ASSORTATIVE MATCHING WITH LARGE FIRMS

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Motivation

- Two cornerstones of analyzing firms in Macro, Labor, IO, Trade,...
  1. Firm size: productive firms are larger and produce more
  2. Sorting of workers: firms compete for skilled workers
- These two aspects are usually treated independently
  2. Matching: one-to-one (e.g. Becker 1973) → extensive margin
- Needed: Trade-off better workers vs. more workers
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- Needed: Trade-off better workers vs. more workers

→ Apply theory to technological change: SBTC vs. QBTC
The Model

Intensive and Extensive Margin

- **Population**
  - Workers of type $x \in X = [\underline{x}, \overline{x}]$, distribution $H^w(x)$
  - Firms of types $y \in Y = [\underline{y}, \overline{y}]$, distribution $H^f(y)$

- **Production of firm $y$** $F(x, y, l_x, r_x)$
  - $l_x$ workers of type $x$, $r_x$ fraction of firm’s resources
  - $F$ increasing in all, concave in last two arguments
  - $F$ constant returns to scale in last two arguments

  $\Rightarrow$ Denote:

  $$f(x, y, \theta) = rF \left( x, y, \frac{l}{r}, 1 \right), \text{ where } \theta = \frac{l}{r}$$
The Model
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  \[ f(x, y, \theta) = rF \left( x, y, \frac{l}{r}, 1 \right), \text{ where } \theta = \frac{l}{r} \]

  • Key assumption: **no peer effects** \( \Rightarrow \) satisfies GS

  \[ \Rightarrow \text{Total output: } \int F(x, y, l_x, r_x)dx \]
The Model
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  \[
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  \]

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- **Preferences**
  - transferable utility (additive in output goods and numeraire)
Hedonic wage schedule $w(x)$ taken as given.

- **Optimization:**
  - Firms maximize: $\max_{l_x, r_x} \int [F(x, y, l_x, r_x) - w(x)l_x]dx$
  \[ \Rightarrow r_x > 0 \text{ only if } \left( x, \frac{l_x}{r_x} \right) = \arg \max f(x, y, \theta) - \theta w(x) \quad (\star) \]

- **Feasible Resource Allocation (market clearing) under PAM:**
  \[ \int_X^{\bar{x}} h_w(s) ds = \int_{\mu(x)}^{\bar{y}} \theta(s) h_f(s) ds \]

- **Competitive Equilibrium:** optimality + market clearing
Assortative Matching

**Proposition (Condition for PAM)**

A necessary condition to have equilibria with PAM is that

\[ F_{xy}F_{lr} \geq F_{yl}F_{xr} \]

holds along the equilibrium path. The reverse inequality entails NAM.
Assortative Matching

\[ F_{xy} F_{lr} \geq F_{yl} F_{xr} \]

- Interpretation \((F_{lr} > 0 \text{ by assumption})\):
  1. \(F_{xy} > 0\): better manager produces more with better workers (Becker)
  2. \(F_{yl} > 0\): better managers can manage more people (as in Lucas)
  3. \(F_{xr} > 0\): better workers produce more with manager time
**Assortative Matching**

\[ F_{xy} F_{lr} \geq F_{yl} F_{xr} \]

- Interpretation \((F_{lr} > 0 \text{ by assumption})\):
  1. \(F_{xy} > 0\): bet. manag. produce more w/ bet. workers (Becker)
  2. \(F_{yl} > 0\): bet. manag., larger span of control (as in Lucas)
  3. \(F_{xr} > 0\): bet. workers produce more w/ manag. time

- Quantity-quality trade-off by firm \(y\) with resources \(r\):
  1. \(F_{xy}\): better manager manages quality workers better vs.
  2. \(F_{yl}\): better managers can manage more people

\[ \Rightarrow \text{Marginal increase of better } \geq \text{ marginal impact of more workers} \]
Sketch of Proof of PAM-Condition

Assume PAM allocation with resources on \((x, \mu(x), \theta(x))\). Must be optimal, i.e., maximizes:

\[
\max_{x, \theta} f(x, \mu(x), \theta) - \theta w(x).
\]

First order conditions:

\[
\begin{align*}
 f_\theta(x, \mu(x), \theta(x)) - w(x) &= 0 \\
 f_x(x, \mu(x), \theta(x)) - \theta(x) w'(x) &= 0
\end{align*}
\]

The Hessian is

\[
Hess = \begin{pmatrix}
 f_{\theta \theta} & f_{\theta x} - w'(x) \\
 f_{\theta x} - w'(x) & f_{xx} - \theta w''(x)
\end{pmatrix}.
\]

Second order condition requires \(|Hess| \geq 0\):

\[
f_{\theta \theta} [f_{xx} - \theta w''(x)] - (f_{\theta x} - w'(x))^2 \geq 0
\]

Differentiate FOC’s with respect to \(x\), substitute:

\[
-\mu'(x) [f_{\theta \theta} f_{xy} - f_{y \theta} f_{x \theta} + f_{y \theta} f_x / \theta] \geq 0
\]

Positive sorting means \(\mu'(x) > 0\), requiring \([\cdot] < 0\) and after rearranging:

\[
F_{xy} F_{lr} \geq F_{yl} F_{xr}
\]
Special Cases

Efficiency Units of Labor

- Skill “=” Quantity: \( F(x, y, l, r) = \tilde{F}(y, xl, r) \Rightarrow F_{xy}F_{lr} = F_{yl}F_{xr} \)
Special Cases

Efficiency Units of Labor

- Skill "=" Quantity: \( F(x, y, l, r) = \tilde{F}(y, xl, r) \quad \Rightarrow \quad F_{xy} F_{lr} = F_{yl} F_{xr} \)

Multiplicative Separability

- \( F(x, y, l, r) = A(x, y) B(l, r) \) sorting if \( \frac{AA_{xy}}{A_x A_y} \frac{BB_{lr}}{B_l B_r} \geq 1 \)

- If \( B \) is CES with elast. of substitution \( \epsilon \): \( \frac{AA_{xy}}{A_x A_y} \geq \epsilon \) (root-sm)
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Becker’s one-on-one matching
- $F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\}$,
- Like inelastic CES ($\epsilon \rightarrow 0$), so sorting if $F_{12} \geq 0$
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- \( F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\} \)
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Sattinger’s span of control model

- \( F(x, y, l, r) = \min \left\{ \frac{r}{t(x, y)}, l \right\} \); write as CES between both arguments
- Our condition converges for inelastic case to log-supermod. in qualities
Special Cases

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Extension of Lucas’ span of control model

• \( F(x, y, l, r) = yg(x, l/r)r \), sorting only if good types work less well together \((-g_1g_{22} \geq -g_2g_{12})\).
Special Cases

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- \( F(x, y, \text{min}\{l, r\}, \text{min}\{r, l\}) = F(x, y, 1, 1) \text{min}\{l, r\} \)
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- \( F(x, y, l, r) = yg(x, l/r)r \), sorting only if good types work less well together \( (-g_1g_{22} \geq -g_2g_{12}) \).

Spatial sorting in mono-centric city:

- \( F(x, y, l, r) = l(xg(y) + v(r/l)) \Rightarrow \text{higher earners in center.} \)
Proposition

Under assortative matching (symmetric distributions of $x, y$)

\[
PAM : \quad \theta'(x) = \frac{F_{yl} - F_{xr}}{F_{lr}}, \quad \mu'(x) = \frac{1}{\theta(x)}, \quad w'(x) = \frac{F_x}{\theta(x)},
\]

\[
NAM : \quad \theta'(x) = -\frac{F_{yl} + F_{xr}}{F_{lr}}, \quad \mu'(x) = \frac{-1}{\theta(x)}, \quad w'(x) = \frac{F_x}{\theta(x)},
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PROPOSITION

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\[ NAM : \quad \theta'(x) = -\frac{F_{yl} + F_{xr}}{F_{lr}}; \quad \mu'(x) = \frac{-1}{\theta(x)}; \quad w'(x) = \frac{F_x}{\theta(x)}, \]

COROLLARY

Under assortative matching, better firms hire more workers if and only if along the equilibrium path

\[ F_{yl} > F_{xr} \text{ under PAM, and } -F_{yl} < F_{xr} \text{ under NAM.} \]
**Application: SBTC vs. QBTC**

- How has technology changed: 1996 → 2010?
- Estimate technological parameters that affect size and sorting

\[ F(x, y, l, 1) = \left( \omega_x x^{\frac{\sigma-1}{\sigma}} + \omega_y y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} l^{\omega_l}. \]

- Distribution of types \( x \) and \( y \) assumed log-normal
- Estimate parameters \( \omega_x, \omega_y, \omega_l, \sigma \) with parameters of type distributions to match 3 moment conditions:
  1. size-wage
  2. size-profits
  3. size distribution
- German administrative data for matched employer-employees
Results
Targeted Moments 1996

Wages-firm size  –  Profits-firm size  –  Firm size distribution
Results
Targeted Moments 2010

Wages-firm size — Profits-firm size — Firm size distribution
Results

Estimated Parameters

\[ F(x, y, l, 1) = \left( \omega_x x^{\frac{\sigma - 1}{\sigma}} + \omega_y y^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}} \]

<table>
<thead>
<tr>
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The Distributions of Worker Types $x$ and Firm Types $y$. 

**Results**

**Estimated Parameters**
**Results**

**Estimated Parameters**

\[ F(x, y, l, 1) = \left( \omega_x x^{\frac{\sigma-1}{\sigma}} + \omega_y y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} l^{\omega_l} \]

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| Distributions |  |  |
|----------------|-----------------|
| \( x \) | \( \mathcal{LN}(2.49, 1.35) \) | \( \mathcal{LN}(2.69, 1.35) \) |
| \( y \) | \( \mathcal{LN}(0.08, 1.57) \) | \( \mathcal{LN}(0.03, 1.54) \) |
\begin{itemize}
  \item $\sigma < 1 \Rightarrow \text{PAM}$
  \item $\sigma \approx 1$, technology can be approximated by the Cobb-Douglas
    \[ F(x, y, l, 1) \approx x^{\omega_x} y^{\omega_y} l^{\omega_l}. \]
    but not $\sigma = 1$: No sorting!
\end{itemize}
**Results**

**Estimated Parameters**

\[ F(x, y, l, 1) = \left( \omega_x x^{\frac{\sigma-1}{\sigma}} + \omega_y y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} l^l \]

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Results

Technological Change

- $\omega_x \uparrow 136\%$: Skill-biased Technological Change (SBTC)
- $\omega_l \uparrow 76\%$: Quantity-biased Technological Change (QBTC)
- $\omega_y$ unchanged
- $(1 - \sigma) \uparrow 14 \times$: Increase in complementarity between $x, y$
Results
Complementarities

\begin{align*}
F_{xy} \\
F_{lr}
\end{align*}
Results
Complementarities

\[ F_{yl} \]

\[ F_{xr} \]
RESULTS
Firm Size, Allocation, Skill Premium

Size Distribution
Allocation
Skill Premium $w'(x)$
**Results**

**Firm Size, Allocation, Skill Premium**

1. There is both SBTC and QBTC
2. FOSD in firm size distribution and shift in allocation
3. Skill premium $\uparrow$, but polarization (Goos-Manning, Autor-Dorn)
4. SBTC and QBTC interact
   - SBTC increases skill premium
   - QBTC decreases skill premium (concave production)

$\rightarrow$ Skill premium increase dampened by QBTC
## Counterfactuals

**1996 economy with one 2010 parameter**

<table>
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<tr>
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<th>Median Firm Size</th>
<th>% change 1996</th>
<th>Average $w'(x)$</th>
<th>% change 1996</th>
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<tr>
<td>1996</td>
<td>11.98</td>
<td></td>
<td>0.019</td>
<td></td>
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<tr>
<td>2010</td>
<td>12.53</td>
<td>4.60 %</td>
<td>0.027</td>
<td>44.06%</td>
</tr>
<tr>
<td>2010 $\omega_x$</td>
<td>14.21</td>
<td>18.66%</td>
<td>0.049</td>
<td>156.90%</td>
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<td>2010 $\omega_y$</td>
<td>11.95</td>
<td>-0.21%</td>
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<tr>
<td>2010 $\omega_l$</td>
<td>14.81</td>
<td>23.65%</td>
<td>0.009</td>
<td>-52.04%</td>
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<tr>
<td>2010 $\sigma$</td>
<td>12.01</td>
<td>0.24%</td>
<td>0.022</td>
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<td>2010 Distributions</td>
<td>12.36</td>
<td>3.20%</td>
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Conclusion

- Assortative matching with large firms: intensive and extensive margin
- A simple condition for sorting; nests many known models
- Equilibrium allocation: system of 3 differential equations
- Application: Technological Change
  1. both SBTC and QBTC
  2. effect of QBTC on skill premium: negative
  3. effect of SBTC on skill premium would have been 4 times larger