

MATCHING INFORMATION

Hector Chade¹ Jan Eeckhout²

¹Arizona State University

²University College London and GSE-UPF

NYU Search and Matching

April 2014

MOTIVATION

- We explore the following matching problem:
 - A population of heterogeneous agents must be partitioned in groups of a given size (not necessarily two)
 - Agents differ in expertise in estimating an unknown variable that is relevant for the performance of the group
 - Expertise \equiv Information technology that generates a signal about the unknown variable of interest
 - A better expert is one with a more informative technology
 - Within a group, agents take a joint action
 - Agents can make monetary transfers among themselves

MOTIVATION

- We address the following standard matching question:
 - What is the optimal sorting of agents into groups?
 - Will *more informed* agents be paired with *more* or *less informed* ones?
- We also shed light on:
 - The role of correlated information on sorting patterns
 - Matching groups of experts with heterogeneous firms
 - Endogenous group size

MOTIVATION

- Remarks:
 - Many Groups
 - Interpretations: Partnerships, groups that are paired with identical firms, groups within an organization
 - Matchmaker can be the planner, or can take place in a decentralized market
 - Applications: financial experts, R&D groups, composition of skilled workers across firms, etc.

MAIN INSIGHT

- Diversification of expertise within groups is optimal
 - Even if in isolation more information is better
 - Matching \Rightarrow information diversification
- In the canonical case of conditionally independent signals, we obtain a stronger result:
 1. Maximally balanced teams are optimal
 2. Strong form of diversification of expertise

RELATED LITERATURE

MATCHING

- Becker (1973) theory of marriage (see also Legros and Newman (2007), and assignment games)
 - Matching problem among heterogeneous men and women
 - Match output depends on their attributes
 - Transferable utility
 - Positive (negative) sorting (PAM or NAM) if supermodular (submodular) payoff
 - Both centralized solution and competitive equilibrium
- Pycia (2012) matching with peer effects
- Kelso and Crawford (1982) labor market model
- Most applications assume attributes are scalars (ordered)
- More general interpretation: stochastic sorting

RELATED LITERATURE

THEORY OF TEAMS

- Marschak-Radner's (1972) theory of teams (see also Cremer (1990), Prat (2002), Lamberson and Page (2011), etc.)
 - Information decentralization: decision makers have heterogeneous information within an organization
 - Need to make decision with common goal
 - What is the optimal decision function given information?
 - Compare different information structures
- Meyer (1984) on fractional assignment
- Olzewski and Vohra (2012) on optimal composition of a team
- Unlike this literature:
 - Many teams that form instead of a team in isolation
 - Matching problem

RELATED LITERATURE

VALUE OF INFORMATION

- The paper is related to three topics in this literature:
 1. Comparison of multivariate normal experiments: Hansen and Torgersen (1974), Shaked and Tong (1990, 1992)
 - In our model, each group runs a multivariate normal experiment
 - Correlation affects informativeness of the signals
 2. Substitute and complementary signals: Borgers, Hernando-Veciana, and Krämer (2010)
 - We provide results for normally distributed signals
 - We cast model as a matching problem
 3. Value of Information: Radner and Stiglitz (1984), Chade and Schlee (2002), Moscarini and Smith (2002)
 - We exploit concavity properties of the quadratic payoff/normal signals problem
 - Shed light on extent to which results generalize

RELATED LITERATURE

PARTITIONING PROBLEMS

- There is a recent literature in discrete optimization on partitioning problems: Chakravarty, Orlin, Rothblum (1985), Anily and Federgruen (1991), Hwang and Rothblum (2012)
 - They focus on problems that deliver consecutive partitions (similar to PAM)
 - Our model does not fit their framework: harder to solve

MODEL

- Agents
 - Finite set I of agents, with $|I| = kN$
 - Set of 'types' $[\underline{x}, \bar{x}]$: function $x : I \rightarrow [\underline{x}, \bar{x}]$ assigns types to agents, where $x(i) \equiv x_i, i = 1, 2, \dots, kN$
 - $\Upsilon = \{x_1, x_2, \dots, x_{kN}\}$ (multiset), assume $x_1 \leq x_2 \leq \dots \leq x_{kN}$
 - Each agent assigned to a group of size k ; there are N groups

MODEL

- Information

- State of the world, prior belief $\tilde{s} \sim \mathcal{N}(\mu, \tau^{-1})$
- Agent x_i observes signal $\tilde{\sigma}_i \sim f(\cdot | s, x_i) = \mathcal{N}(s, x_i^{-1})$
- Informativeness: $x_i \uparrow \Rightarrow$ more informative signals (Blackwell)
- Signals from different partners can be correlated, with pairwise covariance given by $\rho / (x_i x_j)^{0.5}$, for all $i, j, i \neq j$
- $\rho \in (-(k-1)^{-1}, 1)$ to ensure that the covariance matrix is positive semi-definite

MODEL

- Actions and payoffs
 - Group observes signal realizations of all partners \Rightarrow choose joint, signal-contingent action
 - Joint action $a \in \mathbb{R}$ to maximize the group's profit
 - Maximize expected value of $\pi - (a - s)^2$, where $\pi \geq 1/\tau$

GROUP PROBLEM

- A group S with types $\vec{x}^S = (x_1^S, x_2^S, \dots, x_k^S)$
- Choose $a : \mathbb{R}^k \rightarrow \mathbb{R}$ to maximize

$$V(\vec{x}^S) = \max_{a(\cdot)} \pi - \int \cdots \int [a(\vec{\sigma}) - s]^2 f(\vec{\sigma}|s, \vec{x}^S, \rho) h(s) \prod_{i=1}^k d\sigma_i ds$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_k)$, and $f(\vec{\sigma}|s, \vec{x}^S, \rho)$ is the joint density of the signals generated by the members of the group (distributed according to a multivariate normal)

- Denote by $V(\vec{x}^S)$ the maximum expected payoff of group S

MATCHING

- Matching: partition of $\Upsilon = \{x_1, x_2, \dots, x_{kN}\}$ in groups (sub-multisets) of size k
- N elements in each partition
- Transferable utility
- Optimal partition problem
 - Find partition that maximizes $\sum_S V(\bar{x}^S)$
- Remark:
 - Allocation can be decentralized if $k = 2$ (Becker (1973)) or if there is fractional assignment
 - No general result for integer assignment with $k > 2$ since Kelso and Crawford (1982) gross substitutes condition fails

SOLUTION TO THE GROUP PROBLEM

- After observing $\vec{\sigma}$, the posterior density function $h(\cdot | \vec{\sigma}, \vec{x}^S, \rho)$ is normally distributed
- Optimal action solves

$$\max_{a \in \mathbb{R}} \pi - \int (a - s)^2 h(s | \vec{\sigma}, \vec{x}^S, \rho) ds$$

- From FOC,

$$a^*(\vec{\sigma}) = \int s h(s | \vec{\sigma}, \vec{x}^S, \rho) ds = \mathbb{E} [\tilde{s} | \vec{\sigma}, \vec{x}^S, \rho]$$

- Inserting $a^*(\vec{\sigma})$ in objective function we obtain, after algebra,

$$V(\vec{x}^S) = \pi - \int \cdots \int \text{Var}(s | \vec{\sigma}, \vec{x}^S, \rho) f(\vec{\sigma} | \vec{x}^S, \rho) \prod_{i=1}^k d\sigma_i$$

where $f(\vec{\sigma} | \vec{x}^S, \rho) \equiv \int f(\vec{\sigma} | s, \vec{x}^S, \rho) h(s) ds$

SOLUTION TO THE GROUP PROBLEM

- Algebra is as follows: $V(\vec{x}^S)$ equals

$$= \pi - \int \cdots \int \left(\mathbb{E}[\tilde{s}|\vec{\sigma}, \vec{x}^S] - s \right)^2 f(\vec{\sigma}|s, \vec{x}^S, \rho) h(s) \prod_{i=1}^k d\sigma_i ds$$

$$\stackrel{(2)}{=} \pi - \int \cdots \int \left(\int (s - \mathbb{E}[\tilde{s}|\vec{\sigma}, \vec{x}^S])^2 h(s|\vec{\sigma}, \vec{x}^S, \rho) ds \right) f(\vec{\sigma}|\vec{x}^S, \rho) \prod_{i=1}^k d\sigma_i$$

$$\stackrel{(3)}{=} \pi - \int \cdots \int \text{Var}(s|\vec{\sigma}, \vec{x}^S, \rho) f(\vec{\sigma}|\vec{x}^S, \rho) \prod_{i=1}^k d\sigma_i$$

- where:

(2) from $h(s|\vec{\sigma}, \vec{x}^S, \rho) f(\vec{\sigma}|\vec{x}^S, \rho) = h(s) f(\vec{\sigma}|s, \vec{x}^S, \rho)$

(3) replacing the expression for the variance of posterior density

SOLUTION TO THE GROUP PROBLEM

- Easy to compute in the conditionally independent case ($\rho = 0$)
- After observing $\vec{\sigma}$, the posterior density function:

$$h(\cdot | \vec{\sigma}, \vec{x}^S) \sim \mathcal{N}\left(\frac{\mu\tau + \sum_{i=1}^k \sigma_i x_i^S}{\tau + \sum_{i=1}^k x_i^S}, \frac{1}{\tau + \sum_{i=1}^k x_i^S}\right)$$

- Notice that the variance of the posterior is independent of $\vec{\sigma}$
- Therefore

$$V(\vec{x}^S) = \pi - \left(\frac{1}{\tau + \sum_{i=1}^k x_i^S}\right)$$

SOLUTION TO THE GROUP PROBLEM

- More generally, we have the following result:

PROPOSITION

The value function of the group problem is

$$V(\vec{x}^S) = \pi - \left(\frac{1}{\tau + \mathcal{B}(\vec{x}^S, \rho)} \right)$$

where

$$\mathcal{B}(\vec{x}^S, \rho) = \frac{(1 + (k - 2)\rho) \sum_{i=1}^k x_i^S - 2\rho \sum_{i=1}^{k-1} \sum_{j=i+1}^k (x_i^S x_j^S)^{0.5}}{(1 - \rho)(1 + (k - 1)\rho)}$$

- $\mathcal{B}(\vec{x}^S, \rho)$ is the index of informativeness
 - Higher $\mathcal{B}(\vec{x}^S, \rho)$ implies Blackwell-more-informative signals
 - $\mathcal{B}(\vec{x}^S, 0) = \sum x_i^S$, and if $x_i = x$ for all i then $\mathcal{B}(\vec{x}^S, 0) = kx$

SOLUTION TO THE GROUP PROBLEM

- Some special cases of

$$V(\vec{x}^S) = \pi - \left(\frac{1}{\tau + \frac{(1+(k-2)\rho) \sum_{i=1}^k x_i^S - 2\rho \sum_{i=1}^{k-1} \sum_{j=i+1}^k (x_i^S x_j^S)^{0.5}}{(1-\rho)(1+(k-1)\rho)}}} \right)$$

- $\rho = 0$ yields the conditionally independent case
- $k = 2$ yields

$$V(\vec{x}^S) = \pi - \left(\frac{1}{\tau + \frac{x_1^S + x_2^S - 2\rho(x_1^S x_2^S)^{0.5}}{(1-\rho^2)}}} \right)$$

- $x_1^S = x_2^S = \dots = x_k^S = x$ yields

$$V(\vec{x}^S) = \pi - \left(\frac{1}{\tau + \frac{kx}{1+(k-1)\rho}} \right)$$

SOLUTION TO THE GROUP PROBLEM

- The proof is by induction after obtaining the general functional form of the inverse of the covariance matrix
- Sketch:
 - Start with $s \sim N(\mu, 1/\tau)$, $\sigma_1 \sim N(s, 1/x_1)$ and $s|\sigma_1 \sim N(\mu_1, 1/\tau_1)$
 - Show that formula holds for $k = 1$ (trivial)
 - Assume true for $k = n - 1$
 - Find $\sigma_n|\sigma_1, \dots, \sigma_{n-1}, s$ and compute $s|\sigma_1, \dots, \sigma_n$
 - Show formula holds for $k = n$

SOLUTION TO THE GROUP PROBLEM

- A generalization of objective function:
 - Same value function if $\pi - (a - s)^n$, n even (since all odd centered moments of normal are zero)
- A generalization to a class of distributions:
 - Same value function if:
 - Restriction to actions that are weighted averages of signals
 - Joint signal distribution has mean and covariance assumed

CORRELATION AND INFORMATIVENESS

- Let $k = 2$ and $x_1^S = x_2^S = x$
 - Then $\mathcal{B}(\vec{x}^S, \rho) = \frac{2x}{(1+\rho)} \geq 2x = \mathcal{B}(\vec{x}^S, 0)$ if $\rho \leq 0$
 - Negatively (positively) correlated signals are more (less) informative than conditionally independent ones
- Some 'intuition'
 - Consider first extreme cases of $\rho = \pm 1$
 - More generally, $\sigma_2 | \sigma_1, s \sim N\left((1 - \rho)s + \rho\sigma_1, \frac{1-\rho^2}{x}\right)$
 - Correlation reduces variance of second signal
 - Negative correlation makes mean 'more sensitive' to s
- In the general case we have the following result:

PROPOSITION (CORRELATION AND TEAM PRECISION)

- (i) If $\rho < 0$, then $\mathcal{B}(\vec{x}^S, \rho) > \mathcal{B}(\vec{x}^S, 0)$
- (ii) There is a $\hat{\rho}$ s.t. if $0 < \rho < \hat{\rho}$, then $\mathcal{B}(\vec{x}^S, \rho) < \mathcal{B}(\vec{x}^S, 0)$ ($\forall \vec{x}^S$)

OPTIMAL MATCHING PROPERTIES

- The main sorting properties follow from this result:

LEMMA (VALUE FUNCTION PROPERTIES)

Consider any group S with \vec{x}^S :

(i) There exists an interval $(-r, r)$, where r depends on $(\underline{x}, \bar{x}, \tau, k)$ such that if $\rho \in (-r, r)$ then $V(\vec{x}^S)$ is strictly submodular in \vec{x}^S ;

(ii) If $\rho > r$, then $V(\vec{x}^S)$ cannot be supermodular in \vec{x}^S , and it is strictly submodular if τ is sufficiently large;

(iii) If $\rho < -r$, then $V(\vec{x}^S)$ cannot be supermodular in \vec{x}^S unless τ is sufficiently large.

- This result reveals that the value function is:
 - Submodular in many cases
 - Not supermodular in most cases

OPTIMAL MATCHING PROPERTIES

- The properties of the team value function yield:

PROPOSITION (OPTIMALITY OF DIVERSIFICATION)

(i) Diversification within teams is always optimal for values of ρ in a neighborhood of 0.

(ii) Diversification is optimal on an open subset of $[\underline{x}, \bar{x}]^k$ when ρ is positive, and it is always optimal if τ is large enough.

(iii) Diversification is optimal when ρ is negative, so long as τ is not too large.

- This result reveals that the optimal matching:
 - Cannot be PAM except in 'rare' cases
 - Cannot exhibit two 'ordered' teams (except in 'rare' cases)
 - Exhibits 'balanced' expertise assignment across teams

OPTIMAL MATCHING PROPERTIES

- Consider part (i) of the proposition:
 - V strictly submodular on $[\underline{x}, \bar{x}]^k$ for $\rho \in (-r, r)$
 - Thus, PAM does not maximize $\sum V(\vec{x}^S)$
 - Given PAM, swap (e.g.) the best expert in one group with the worst expert in the other group
 - By strict submodularity, the objective function increases
 - Implication \rightarrow the optimal matching will consist of teams with diversified composition of expertise
 - No team can have all members with uniformly higher types than any other team

OPTIMAL MATCHING PROPERTIES

- Consider part (ii) of the proposition:
 - V cannot be supermodular on $[\underline{x}, \bar{x}]^k$ if $\rho > 0$
 - There is an open set around $x_1 = x_2 = \dots = x_k$ such that V is submodular in that set
 - Thus, in that region one can do some profitable swapping if types belong to it
 - Diversification can 'sometimes' occur (but PAM cannot occur for all multisets Υ , or for $0 < \rho < r$)
 - For each $\rho > 0$, diversification occurs if τ is large enough

OPTIMAL MATCHING PROPERTIES

- Consider part (iii) of the proposition:
 - If $\rho < 0$, then V supermodular if τ is large enough
 - Partition generates $(\mathcal{B}_1, \dots, \mathcal{B}_N)$ with 'mean' and 'variance'
 - $\mathcal{B}(\bar{x}_S, \rho)$ is supermodular in \bar{x}_S when $\rho < 0$
 - Hence, PAM maximizes $\sum \mathcal{B}_S$
 - As $\tau \rightarrow \infty$, $V = \pi - \frac{1}{\tau + \mathcal{B}} \Rightarrow -V_{\mathcal{B}\mathcal{B}}/V_{\mathcal{B}} \rightarrow 0$
 - Thus planner behaves as if he maximizes $\sum \mathcal{B}_S$ when τ is large enough, i.e., PAM is optimal
 - (Similar intuition applies to $\rho > 0$ and τ large when $k = 2$)
- Remark:
 - How large should τ be for PAM?
 - When $k = 2$, for each $\rho > 0$, τ should be strictly bigger than $4\bar{x}$ (two times the precision of the best team possible)
 - Thus, information about s is 'very' precise to begin with
 - If $\tau \leq 4\bar{x}$, V is strictly submodular and diversification ensues

OPTIMAL MATCHING PROPERTIES

- We can say more if $k = 2$
 - $V(\cdot)$ submodular \Rightarrow Negative Assortative Matching
 - If types $x_1 > x_2 \geq x_3 > x_4$, then total payoff maximized if $\{x_1, x_4\}$ and $\{x_2, x_3\}$
 - x_4 can outbid x_2 and x_3 when competing for x_1
 - For $k = 2$, optimal matching is straightforward:
 1. x_1 with x_{2N}
 2. x_2 with x_{2N-1}
 - ...
 - N . x_N with x_{N+1}
 - Similarly for PAM and supermodularity
 - Optimal matching can be decentralized as outcome of Walrasian Equilibrium

THE CONDITIONAL INDEPENDENT CASE (CIC)

- From now on we focus on the canonical case with conditionally independent signals
- Recall that group value functions in this case is

$$V(\vec{x}^S) = \pi - \left(\frac{1}{\tau + \sum_{i=1}^k x_i^S} \right)$$

- Consistent with previous result, it is strictly submodular in \vec{x}^S
- Since $V(\cdot)$ depends on \vec{x}^S only through the sum, we define

$$v\left(\sum x_i^S\right) \equiv V\left(\vec{x}^S\right)$$

CIC: DISCRETE ASSIGNMENT

- Given the value of a group $v(\sum x_i^S)$, what is the optimal sorting of agents into groups?
- That is, we want the partition that maximizes $\sum_S v(\sum x_i^S)$
 - Sorting with $k > 2$ is much more complex
 - $v(\sum x_i^S)$ submodular \Rightarrow PAM not optimal
 - But exact sorting pattern is not obvious
 - Clear: optimal matching entails diversification within groups \Rightarrow balanced teams

CIC: DISCRETE ASSIGNMENT

- Notice that:

- Every partition has the same sum $\sum_{S=1}^N \sum_{i=1}^k x_i^S = X$
- Objective function is (strictly) Schur-concave on vector of groups precision (sums of members' precisions) partially ordered by majorization
 - $x = (x_1, x_2, \dots, x_N)$ majorizes $x' = (x'_1, x'_2, \dots, x'_N)$ ($x \succ x'$) if $\sum_{\ell=1}^m x_{[\ell]} \geq \sum_{\ell=1}^m x'_{[\ell]}$ for all m , with $\sum_{\ell=1}^N x_{[\ell]} = \sum_{\ell=1}^N x'_{[\ell]}$
 - $x_{[\ell]}$ is the ℓ -th largest coordinate of the vector x
 - Majorization can be thought of as a notion of similarity/dispersion of vectors
 - $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is Schur concave if $x \succ x'$ implies $f(x') \geq f(x)$
 - $f(x_1, x_2, \dots, x_N) = \sum_{i=1}^N g(x_i)$ is strictly Schur concave if g is strictly concave

CIC: DISCRETE ASSIGNMENT

- Optimal matching in CIC:

PROPOSITION (MAXIMALLY BALANCED TEAMS)

Assume conditionally independent signals.

- (i) *The optimal matching must be an element of the set of partitions whose team precision vectors (X_1, X_2, \dots, X_N) are majorized by those generated by all the remaining partitions.*
- (ii) *If a team precision vector is majorized by the precision vectors of all the feasible partitions of the agents, then its associated partition is the optimal matching.*

CIC: DISCRETE ASSIGNMENT

- If there is a partition that is majorized by all the other ones, then it is optimal
 - *Solution is a partition with 'lowest spread' in group precision*
→ *maximum diversification*
 - Clearly, if there is a partition with $\sum x^S = X/N$ for all S , this is the solution
 - Still need to prove that there is a 'minorizing' partition
 - True if $N = 2$ or $k = 2$
- Weaker: Find partitions majorized by the remaining ones
 - Solution is in the set containing those partitions
- Proposition holds for a class of matching problems with $\sum v(\sum_i x_i)$ strictly Schur concave

CIC: DISCRETE ASSIGNMENT

- Is there an algorithm to find the optimal partition?
 - Hard problem in general (except for $k = 2$)
 - With $N = 2$, equivalent to NP-hard 'number partitioning problem' (Garey and Johnson (1978), Mertens (2006))
 - With $N = 3$, equivalent to strong NP-complete 3-partition problem (Garey and Johnson (1978))
 - Related: problem is a variant of submodular welfare maximization problem (Vondrak (2007)), which is NP-hard
 - Vondrak (2007) describe an approximation algorithm that captures $1 - 1/e$ of optimal value
- Example of failure of a greedy algorithm:
 - 8 agents with types are 1, 3, 6, 10, 12, 15, 20, 23; $N = 2$
 - Greedy: $\{23, 12, 10, 1\}$, $X_1 = 46$; $\{20, 15, 6, 3\}$, $X_2 = 44$
 - Optimal: $\{23, 15, 6, 1\}$, $X_1 = 45$; $\{20, 12, 10, 3\}$, $X_2 = 45$
 - Flexible k does not solve the problem

CIC: FRACTIONAL ASSIGNMENT

- No integer restriction
- Agents can be fractionally assigned to multiple groups
 - Agent's precision proportionally re-scaled according to fraction assigned to the group
- Interpretation: Time dedication to a group given time 'budget'
- Another interpretation: Approximates discrete solution with large but finite teams
- Assumption: s independent across teams (to avoid information 'spillovers')

CIC: FRACTIONAL ASSIGNMENT

- $x(I) = \{x_1, \dots, x_J\}$ set of *distinct* types; $m_j \neq$ type x_j
- Let $X = \sum_{j=1}^J m_j x_j$ and $\sum_{j=1}^J m_j = kN$
- $\mu_{jn} \geq 0$ the fractional assignment of type- j agents to group n
- The optimal fractional assignment problem solves:

$$\begin{aligned} \max_{\{\mu_{jn}\}_{j,n}} \quad & \sum_{n=1}^N v \left(\sum_{j=1}^J \mu_{jn} x_j \right) \\ \text{s.t.} \quad & \sum_{n=1}^N \mu_{jn} = m_j \quad \forall j \\ & \sum_{j=1}^J \mu_{jn} = k \quad \forall n \\ & \mu_{jn} \geq 0 \quad \forall j, n \end{aligned}$$

CIC: FRACTIONAL ASSIGNMENT

- Let $X_n \equiv \sum_{j=1}^J \mu_{jn} x_j \Rightarrow \sum_{n=1}^N X_n = X$.
- The 'relaxed problem':

$$\begin{aligned} \max_{\{X_n\}_{n=1}^N} & \quad \sum_{n=1}^N v(X_n) \\ \text{s.t.} & \quad \sum_{n=1}^N X_n = X \end{aligned}$$

- Solution is $X_n = \frac{X}{N}, \forall n$
- Maximal balance achieved: Equal precision teams

CIC: FRACTIONAL ASSIGNMENT

- If $\{\mu_{jn}\}$ implements the solution to the relaxed problem, then it solves the original problem.
- We have the following result:

PROPOSITION (PERFECT DIVERSIFICATION)

Any solution to the fractional assignment problem equalizes X_n across all groups, i.e., entails maximum diversification.

It can be implemented using $\mu_{jn} = m_j/N$ for all j, n .

- Solution is not unique except when $J = 2$
- Example: 1, 2, 3, 4, 5, 5
 1. $\{\frac{1}{2}1, \frac{1}{2}2, \frac{1}{2}3, \frac{1}{2}4, \frac{2}{2}5\} \Rightarrow$ both: $\pi - \frac{1}{\tau+10}$
 2. $\{1, 4, 5\}, \{2, 3, 5\} \Rightarrow$ both: $\pi - \frac{1}{\tau+10}$
- Unique symmetric solution

CIC: FIRM HETEROGENEITY

- N heterogeneous firms $y_1 \leq y_2 \leq \dots \leq y_N$
- Match payoff when firm y_S matches with team $\sum x_i^S$ is

$$y_S \cdot v \left(\sum x_i^S \right)$$

- Notice that match payoff is supermodular in $(y_S, \sum x_i^S)$
- Optimal matching properties are:
 - PAM between firms and groups (better firms match with better teams)
 - Diversity within groups (heterogeneous experts in each group)

CIC: FIRM HETEROGENEITY

- We can pin down exact solution with fractional assignment
- Optimal assignment of firms to groups solves

$$\begin{aligned} \max_{X_1, X_2, \dots, X_N} \quad & \sum_{n=1}^N y_n v(X_n) \\ \text{s.t.} \quad & \sum_n X_n = X \end{aligned}$$

- From FOC, we obtain

$$X_n = \frac{y_n^{0.5}}{\sum_{n=1}^N y_n^{0.5}} (\tau N + X) - \tau$$

CIC: FIRM HETEROGENEITY

- We have the following result:

PROPOSITION (HETEROGENEOUS FIRMS)

Firms with higher types are matched with higher precision teams ($X_1 \leq X_2 \leq \dots \leq X_N$).

An increase in $\tau \uparrow X_n$ iff $n \geq n^$, and $\uparrow X_n - X_{n-1}$ for all n*

An increase in $X \uparrow X_n$ and $\uparrow X_n - X_{n-1}$ for all n

An increase in the spread of (y_1, y_2, \dots, y_N) increases (decreases) the precision of teams above (below) $1 \leq \hat{n} \leq N$

There is a fractional assignment rule $\{\mu_{jn}\}$ that implements the optimal (X_1, X_2, \dots, X_N)

CIC: FIRM HETEROGENEITY

- Example:
 - $N = 2, y_1 = 1, y_2 = y \geq 1, \tau = 0, \Upsilon = \{5, 5, 20, 20\}$
 - Then

$$X_2 = \frac{y^{0.5}}{y^{0.5} + 1} X$$

$$X_1 = \frac{1}{y^{0.5} + 1} X$$

- Easy to show that
 - If $y = 1$ then $X_2 = X_1$ and NAM, i.e. $\{5, 20\}, \{5, 20\}$
 - If $y = 16$ then $X_2 = 40, X_1 = 10$, and PAM, i.e., $\{5, 5\}, \{20, 20\}$
 - If $1 \leq y < 16$ there is diversification within groups
 - Fractional assignment rule: E.g., $\mu_{20,2} = 1 + (1/15)(X_2 - 25)$

CIC: FRACTIONAL ASSIGNMENT

ENDOGENOUS k

- One way to endogenize k is to assume a cost function $c(N)$ that is strictly increasing and convex in N
 - FOC that determines optimal N is

$$\pi - c'(N) = N(N\tau + X)^{-1} + NX(N\tau + X)^{-2}$$

- This yields N ; since l is fixed, $k = |l|/N$ is determined
- Another way to endogenize k is as follows:
 - Assume that identical firms are locations where groups of size k form (reinterpretation) under symmetric solution
 - Free entry of firms, entry cost $F > 0$, wages w_j
 - For any N , $w_j = v'(X_n)x_j$, where $v'(X_n) = (\tau + X_n)^{-2}$
 - Zero profit condition, $X_n = X/N$, and $\sum_j m_j x_j = X$ yield

$$\pi - F = N(N\tau + X)^{-1} + NX(N\tau + X)^{-2}$$

CIC: FRACTIONAL ASSIGNMENT

ENDOGENOUS k

- Either way of endogeneizing k yields the following result

PROPOSITION (ENDOGENOUS GROUP SIZE)

There exists a unique value of N and thus of $k = |I|/N$.

The equilibrium group size $k \downarrow$ in τ , and \downarrow in X

CIC: DECENTRALIZED MARKET SOLUTION

- Thus far, we have focused almost exclusively on the optimal matching problem
- If $k = 2$, then it is easy to decentralize the model if we think of it as a two-sided matching problem
 - Standard results from assignment games (nonempty core, competitive equilibrium existence)
- For $k > 2$, we do not have a decentralization result:
 - Consider matching groups of experts with identical firms
 - Problem does not satisfy gross substitutes condition of Kelso and Crawford (1982) (or Gul and Stacchetti (1999))
- With fractional assignment, easy decentralization result

CIC: DECENTRALIZED MARKET SOLUTION

FAILURE OF GROSS SUBSTITUTES CONDITION

- The firm solves:

$$\max_{A \subseteq \mathcal{I}} v \left(\sum_{i \in A} x_i \right) - \sum_{i \in A} w(x_i)$$

- Let $D(w)$ be the set of solutions
 - GS: If $A^* \in D(w)$ and $w' \geq w$, then there is a $B^* \in D(w')$ such that $T(A^*) \subseteq B^*$, $T(A^*) = \{i \in A^* | w(x_i) = w'(x_i)\}$.
- The following example shows that GS fails in our model.
 - Firm and experts 1, 2, and 3, with $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$
 - Assume $\pi = \tau = 1$
 - If $w = ((1/12) - \varepsilon, 1/12, 1/6)$, then $A^* = \{1, 2\}$
 - Let $w = ((1/12) - \varepsilon, 1/6, 1/6)$, so that only the wage of expert 2 has increased. If $\varepsilon < 1/30$, then $B^* = \{3\}$
 - Hence, GS fails since $T(A^*) = \{1\} \not\subseteq B^*$
 - Same example shows that if the firm were constrained to hire at most two experts, GS would still fail

CIC: DECENTRALIZED MARKET SOLUTION

- With fractional assignment, given prices w_j , firm n solves:

$$\begin{aligned} \max_{\mu_{jn}} \quad & v \left(\sum_j \mu_{jn} x_j \right) - \sum_j \mu_{jn} w_j \\ \text{s.t.} \quad & \sum_j \mu_{jn} = k \end{aligned}$$

- There are J first order conditions for all N firms:

$$\text{FOC}_{jn} : v' \left(\sum_j \mu_{jn} x_j \right) x_j - w_j + \phi_n = 0, \forall j, n$$

- Coincides with the planner's solution

$$\text{FOC}_{jn} : v' \left(\sum_j \mu_{jn} x_j \right) x_j + \lambda_j + \phi_n = 0, \forall j, n$$

DISCUSSION

MODELING ASSUMPTIONS

- How restrictive is the normal/quadratic-payoff group problem?
 - Affords analytical solutions
 - Widely used (teams a la Marschak-Radner, global games, etc.)
 - Beyond this set up:
 - Binary case
 - Other canonical model with submodular value function: probabilistic information arrival
 - Stumbling block: Non-concavity in value of information (Radner-Stiglitz (1984), Chade-Schlee (2002))
- How general are our matching results?
 - More general than problem of matching information
 - The CIC shows that it holds for any matching problem with V concave in the sum of types $\sum_i x_i^S$

DISCUSSION

STOCHASTIC SORTING

- Matching information is an example of stochastic sorting
 - Type is a family $\{f(\sigma_i|s, x_i)\}_s$ indexed by x_i
 - Blackwell informativeness orders the distributions
 - 'Matching distributions'
- Other problems with stochastic sorting
- Becker (1973) with uncertainty: $f(\sigma_i|x_i)$, $v(\vec{\sigma})$
 - $f(\sigma_i|\cdot)$ ordered by FOSD
 - $f(\sigma_i|\cdot)$ ordered by MPS
 - Results for this case
- Legros-Newman (2007) with NTU and uncertainty
 - Risk sharing problems with FOSD and MPS
 - Examples and some results

CONCLUDING REMARKS

- Competition \Rightarrow diversification of information within firm
 - Marschak-Radner: recognize individuals have *different*, decentralized information
 - Matching: want to choose agents with different precision
- Interpret maximum within group spread as:
 1. Information diversification: driven by the market forces
 2. Endogenous formation of hierarchies: each firm endogenously generates a hierarchy in precision
- Extensions:
 1. Algorithms
 2. Allowing for different group sizes
 - $\rho = 0$, six agents, 2, 2, 7, 7, 8, 10
 - If $k = 3$, then $\{2, 7, 10\}, \{2, 7, 8\}$, with $X_1 = 19$ and $X_2 = 17$
 - If different sizes allowed, then $\{2, 2, 7, 7\}, \{8, 10\}$, with $X_1 = X_2 = 18$, a strict improvement
 - Not an issue with fractional assignment
 3. Non transferable utility
 4. Decentralization without gross substitutes