ASSORTATIVE MATCHING WITH LARGE FIRMS
SPAN OF CONTROL OVER MORE VERSUS BETTER WORKERS

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Motivation

• Background:
  • Matching: one-to-one (e.g. Becker 1973) $\rightarrow$ extensive margin
  • Macro / Labor / Trade / Urban / Devel: intensive margin
  • Intensive Margin $\Rightarrow$ Firm Size

• Trade-Off: better workers vs. more workers
  • managerial time: “span of control”: Sattinger 75, Lucas 78
  • assignment of land, of “distance”, of assets...
Motivation

• Goals:

1. Capture factor intensity in tractable manner (no peer effects)
2. Sorting condition: complementarity quality vs. quantity
3. Characterize firm size, assignment, wages
4. Introduce frictions: unemployment across skills and firm size

Economic Relevance

1. Characterizing production technology across industries: Walmart vs. mom-&-pop store; consulting and law firms;...
2. Misallocation debate: output difference across economies

Firm heterogeneity in productivity → differences in $K$, $p$, $A$ (Restuccia-Rogerson (08), Hsieh-Klenow (10),...)

Intensive margin and heterogeneity

Also worker heterogeneity ⇒ skill (mis)allocation and human capital distribution matter
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    • Intensive margin and heterogeneity
    • Also worker heterogeneity $\Rightarrow$ skill (mis)allocation and human capital distribution matter
Motivation

Resources / Firms

Workers

\[ \text{Output} \quad F(x_1, y) \quad \text{qualities}, \quad l_{1}, r_{1} \quad \text{quantities} \]

\[ h_{f 1}, h_{f 2}, h_{w 1}, h_{w 2} \]
Motivation

Resources / Firms

Workers

\[ \text{Prod: } f(x_1, y, l_1) = F(x_1, y, l_1, 1) + F(x_2, y, l_2, r_2) \]
Motivation

Resources / Firms

Workers

Prod: $f(x_1, y, l_1)$
Motivation

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Prod: $f(x_1, y, l_1)$ ; Prod: $f(x_2, y, l_2)$
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Prod: $F(\underbrace{x_1, y, l'}_{\text{qualities}}, \underbrace{l', r'}_{\text{quantities}}) + F(x_2, y, l'', r'')$
The Model

- **Population**
  - Workers of type $x \in X = [x, \bar{x}]$, distribution $H^w(x)$
  - Firms of types $y \in Y = [y, \bar{y}]$, distribution $H^f(y)$

- **Production of firm** $y$: $F(x, y, l_x, r_x)$
  - $l_x$ workers of type $x$, $r_x$ fraction of firm’s resources
  - $F$ increasing in all, concave in last two arguments
  - $F$ constant returns to scale in last two arguments
  \[\Rightarrow\] Denote: $f(x, y, \theta) = rF(x, y, \frac{l}{r}, 1)$, where $\theta = \frac{l}{r}$
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  - Could allow for $\neq$ resources: $F(x, y, l, r) = \tilde{F}(x, y, l, rT(y))$
  - Key assumption: no peer effects
  $\Rightarrow$ Total output: $\int F(x, y, l_x, r_x)dx$
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- **Preferences**
  - Transferable utility (additive in output goods and numeraire)
**Literature**

**Special Cases**

- **Becker 73:** \( l_{ji} = r_{ij} \rightarrow F(x, y, \min\{l, r\}, \min\{l, r\}) \)
- **Sattinger 75:** \( l_{ji} \leq \frac{r_{ij}}{t(x_i, y_i)} \rightarrow F = \min \left\{ l, \frac{r}{t(x, y)} \right\} \)
- **Garicano 00:** \( l \leq \frac{r}{t(x)} \rightarrow F = y \min \left\{ l, \frac{r}{t(x)} \right\} \)
- **Lucas 78:** Worker input independent of skill \( F = yg(l) \)
- **Rosen 74:** more general; existence
  (also, Kelso-Crawford 82, Cole-Prescott 97, Gul-Stacchetti 99, Milgrom-Hatfield 05)
- **Roy 51:** \( l_{ji} = r_{ij} \) & no factor intensity
- **Roy 51+CES:** particular functional form for decreasing return
- **Frictional Markets:** one-on-one matching, competitive search
  (Shimer-Smith 00, Atakan 06, Mortensen-Wright 03, Shi 02, Shimer 05, Eeckhout-Kircher 10)
The Model

Hedonic wage schedule $w(x)$ taken as given.

- **Optimization:***

- **Feasible Resource Allocation:***

- **Competitive Equilibrium***
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- **Optimization**:
  - Firms maximize: $\max_{l_x, r_x} \int [F(x, y, l_x, r_x) - w(x)l_x] dx$
  - Implies: $r_x > 0$ only if $\left( x, \frac{l_x}{r_x} \right) = \arg \max f(x, y, \theta) - \theta w(x)$  

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  - Firms maximize: \( \max_{l_x, r_x} \int [F(x, y, l_x, r_x) - w(x) l_x] \, dx \) s.t. \( \int r_x \, dx \leq 1 \)
  - Implies: \( r_x > 0 \) only if \( \left( x, \frac{l_x}{r_x} \right) = \arg \max f(x, y, \theta) - \theta w(x) \) (\( \star \))

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- **Feasible Resource Allocation:**
  - $\mathcal{R}(x, y, \theta)$: resources to any $x' \leq x$ by any $y' \leq y$ with $\frac{l_{x'}}{r_{x'}} \leq \theta$.
    1. Resource feasibility $[\mathcal{R}(y|X, \Theta) \leq H^f(y) \forall y]$
    2. Worker feasibility $[\int_{\theta \in \Theta} \int_{x' \leq x} \theta d\mathcal{R}(\theta, x'|Y) \leq H^w(x) \forall x]$

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- **Competitive Equilibrium** is a tuple $(w, \mathcal{R})$ s.t.
  1. Optimality Cond. $[(x, y, \theta) \in \text{supp } \mathcal{R}$ only if it satisfies (⋆)]
  2. Market Clearing $[\int \theta d\mathcal{R}(\theta|x, Y) \leq h^w(x), \ "\equiv\" \text{ if } w(x) > 0, \ \forall x]$
**Assortative Matching**

**Definition (Assortative Matching)**

Allocation \( \mathcal{R} \) entails positive (negative) sorting if for any \( x \) and \( x' \) with \( x < x' \) it holds that \((x, y, \theta) \in \mathcal{R}\) and \((x', y', \theta') \in \mathcal{R}\) only if \( y' \geq y \) (only if \( y' \leq y \)).

Allocation \( \mathcal{R} \) entails differential positive (negative) sorting its support only comprises points \((x, \mu(x), \theta(x))\) with \( \mu'(x) > 0 \) (< 0).

**Main Result:**

Proposition (Condition for PAM)

A necessary condition to have equilibria with PAM for any arbitrary distribution of types is

\[
F_{12} F_{34} \geq F_{23} F_{14}
\]

for all \((x, y, l, r)\). The strict inequality is also sufficient, and guarantees that no other equilibria exist. The reverse inequality is necessary and sufficient for NAM.
Definition (Assortative Matching)

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- Interpretation \((F_{34} > 0\) by assumption):
  1. \(F_{12} > 0\): bet. manag. produce more w/ bet. workers (Becker)
  2. \(F_{23} > 0\): bet. manag., larger span of control (as in Lucas)
  3. \(F_{14} > 0\): bet. workers produce more w/ manag. time (school?)
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• Interpretation (\( F_{34} > 0 \) by assumption):
  1. \( F_{12} > 0 \): better manager produces more with better workers (Becker)
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• Quantity-quality trade-off by firm \( y \) with resources \( r \):  
  1. \( F_{12} \): better manager manages quality workers better vs. 
  2. \( F_{23} \): better managers can manage more people 
  \( \Rightarrow \) Marginal increase of better \( \geq \) marginal impact of more workers
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- **Quantity-quality trade-off by firm** \(y\) **with resources** \(r\):
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\[ \Rightarrow \text{Marginal increase of better} \geq \text{marginal impact of more workers} \]

- **Examples**: technological differences across industries, establishments

1. Walmart vs. mom-&-pop store: low \(x\), high \(y\), high \(\theta, \theta' < 0\)
   \[ \Rightarrow F_{23} > 0, F_{14} > 0, F_{12} \text{ not too large} \Rightarrow \text{NAM} \]
2. Law firm, Mgt Consulting: high \(x\), high \(y\), low \(\theta, \theta' > 0\)
   \[ \Rightarrow F_{14} > 0, F_{23} > 0, F_{12} \text{ large} \Rightarrow \text{PAM} \]
Sketch of Proof of PAM-Condition

Assume PAM allocation with resources on \((x, \mu(x), \theta(x))\). Must be optimal, i.e., maximizes:

\[
\max_{x, \theta} f(x, \mu(x), \theta) - \theta w(x).
\]

First order conditions:

\[
\begin{align*}
f_{\theta}(x, \mu(x), \theta(x)) - w(x) &= 0 \quad (1) \\
f_x(x, \mu(x), \theta(x)) - \theta(x)w'(x) &= 0, \quad (2)
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The Hessian is

\[
Hess = \begin{pmatrix}
f_{\theta\theta} & f_{x\theta} - w'(x) \\[2pt] f_{x\theta} - w'(x) & f_{xx} - \theta w''(x) \end{pmatrix}.
\]

Second order condition requires \(|Hess| \geq 0\):

\[
f_{\theta\theta}[f_{xx} - \theta w''(x)] - (f_{x\theta} - w'(x))^2 \geq 0.
\]

Differentiate (1) and (2) with respect to \(x\), substitute:

\[-\mu'(x)[f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_{x}/\theta] \geq 0
\]

Positive sorting means \(\mu'(x) > 0\), requiring \([\cdot] < 0\) and after rearranging:

\[
F_{12}F_{34} \geq F_{23}F_{14}.
\]
$F_{12}F_{34} > F_{23}F_{14}: \textsc{Graphical}$

Budget Set: $D = \{(x, l)|lw(x) \leq M\}$

Iso-output Curve: $i_y = \{(x, l)|F(x, y, l, 1) = \Pi\}$

Slope of Iso-output Curve: $\frac{\partial l}{\partial x} = -\frac{F_1(x, y, l, 1)}{F_3(x, y, l, 1)}$.

Fix $F_{23} > 0$ and consider better firm:

- If $F_{12} \approx 0$, higher $y$ has flatter slope (numerator is constant).
- If $F_{12} \gg 0$, then higher $y$ will have steeper slope.
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Assume $F_{12}F_{34} > F_{23}F_{14}$ but negative sorting. Then improved output after re-sorting.
Efficiency: Gains from “Re-sorting”

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Special Cases

Efficiency Units of Labor

- Skill “=” Quantity: $F(x, y, l, r) = \tilde{F}(y, xl, r) \Rightarrow F_{12}F_{34} = F_{23}F_{14}$
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Multiplicative Separability

- $F(x, y, l, r) = A(x, y)B(l, r)$ sorting if $\frac{AA_{12}}{A_1A_2} \frac{BB_{12}}{B_1B_2} \geq 1$
- If $B$ is CES with elast. of substitution $\epsilon$: $\frac{AA_{12}}{A_1A_2} \geq \epsilon$ (root-sm)

Becker's one-on-one matching

- $F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\}$ like inelastic CES ($\epsilon \to 0$) so sorting if $F_{12} \geq 0$

Sattinger's span of control model

- $F(x, y, l, r) = \min\{rt(x, y), l\}$; write as CES between both arguments
- Our condition converges for inelastic case to log-supermod. in qualities

Extension of Lucas' span of control model

- $F(x, y, l, r) = yg(x, l/r)\frac{r}{l}$, sorting only if good types work less well together ($g_2 g_{12} \geq g_1 g_{22} + g_{1} g_{2}/\theta$).

Spacial sorting in mono-centric city:

- $F(x, y, l, r) = l(xg(y) + v(r/l)) \Rightarrow$ higher earners in center.
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**Becker’s one-on-one matching**
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Extension of Lucas’ span of control model
- $F(x, y, l, r) = yg(x, l/r)r$, sorting only if good types work less well together $(g_2g_{12} \geq g_1g_{22} + g_1g_2/\theta)$. 
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- If \( B \) is CES with elast. of substitution \( \epsilon: \frac{AA_{12}}{A_1A_2} \geq \epsilon \) (root-sm)

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- \( F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\} \),
- Like inelastic CES (\( \epsilon \rightarrow 0 \)), so sorting if \( F_{12} \geq 0 \)

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Spatial sorting in mono-centric city:

- \( F(x, y, l, r) = l(xg(y) + v(r/l)) \Rightarrow \) higher earners in center.
**Proposition**

Under assortative matching (symmetric distributions of $x, y$):

**PAM** : \[ \theta'(x) = \frac{F_{23} - F_{14}}{F_{34}}; \quad \mu'(x) = \frac{1}{\theta(x)}; \quad w'(x) = \frac{F_1}{\theta(x)}, \]

**NAM** : \[ \theta'(x) = -\frac{F_{23} + F_{14}}{F_{34}}; \quad \mu'(x) = \frac{-1}{\theta(x)}; \quad w'(x) = \frac{F_1}{\theta(x)}, \]
Firm Size, Assignment, Wages

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**PAM** : \[ \theta'(x) = \frac{F_{23} - F_{14}}{F_{34}}; \quad \mu'(x) = \frac{1}{\theta(x)}; \quad w'(x) = \frac{F_1}{\theta(x)}, \]

**NAM** : \[ \theta'(x) = -\frac{F_{23} + F_{14}}{F_{34}}; \quad \mu'(x) = -\frac{1}{\theta(x)}; \quad w'(x) = \frac{F_1}{\theta(x)}, \]

**Proof:** \( \mu' \) from market clearing: \[ H_w(\bar{x}) - H_w(x) = \int_{\mu(x)}^{\bar{y}} \theta(\tilde{x})h_f(\tilde{x})d\tilde{x} \]

\( \theta' \) from FOC: \( f_\theta = w(x) \) and \( f_x/\theta = w' \), diff. and subst. \( \mu' \).
**Firm Size, Assignment, Wages**

**Proposition**

*Under assortative matching (symmetric distributions of x, y):*

\[
PAM : \quad \theta'(x) = \frac{F_{23} - F_{14}}{F_{34}}; \quad \mu'(x) = \frac{1}{\theta(x)}; \quad w'(x) = \frac{F_1}{\theta(x)},
\]

\[
NAM : \quad \theta'(x) = -\frac{F_{23} + F_{14}}{F_{34}}; \quad \mu'(x) = \frac{-1}{\theta(x)}; \quad w'(x) = \frac{F_1}{\theta(x)},
\]

**Proof:** \( \mu' \) from market clearing: \( H_w(\bar{x}) - H_w(x) = \int_{\mu(x)}^{\bar{y}} \theta(\tilde{x}) h_f(\tilde{x}) d\tilde{x} \)

\( \theta' \) from FOC: \( f_{\theta} = w(x) \) and \( f_x/\theta = w' \), diff. and subst. \( \mu' \).

**Corollary**

*Under assortative matching, better firms hire more workers if and only if along the equilibrium path*

\( F_{23} > F_{14} \) under PAM, and \( -F_{23} < F_{14} \) under NAM.
**Proposition**

*Under assortative matching*

\[ \mathcal{H}(x) = \frac{h_w}{h_f} \]

**PAM** : \[ \theta'(x) = \frac{\mathcal{H}(x)F_{23} - F_{14}}{F_{34}}; \quad \mu'(x) = \frac{1}{\theta(x)}\mathcal{H}(x); \quad w'(x) = \frac{F_1}{\theta(x)}, \]

**NAM** : \[ \theta'(x) = -\frac{\mathcal{H}(x)F_{23} + F_{14}}{F_{34}}; \quad \mu'(x) = \frac{-1}{\theta(x)}\mathcal{H}(x); \quad w'(x) = \frac{F_1}{\theta(x)}, \]

**Proof:** \( \mu' \) from market clearing: \( H_w(\tilde{x}) - H_w(x) = \int_{\mu(x)}^{\tilde{y}} \theta(\tilde{x})h_f(\tilde{x})d\tilde{x} \)

\[ \theta' \] from FOC: \( f_{\theta} = w(x) \) and \( f_x/\theta = w' \), diff. and subst. \( \mu' \).

**Corollary**

*Under assortative matching, better firms hire more workers if and only if along the equilibrium path*

\[ \mathcal{H}(x)F_{23} > F_{14} \text{ under PAM, and} \quad -\mathcal{H}(x)F_{23} < F_{14} \text{ under NAM.} \]
Firm Size under PAM

\[ F_{23} > F_{14} \]

- Firm size increasing depends on relative strength of
  1. \( F_{23} \): span of control
  2. \( F_{14} \): resource intensity of labor

- If marginal impact of output from firm \( y \)' span of control is larger than worker \( x \)'s marginal impact of resources \( \Rightarrow \) high productivity
  firms are larger
**Firm Size under PAM**

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• If marginal impact of output from firm \( y \)’ span of control is larger than worker \( x \)’s marginal impact of resources \( \Rightarrow \) high productivity firms are larger

• Special case: Lucas 78
General Capital, Monopolistic Competition

- General Capital:
  - \( F(x, y, l, r) = \max_k \hat{F}(x, y, l, r, k) - ik \); Sorting cond. on max
    \[
    \hat{F}_{12}\hat{F}_{34}\hat{F}_{55} - \hat{F}_{12}\hat{F}_{35}\hat{F}_{45} - \hat{F}_{15}\hat{F}_{25}\hat{F}_{34} \geq \hat{F}_{14}\hat{F}_{23}\hat{F}_{55} - \hat{F}_{14}\hat{F}_{25}\hat{F}_{35} - \hat{F}_{15}\hat{F}_{23}\hat{F}_{45}
    \]
General Capital, Monopolistic Competition

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  \]

- Monopolistic Competition in the Output Market:
  - consumers have CES preferences with substitution \( \rho \)
  - sales revenue of firm \( y \): \( \chi F(x, y, l, 1)^\rho \)
  - Sorting condition
    \[
    \left[ \rho \tilde{F}_{12} + (1 - \rho)(\tilde{F}) \frac{\partial^2 \ln \tilde{F}}{\partial x \partial y} \right] \left[ \rho \tilde{F}_{34} - (1 - \rho)\tilde{F} \frac{\partial^2 \ln \tilde{F}}{\partial l^2} \right]
    \geq \left[ \rho \tilde{F}_{23} + (1 - \rho)\tilde{F} \frac{\partial^2 \ln \tilde{F}}{\partial y \partial l} \right] \left[ \rho \tilde{F}_{14} + (1 - \rho)\left( l\tilde{F}_{13} - \tilde{F} \frac{\partial^2 \ln \tilde{F}}{\partial x \partial r} \right) \right].
    \]
  - independent of \( \chi \)
  - our condition under \( \rho = 1 \), log-sm when production linear in \( l \).
Search, Firm Size and Sorting

- Existing literature on search and firm size: identical workers (Smith 99, Acemoglu-Hawkins 06, Mortensen 09, Kaas-Kircher 10, Helpman-Itsphoki-Redding 10, Menzio-Moen 10, ...).

- Vacancy filling prob \( m(q) \). Job finding prob \( m(q)/q \). Post \((x, \nu_x, \omega_x)\)

\[
\max_{r_x, l_x, \omega_x, \nu_x} \int [F(x, y, l_x, r_x) - l_x\omega_x - \nu_x c] \, dx
\]

s.t. \( l_x = \nu_x m(q_x) \); and \( \omega_x m(q_x)/q_x = w(x) \).
Search, Firm Size and Sorting

- Existing literature on search and firm size: identical workers (Smith 99, Acemoglu-Hawkins 06, Mortensen 09, Kaas-Kircher 10, Helpman-Itskhoki-Redding 10, Menzio-Moen 10,...).

- Vacancy filling prob $m(q)$. Job finding prob $m(q)/q$. Post $(x, v_x, \omega_x)$

$$
\max_{r_x, l_x, \omega_x, v_x} \int [F(x, y, l_x, r_x) - l_x \omega_x - v_x c] \, dx
$$

s.t. $l_x = v_x m(q_x)$; and $\omega_x m(q_x)/q_x = w(x)$.

- Two equivalent formulations:
  1. $\max_{s_x, r_x} \int [G(x, y, s_x, r_x) - w(x)s_x] \, dx$, where
     $$
     G(x, y, s_x, r_x) = \max_{v_x} [F(x, y, v_x m(s_x/v_x), r_x) - v_x c].
     $$
  2. $\max_{r_x, l_x, v_x} \int [F(x, y, l_x, r_x) - C(x, l_x)] \, dx$, where
     $$
     C(x, l_x) = \min_{v_x, q_x} cv_x + q_x v_x w(x) \text{ s.t. } l_x = v_x m(q_x).
     $$
Search, Firm Size and Sorting

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\[
\max_{r_x, l_x, \omega_x, v_x} \int [F(x, y, l_x, r_x) - l_x \omega_x - v_x c] \, dx \\
\text{s.t. } l_x = v_x m(q_x); \quad \text{and} \quad \omega_x m(q_x)/q_x = w(x).
\]

- Two equivalent formulations:
  1. \( \max_{s_x, r_x} \int [G(x, y, s_x, r_x) - w(x)s_x] \, dx \), where
     \[
     G(x, y, s_x, r_x) = \max_{v_x} [F(x, y, v_x m(s_x/v_x), r_x) - v_x c].
     \]
  2. \( \max_{r_x, l_x, v_x} \int [F(x, y, l_x, r_x) - C(x, l_x)] \, dx \), where
     \[
     C(x, l_x) = \min_{v_x, q_x} cv_x + q_x v_x w(x) \text{ s.t. } l_x = v_x m(q_x).
     \]

- Check sorting, compute \( w(x) \) as in previous part.
- Determine unemployment. FOC

\[
w(x)q_x = \frac{\eta(q)}{1 - \eta(q)c}
\]
Search, Firm Size and Sorting

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- Vacancy filling prob \(m(q)\). Job finding prob \(m(q)/q\). Post \((x, v_x, \omega_x)\)

\[
\max_{r_x, l_x, \omega_x, v_x} \int [F(x, y, l_x, r_x) - l_x \omega_x - v_x c] \, dx
\]

s.t. \(l_x = v_x m(q_x)\); and \(\omega_x m(q_x)/q_x = w(x)\).

- Two equivalent formulations:
  1. \(\max_{s_x, r_x} \int [G(x, y, s_x, r_x) - w(x)s_x] \, dx\), where

\[
G(x, y, s_x, r_x) = \max_{v_x} [F(x, y, v_x m(s_x/v_x), r_x) - v_x c].
\]
  2. \(\max_{r_x, l_x, v_x} \int [F(x, y, l_x, r_x) - C(x, l_x)] \, dx\), where

\[
C(x, l_x) = \min_{v_x, q_x} cv_x + q_x v_x w(x) \text{ s.t. } l_x = v_x m(q_x).
\]

- Check sorting, compute \(w(x)\) as in previous part.

- Determine unemployment. FOC (simple closed form with const. elasticity \(\alpha\))

\[
w(x)q_x = \frac{\eta(q)}{1 - \eta(q)} c = \frac{1 - \alpha}{\alpha} c
\]
Unemployment, Firm Size and Sorting

Proposition

The unemployment rate is falling in worker skills.

- $\eta(q)$ weakly decreasing $\Rightarrow q$ decreasing in $x$
** Proposition **

The unemployment rate is falling in worker skills.

- \( \eta(q) \) weakly decreasing \( \Rightarrow q \) decreasing in \( x \)

** Proposition **

The vacancy rate is ambiguous in firm size.

- Consider PAM (likewise for NAM)
- Vacancies \((1/q)\) increasing in \( x \)
- Firm size ambiguous in \( y \) : \( F_{23} \geq F_{14} \)
CONCLUSION

This work:

- Lay out a matching model with factor intensity
- Derive tractable sorting condition \((F_{12}F_{34} \geq F_{14}F_{23})\)
- Characterize equilibrium firm size \((F_{23} > F_{14})\), assignment and wages
- Search frictions: relation unemployment, skill and firm size
Conclusion

This work:
- Lay out a matching model with factor intensity
- Derive tractable sorting condition ($F_{12}F_{34} \geq F_{14}F_{23}$)
- Characterize equilibrium firm size ($F_{23} > F_{14}$), assignment and wages
- Search frictions: relation unemployment, skill and firm size

Economic Relevance & Applications in trade/macro/labor...:
- Mismatch debate: worker heterogeneity matters
- Comparative statics: impact of aggregate fluctuations
- Empirical: How does unemployment change across skills/firm size?
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01, ...)

- Searchers per vacancy: $q = s/v$
- Vacancy filling prob: $m$
- Job finding prob: $m/q$
Wage posting and frictional hiring \( (\text{Peters 90, Burdett-Shi-Wright 01, ...}) \)

- Searchers per vacancy: \( q = s/v \)
- Vacancy filling prob: \( m \)
- Job finding prob: \( m/q \)
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

- Searchers per vacancy: $q = s/v$
- Vacancy filling prob: $m = \frac{3}{4}$
- Job finding prob: $m/q = \frac{3}{4}$
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01, ...)

- Searchers per vacancy: \( q = s/v \)
- Vacancy filling prob: \( m \)
- Job finding prob: \( m/q \)
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

- Searchers per vacancy: $q = s/v$
- Vacancy filling prob: $m(q) \rightarrow 1 - e^{-q}$
- Job finding prob: $m/q \rightarrow \frac{1 - e^{-q}}{q}$
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

- Searchers per vacancy: \( q = \frac{s}{v} \)
- Vacancy filling prob: \( m(q) \rightarrow 1 - e^{-q} \)
- Job finding prob: \( \frac{m}{q} \rightarrow \frac{1 - e^{-q}}{q} \)
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

- Searchers per vacancy: \( q = s/v \)
- Vacancy filling prob: \( m(q) \rightarrow 1 - e^{-q}, m' \)
- Job finding prob: \( m/\omega q \rightarrow \frac{1-e^{-q}}{q}, \omega' \frac{m(q')}{q'} = \omega \frac{m(q)}{q} = w(x) \)