* IC moves less when submodularity



Assortative Matching

Graphical Interpretation

- Need root-supermodularity for IC to move "far enough"



Assortative Matching

Comparing Logs and Roots

Competition
supermodularity
$\Rightarrow$ +AM
submodularity
$\Rightarrow$ –AM

Dec. Price Comp
root-supermodularity
$\Rightarrow$ +AM
sub- and modularity
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Random Search
log-supermodularity
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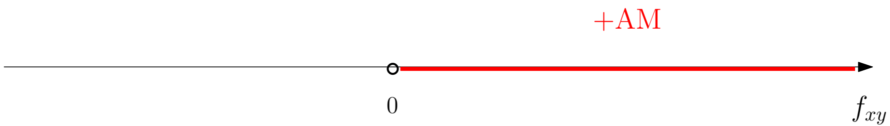

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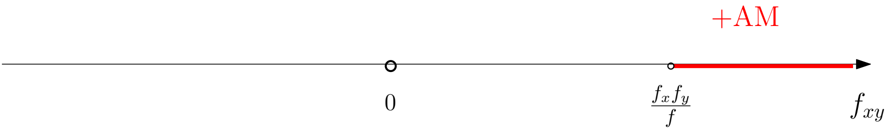

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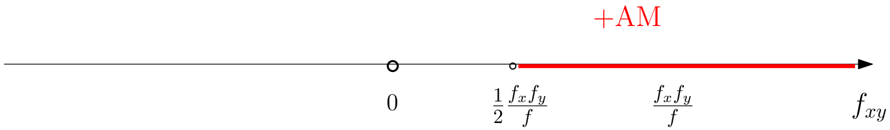

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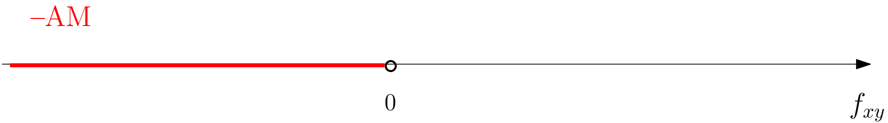

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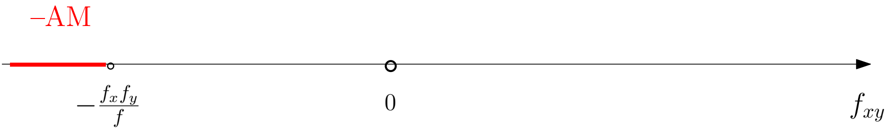

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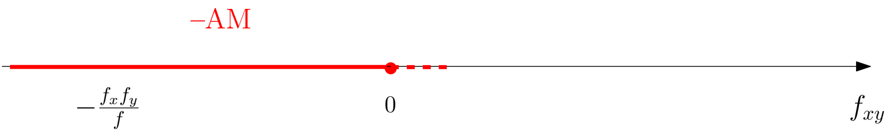

EXISTENCE

PROPOSITION

If $f(x, y)$ is $\bar{n}$-root-supermodular (or nowhere $\underline{n}$-rs), then there exists an equilibrium for all type distributions.

PROOF.

- construct equilibrium, monotonically increasing (+AM)
- solution to FOCs satisfies system of 2 differential equations in λ and μ with the appropriate boundary conditions
- verify SOCs along equilibrium allocation μ^*
- establish this is a global maximum by considering different solutions to the FOCs and showing that none other exist

□

Efficiency

+AM constrained efficient under root-supermodularity

Distribution for buyers: $D_b : \mathcal{X} \times \mathcal{Y} \to [0, 1]$
Distribution for sellers: $D_s : \mathcal{X} \times \mathcal{Y} \to [0, S(\bar{y})]$

Planner's program:

$$\max_{D_b, D_s, \lambda^P} \int m(\lambda^P(x, y)) f(x, y) dD_s$$

s.t. $\displaystyle\int_{\mathcal{A} \times \mathcal{Y}} dD_b \leq \int_{\mathcal{A}} dB \; \forall \mathcal{A} \subset \mathcal{X}$ and $\displaystyle\int_{\mathcal{X} \times \mathcal{A}} dD_s \leq \int_{\mathcal{A}} dS \; \forall \mathcal{A} \subset \mathcal{Y}$

$$\int_{\mathcal{A}} \lambda^P(\cdot, \cdot) dD_s \leq \int_{\mathcal{A}} dD_b \; \forall \mathcal{A} \subset \mathcal{X} \times \mathcal{Y}$$

Under our root-supermodularity conditions for PAM and NAM:

- solution coincides with decentralized equilibrium
- Hosio's per (x,y) market, Root-SM to connect them

Prices

The equilibrium price schedule under PAM satisfies

$$p'(y) = \underbrace{f_y}_{Becker(1973)} + \underbrace{\left(\eta_q f_x - \frac{b}{s}\eta_m f_y\right) a}_{Compensation\ through\ trading\ probabilities}$$

η_q elasticity of q (likewise for m), b/s density of buyers to density of sellers along equilibrium path

Insights:

1. Prices might be non-monotone
2. Sufficient condition for monotonicity: $b(x)/s(y) < 1 \ \forall\ x, y$
3. But: expected payoffs are monotonic: $U'(x) = qf_x > 0$

Extensions and Robustness

Entry of Firms

Entry at cost $C(y)$

Induces a particular type distribution. Combined with a particular matching function (urnball) Shi (2001) derives

$$\frac{ff_{xy}}{f_x f_y} > \frac{Cf_y(f_y - C_y)}{C_y(fC_y - Cf_y)}$$

No dependence on matching technology? Reconcile $RHS = a(\lambda)$? Economic Interpretation?

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Can be replicated by substituting free entry and Hosios into

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The general result highlights exactly the interplay between complementarities in production vs complementarities in matching (e.g. under CES RHS is constant).

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Can be replicated by substituting free entry and Hosios into

$$\frac{ff_{xy}}{f_x f_y} \geq a(\lambda(y)) \Leftrightarrow \frac{ff_{xy}}{f_x f_y} > -\ln(1 - \frac{C_y}{f_y})^{-1} + 1 - \frac{f_y}{C_y}$$

The general result highlights exactly the interplay between complementarities in production vs complementarities in matching (e.g. under CES RHS is constant).

Extensions and Robustness

The Class of CES Matching Functions

$$\begin{aligned} m(\lambda) &= (1 + k\lambda^{-r})^{-1/r} \\ [M(\beta, \sigma) &= (\beta^r + k\sigma^r)^{-1/r}\beta\sigma] \end{aligned}$$

$r > 0,\ k > 1,\ a(\lambda) = (1 + r)^{-1}$ constant

Proposition: Fix the type distributions. There is

- $+AM$ if f is n-root-supermodular; $(n = \frac{1+r}{r})$
- $-AM$ if f is nowhere n-root-supermodular; $(n = \frac{1+r}{r})$

Corollary: CES with elasticity e, then PAM under:

1. Supermodularity if $e = 0$ (Leontief);
2. Square-Root-Supermodularity if $e = \frac{1}{2}$ (Telegraph Line);
3. Log-Supermodularity if $e = 1$ (Cobb-Douglas).

EXTENSIONS AND ROBUSTNESS

GENERAL PAYOFFS & DYNAMIC FRAMEWORK

Dynamic Framework:

$$\max_{\lambda \in \overline{\mathbb{R}}_+} m(\lambda)\left[1-\delta\left(1-m(\lambda)\right)\right]^{-1} p$$

$$\text{s.t.} \qquad q(\lambda)\left[1-\delta\left(1-q(\lambda)\right)\right]^{-1}\left(f(x,y)-p\right) = U(x)$$

Necessary and sufficient condition for +AM:

$$f_{xy}(x,y) \geq A(\lambda,\delta)a(\lambda)\frac{f_x(x,y)f_y(x,y)}{f(x,y)}$$

where

1. $A(\lambda,\delta) \in [0,1]$
2. $\lim_{\lambda\to 0} A(\lambda,\delta) = 1$ for all $\delta \in [0,1)$,
3. $\lim_{\delta\to 1} A(\lambda,\delta) = 0$ for all $\lambda > 0$.

Extensions and Robustness

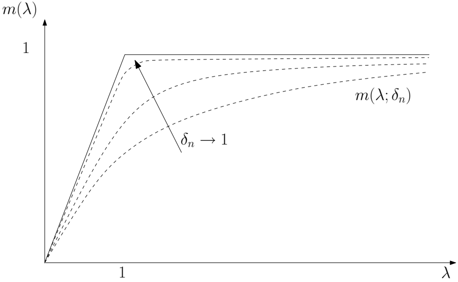
Vanishing Frictions

Two approaches to vanishing frictions:

over time $\delta \to 1$, or change in matching function

- root-supermodularity necessary for +AM for any frictions
- but necessary only at vanishing set of types

Illustration: changing matching function



CONCLUSION

- Complementarities are a source of high productivity in many environments (goods, labor, neighborhood,...)
- Imperfections in trade, but prices play allocative role
- Role of prices: ex-ante sorting, reduces frictions

- Highlights the interplay between frictions and match value:
  1. Match Value: tendency for +AM (if supermodular)
  2. Frictions: tendency for –AM (a-modular $\Rightarrow$ –AM)

- simple trade-off: Becker vs Elasticity in Matching
- root-supermodular: point where effect (1) outweighs (2)

Seller y:

$$max_{p \in \mathcal{P}} m(\lambda(p, y))p(y) \tag{3}$$

where $\lambda(y, p)$, satisfies buyer optimization

$$\begin{aligned} U(x) &= q(\lambda(p, y))[f(x, y) - p(y)], \text{ for } x = \mu^\star(y) \\ U(x') &\geq q(\lambda(p, y))[f(x', y) - p(y)], \text{ for all } x' \end{aligned}$$

Seller y's problem is equivalent to ($p \to \lambda$ and set p s.t. attract x that gives highest possible λ; cf. Competing Mechanisms):

$$\max_{x,p,\lambda} \pi = m(\lambda)p$$

$$\text{s.t.} \qquad \frac{m(\lambda)}{\lambda}[f(x, y) - p] = U(x).$$

Equivalence of the two problems. Fix p, then program (4) and

$$\max_{x,\lambda} \pi = m(\lambda)p$$

$$\text{s.t.} \qquad \frac{m(\lambda)}{\lambda}[f(x, y) - p] = U(x).$$

Assortative Matching

Comparing Logs and Roots

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