Assortative Learning*

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Abstract

Because of sorting, more skilled workers are more productive in higher type firms. They also learn at different rates about their productivity and therefore expect different wage paths across firms. We show that under strict supermodularity there is always positive assortative matching: differential learning is always dominated by the impact of productivity. Surprisingly, this holds even if learning is faster in the low type firm. The key assumption driving this result is that this is a pure Bayesian learning model. We also derive a new equilibrium condition in this class of continuous time models in addition to the common smooth-pasting and value-matching conditions. This *no-deviation condition* captures sequential rationality and results in a restriction on the second derivative of the value function.

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JEL. D83. C02. C61.

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1 Introduction

High ability workers sort into more productive jobs. Due to complementarities in production, their higher marginal product allows them to command higher wages. The Beckerian model of assortative matching is very well suited to explain those patterns of sorting. Unfortunately, it is mute on the issue of turnover of workers between different jobs. Instead, the Jovanovic (1979) learning model has long been the canonical framework for analyzing turnover in the labor market[1] over the life cycle. Workers and firms learn about match-specific human capital and will tend to stay in a match if learning reveals the match is good. Experimentation occurs early on which leads to decreasing turnover over the life cycle. Because in Jovanovic (1979) learning is about the match and not about the worker, there is neither worker heterogeneity nor sorting. In this paper, we offer a unified approach of learning and sorting. We establish a solution method for a market equilibrium in a continuous time economy with multiple learning opportunities (multi-armed bandit) and derive a no-deviation condition, a condition hitherto unknown. We show that under supermodularity, positive assortative matching obtains in equilibrium, even if learning rates differ across firms.

In the labor market, the learning experiences of workers are most likely to differ across different firms. Starting in a top law firm or a multinational will induce different paths of information revelation than working in a local family business. The worker now faces a trade-off between different experimentation experiences: take a lower wage at a high productivity firm where information may be revealed at a different rate or accept higher wage and learn more slowly. It is intuitive that sorting and learning are intimately connected.

Modeling the labor market as a multi-armed bandit problem and solving it is challenging. Most existing learning models and continuous time games are tractable because they are essentially one-armed bandit problems with a fixed outside option that acts as an absorbing state. One-armed bandit problems typically have attractive properties, including reservation strategies. Instead, multi-armed bandits in general do not have reservation strategies when arms are correlated, even if the learning rate is the same across firms.[2] But our labor market is not exactly identical to the canonical bandit problem. First, there are a continuum of experimenters, and as a result of two-sided heterogeneity, deviations and off-equilibrium path beliefs non-trivially affect equilibrium. Second, because of competitive wage determination à la Jovanovic (1979), the payoffs are endogenous. Finally, because workers learn about general human capital instead of match-specific human capital, the arms are positively correlated.

We find that it is the combination of competitive wage determination (endogenous payoffs)

[1] Of course, also the search model inherently exhibits turnover, but with observable types turnover is constant over the life cycle. Moscarini (2005) brings together search and learning in the Jovanovic framework.

[2] See for example Chernoff (1968). Only with multiple *independent* arms are reservation strategies guaranteed, and the Gittins index policy (in discrete time) is optimal.

and the incentives needed to avoid a deviation that give rise to a new condition which we call the *no-deviation condition*. This condition must be satisfied in addition to the common equilibrium conditions of value-matching and smooth-pasting. The no-deviation condition can be interpreted as the continuous time version of the one-shot deviation principle.[3] We prove that the no-deviation condition implies that the second derivative of worker's value function at the cut-off belief is the same in the high type as well as in the low type firms. Recall that value matching requires that at the cut-off the worker's value functions take the same value in both firms, the smooth-pasting condition requires that the first derivative is the same, and now the no-deviation requires equal second derivatives as well.

We show that supermodularity of the production technology is a necessary and sufficient condition for positive assortative matching, and that the equilibrium allocation is unique. Those workers with the highest beliefs about their ability will in equilibrium sort into those firms that are most productive. Moreover, we can analytically solve for the equilibrium allocation in terms of the cut-off belief, and we derive in closed form the stationary distribution of beliefs.

While in most of the analysis we consider common variance across firms, it turns out that the sorting result holds for different learning rates (noise) across firms, even if the rate of learning is slower in the high type firm. It is conceivable that with supermodularity and a learning rate no smaller in high types firms there will be positive sorting. The high type firm is both superior in the learning rate and in productive efficiency. But if high type firms learn at a sufficiently slower rate (the noise is sufficiently high), then the signal-to-noise ratio in the high type firm may well be lower. The reason why this nonetheless does not affect the learning is that the value of learning also depends on the degree of convexity of the value function (from Ito's Lemma), in addition to the signal-to-noise ratio. But by the no-deviation condition, at the cut-off belief, the degree of convexity is the same in both firms and therefore the equilibrium value of learning is the same, no matter the difference in signal-to-noise ratios. Key here is that wages are endogenous and determined competitively. That is why this property does not necessarily hold in the canonical multi-armed bandit problem.

We analyze the planner's problem and show that a planner's stationary allocation coincides with the decentralized equilibrium allocation, even if learning rates differ across different firms. This is surprising since there is a market incompleteness: wages are spot market prices only and cannot be made contingent on future realizations. It turns out that the efficiency result and proof crucially hinges on the martingale property inherent in Bayesian learning. The martingale property implies that no matter how fast workers learn, the expected beliefs about their ability will stay the

[3]The idea of sequential rationality is of course not new and has also been employed in continuous time games by Sannikov (2007) who uses the concept of self generation. And Cohen and Solan (2009) use dependence of strategies on a small interval dt to restrict the set of Markovian strategies, in the spirit of our dt-shot deviation. It is precisely the one-shot deviation *in conjunction* with endogenous payoffs that leads to the equalization of the second derivative of the value functions.

same. Since under strict supermodularity, the differential in expected output between working in high and low productivity firms is monotonically increasing in the likelihood that the worker has high ability, reallocating a group of low belief workers to a better match will decrease expected outputs no matter how fast they learn.

We extend our analysis of Bayesian learning to allow for *observable* human capital accumulation. This adds realism in the sense that workers learn on the job and increase their productivity with tenure, yet we do not resort to non-Bayesian updating. Now cut-off types that characterize the equilibrium allocation depend on the degree of observable experience, and beliefs continue to follow a martingale process. The properties of our equilibrium extend to this more general human capital accumulation case.

The motivation of our analysis and the results are obviously closest related to the labor market learning literature (Jovanovic (1979, 1984), Harris and Holmström (1982), Moscarini (2005) and Papageorgiou (2009)).[4] Yet, there is a close relation to both the experimentation literature (Bolton and Harris (1999), Keller, Rady, and Cripps (2005), Strulovici (2010)) and the literature on continuous time games (Sannikov (2007, 2008), Faingold and Sannikov (2009)). Most models of learning have a finite set of players and have an absorbing state. Ours has a continuum of agents and there is learning in all states. Moreover, it is essentially a competitive model with equilibrium prices and therefore payoffs from learning are endogenous.

The idea of analyzing a matching model where the current allocation determines the future type is first explored in Anderson and Smith (2010). They find the opposite result of ours: positive assortative matching fails even under supermodularity. They analyze a two-sided matching model of reputations with imperfect information about both matched types.[5] Our setup differs substantially, but the main difference is in the information extraction. Their agents infer the type of each of the matched partners from the realization of a *joint* signal.[6]

Another key characteristic of our model is that it is a pure Bayesian learning model where beliefs follow a martingale. In Section 8 we show that our result holds for Bayesian updating processes other than the Brownian motion (we extend our result to a generalized Lévy process), and we also establish that positive assortative matching can fail if the updating process is not Bayesian (this can be interpreted for example as a technology of *unobserved* human capital accumulation in

[4] Papageorgiou (2009) analyzes a learning model with heterogeneity. He estimates the version of Moscarini's search model with two-sided heterogeneity. With search frictions, wage setting is non-competitive and as a result, the no-deviation condition is not imposed in addition to value matching and smooth pasting. Nonetheless, his findings provide us with realistic estimates of the labor market characteristics of our model. See also Groes, Kircher and Manovskii (2009) for estimates of a different learning model.

[5] Our model is more closely related to the standard firm-worker model to which they compare their two-sided model in the discussion. There is only a one-sided inference problem in that model and they find that positive assortative matching arises for extreme beliefs $p = 0$ and 1, but conjecture it does not in the interior.

[6] The difficulty is to account for agents switching partners. Anderson and Smith (2010) resolve this by assuming symmetric learning in discrete time. Both sides of the market update in an identical fashion and under PAM their new matched partner coincides exactly with the updated type of their old partner. As a result, in a candidate PAM equilibrium there is never any switching.

addition to the information extraction).

2 The Model Economy

Population of Firms and Workers. The economy is populated by a unit measure of workers and a unit measure of firms. Both firms and workers are *ex ante* heterogeneous. The firm's type $y \in \{H, L\}$ represents its productivity. The type y is observable to all agents in the economy. The fraction of H type firms is π and all firms are infinitely lived. The worker ability $x \in \{H, L\}$ is not observable, both to firms and workers, i.e., information is symmetric.[7] Nonetheless, both hold a common belief about the worker type, denoted by $p \in [0, 1]$. Upon entry, a newly born worker is of type H with probability p_0 and of type L with probability $1 - p_0$. Workers die with exogenous probability δ. New workers are born at the same rate.[8]

Preferences and Production. Workers and firms are risk-neutral and discount future payoffs at rate $r > 0$. Utility is perfectly transferable. Output is produced in pairs of one worker and one firm (x, y). Time is continuous. Positive output produced consists of a divisible consumption good and is denoted by μ_{xy}. We assume that more able workers are more productive in any firm, $\mu_{Hy} \geq \mu_{Ly}, \forall y$ and refer to it as worker monotonicity. While it is often useful, we do not in general assume firm monotonicity, which would be $\mu_{xH} \geq \mu_{xL}, \forall x$. Strict supermodularity is defined in the usual way:

$$\mu_{HH} - \mu_{LH} > \mu_{HL} - \mu_{LL}, \tag{1}$$

and with the opposite sign for strict submodularity. In the entire paper, we will refer to strict supermodularity when we just mention supermodularity, likewise for submodularity.

Information. Because worker ability is not observable to both the worker and the firm, parties face an information extraction problem. They observe a noisy measure of productivity, denoted by X_t. Cumulative output is assumed to be a Brownian motion with drift μ_{xy} and common variance σ^2

$$X_t = \mu_{xy} t + \sigma Z_t \tag{2}$$

where Z_t is a standard Wiener process and as a result, X_t is normally distributed with mean $\mu_{xy} t$ and variance $\sigma^2 t$. By Girsanov's Theorem the probability measures over the paths of two diffusion processes with the same volatility but different bounded drifts are equivalent, that is, they have the same zero-probability events. Since the volatility of a continuous-time diffusion process is effectively

[7] This substantially simplifies the problem at hand. With private signals Cripps, Ely, Mailath and Samuelson (2008) show that with a finite signal space there will be common learning, but not necessarily with an infinite signal space as is the case in our model here.

[8] Without death, we know the posterior belief will converge with probability one to $p = 1$ or $p = 0$. Death here actually acts as a shuffling device to guarantee a non-trivial stationary distribution of posterior beliefs.

observable, the worker's type could be learned directly from the observed volatility if σ depends on workers' types.[9]

Equilibrium. We consider a stationary competitive equilibrium in this economy. With two types of firms and a continuum of p's in this market, take a competitive wage schedule $w_y(p)$ as given which specifies wage for every possible type p worker working in firm y.[10] Denote by V_y the stationary discounted present value of the competitive profits for firm y. The flow profit can be written as rV_y.[11] Now we are ready to define the notion of competitive equilibrium:

Definition 1 *A stationary competitive equilibrium consists of a competitive wage schedule* $w_y(p) = \mu_y(p) - rV_y$*, where* $\mu_y(p) = p\mu_{Hy} + (1-p)\mu_{Ly}$ *denotes worker* p*'s expected productivity in firm* $y = H, L$ *and worker* p *chooses the firm* y *with the highest discounted present value. The market clears such that the measure of workers in* L *firms is* $1 - \pi$ *and the measure of workers in* H *firms is* π*.*

We would like to point out several things about this definition. First, the definition of competitive equilibrium implies identical types will obtain the same payoff. A firm y earns the same flow profit for every p. Our notion of competitive equilibrium puts restrictions on the off-equilibrium prices, as does the Beckerian definition of a matching equilibrium. Although type p worker is not employed by firm y on the equilibrium path, the hypothetical wage is still $w_y(p) = \mu_y(p) - rV_y$ to guarantee the firm cannot make or lose money if the employment suddenly happens. Second, our wage definition concerns a spot market wage and captures the idea that firms cannot commit to future actions or realizations (see also Hörner and Samuelson (2009) for a model of experimentation in the presence of spot market contracts). Together with sequential rationality, this therefore requires that the wage contract is self-enforcing. We believe this is realistic since it is consistent with the at-will employment doctrine in which parties are free to terminate employment relations with no liability. Our spot market wage assumption is in contrast with Anderson and Smith (2010), who parse the wage into a static wage plus a dynamic human capital effect. Their wage setting process therefore corresponds to the Pareto efficient allocation. Third, like all price taking economies, the wage schedule essentially transforms our problem into a decision problem for the workers.

[9] However, we can allow σ to be firm-specific. In section 8 we analyze the general case of firm-dependent σ_y.

[10] Bergemann and Välimäki (1996) and Felli and Harris (1996) consider a two-firm, one-worker/buyer model with strategic price setting in a world with independent arms. With ex ante heterogeneous firms and workers and correlated arms, we instead focus on competitive price setting which is closest in spirit to the Beckerian benchmark.

[11] Notice since there is no free entry, V_y need not to be zero. We could model free entry as long as in equilibrium there is a non-degenerate distribution of firm types in the economy. We consider this does not add to the insights of our model.

3 Preliminaries

3.1 Benchmark: No Learning

Workers differ in the common beliefs p of being a high type. We shut down learning so that beliefs are invariant. This can be viewed as a special case of the learning model with the variance σ^2 going to infinity. We assume that there is no birth or death so we essentially have a static problem. Suppose without loss of generality that p is uniformly distributed on $[0, 1]$. We continue to maintain the assumption that the worker does not know her true type or that she has no private information about it. Denote $\mu_y(p) = p\mu_{Hy} + (1-p)\mu_{Ly}$ for $y = H, L$ and r as the discount rate.

Under the above notion of competitive equilibrium, it is easy to verify the following claim (All of the results in this paper are in the sense of "almost surely" because we allow a zero measure of agents to behave differently):

Claim 1 *Under strict supermodularity, PAM is the unique (stationary) competitive equilibrium allocation: H firms match with workers $p \in [1-\pi, 1]$, L firms match with workers $p \in [0, 1-\pi)$. The opposite (NAM) holds under strict submodularity: H firms match with workers in $[0, \pi)$.*

Since there is no learning, essentially this result is identical to Becker's (1973) result, but with uncertainty. Noteworthy about this version of Becker is that even though for PAM there is supermodularity of the ex-post payoffs ($\mu_{HH} + \mu_{LL} > \mu_{HL} + \mu_{LH}$), there need not be monotonicity in expected payoffs, i.e., $\mu_H(1-\pi)$ may be smaller than $\mu_L(1-\pi)$. In fact, that will be reflected in the firm's equilibrium payoffs: $V_H \geq V_L$ if and only if $\mu_H(1-\pi) \geq \mu_L(1-\pi)$.

As in Becker, the equilibrium allocation is unique, but there may be multiple splits of the surplus. In the case of PAM, we only require at the cutoff type $\underline{p} = 1-\pi$ that $w_H(\underline{p}) = w_L(\underline{p})$. There are multiple equilibrium payoffs if the surplus of a match between L and $p = 0$ is positive. Instead, if $\mu_L(0) = 0$,[12] there is a unique equilibrium payoff.

3.2 Belief Updating

In the presence of learning we can now derive the beliefs and subsequently the value functions. The posterior belief p_t that the worker has a high productivity is a sufficient statistic for the output history. Now, we can use the following well-known result: conditional on the output process $(X_t)_{t\geq 0}$, $(p_t)_{t\geq 0}$ is a martingale diffusion process. Moreover, this process can be represented as a Brownian motion. Based on the framework of our model, denote $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma,\ y = H, L$, $\Sigma_y(p) = \frac{1}{2}p^2(1-p)^2 s_y^2$ and then we get:

[12] And there is limited liability, i.e., workers and firms cannot receive negative payoffs.

Lemma 1 *(Belief Consistency) Consider any worker who works for firm y between t_0 and t_1. Given a prior $p_{t_0} \in (0,1)$, the posterior belief $(p_t)_{t_0<t\leq t_1}$ is consistent with the output process $(X_{y,t})_{t_0<t\leq t_1}$ if and only if it satisfies*

$$dp_t = p_t(1-p_t)s_y d\bar{Z}_{y,t}$$

where

$$d\bar{Z}_{y,t} = \frac{1}{\sigma}[dX_{y,t} - (p_t\mu_{Hy} + (1-p_t)\mu_{Ly})dt].$$

The proof of this Lemma is in Faingold and Sannikov (2007) or Daley and Green (2008). The basic idea behind the proof is a combination of Bayes' rule and Ito's lemma. Given the period t posterior belief p_t and dX_t, we know the posterior belief at period $t+dt$ is:

$$p_{t+dt} = \frac{p_t \exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\}}{p_t \exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\} + (1-p_t)\exp\{-\frac{[dX_t-\mu_{Ly}dt]^2}{2\sigma^2 dt}\}}.$$

Hence,

$$dp_t = p_{t+dt} - p_t = p_t(1-p_t)\frac{\exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\} - \exp\{-\frac{[dX_t-\mu_{Ly}dt]^2}{2\sigma^2 dt}\}}{p_t \exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\} + (1-p_t)\exp\{-\frac{[dX_t-\mu_{Ly}dt]^2}{2\sigma^2 dt}\}}.$$

Apply Ito's Lemma and we obtain the above result.

Lemma 1 establishes that dp depends on three elements: $p(1-p)$, which peaks at $1/2$; the signal-to-noise ratio of output, $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma$ and $d\bar{Z}_y$, the normalized difference between realized and unconditionally expected flow output, which is a standard Wiener process with respect to the filtration $\{X_{y,t}\}$. Obviously, beliefs move faster the more uncertainty about worker's quality (p close to $1/2$); the less variation in the output process (smaller σ) and the larger the productivity difference (higher $\mu_{Hy} - \mu_{Ly}$).

Learning considerations will change the beachmark results. Moreover, supermodularity not only affects the value of the static output created as in the standard Beckerian model, but it also has dynamic effect by changing the speed of learning. For example, under supermodularity ($\mu_{HH} - \mu_{HL} > \mu_{LH} - \mu_{LL}$), the learning speed is faster in the high type firm, which is especially significant for p close to $1/2$. Intuitively speaking, learning makes it more attractive to match with a high type firm even though statically it is better for her to match with a low type firm without learning.

3.3 Value Functions

Consider any interval for the posterior belief $p \in [p_1, p_2]$ where the worker accepts the offer from a type y firm, then the value function is given by[13]:

$$rW_y(p) = \mu_y(p) - V_y + \Sigma_y(p)W_y^{''}(p) - \delta W_y(p), \tag{3}$$

from Ito's Lemma. The term $\mu_y(p) - V_y$ is equal to the flow wage payoff and corresponds to the deterministic component of the diffusion $X_{y,t}$, and the term $\Sigma_y(p)W_y^{''}(p)$ is the second-order term from the transformation W of the diffusion process $X_{y,t}$. First-order and all higher-order terms vanish as the time interval shrinks to zero. The general solution to this differential equation is:

$$W_y(p) = \frac{\mu_y(p) - V_y}{r + \delta} + k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y} + k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}, \tag{4}$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

First notice that the boundedness of the value function implies that if 0 is included in the domain, then $k_{y1} = 0$ and if 1 is included in the domain, then $k_{y2} = 0$. If not, with $\alpha_y > 1$ the value of W shoots off to infinity. Second, $\Sigma_y(p)W_y''(p)$ is the value of learning and this is an option value in the sense that the worker has the choice to change his job as he learns his type p. It is easy to verify that this value is zero if the worker never changes his job.[14] From the Martingale property of the Brownian motion, at any p the expected value of p in the next time interval is equal to p. There is as much good news as bad news to be expected in the next period. It is the option value of switching to a *more suitable* match that generates the value of learning. Equation (4) implies that this option value can be decomposed into two parts: $k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y}$ ($k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}$) denotes the option value of switching to a *more suitable* match when p goes down (up). The option value $k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y}$ ($k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}$) must be zero if 0 (1) is included since no switch happens as p goes down (up).

4 Analysis and Results

4.1 Characterization of the Equilibrium Allocation

Now consider any candidate stationary equilibrium where a type p worker switches from firm y to y'. Since the worker is essentially facing a two-armed bandit problem given the wage schedule,

[13]Note that we critically need the assumption that the worker does not have any private information about his type. If this assumption is violated, the worker's value functions could not be written like this.

[14]In that case, p can take both the values 0 and 1. So the boundedness of the value function requires that both k_{y1} and k_{y2} are zero and hence $W_y''(p) = 0$ for every p.

optimality in stopping time requires the value-matching condition (the worker gets the same value at the cutoff) and the smooth-pasting condition (the marginal of both value functions is identical) (see Dixit (1993)). For example, if for $p \in [p_1, p_2)$, the worker works in the low type firm and for $p \in [p_2, p_3)$, the worker works in the high type firm, then we must have:[15]

$$W_L(p_2) = W_H(p_2) \quad \text{and} \quad {W_L}'(p_2) = {W_H}'(p_2). \tag{5}$$

Notice that workers are price takers. As a result, there is no strategic interaction between players where equilibrium solves for the fixed point of individual strategies. It is also important to point out that both the value-matching condition and the smooth-pasting condition are on-equilibrium path conditions. They have nothing to do with the off-equilibrium path (i.e., instead of accepting offers from low type firms, workers with $p \in [p_1, p_2)$ are tempted to accept offers from high type firms). In the following lemmas we characterize the value functions establishing convexity and monotonicty:

Lemma 2 *The equilibrium value functions W_y are strictly convex for $p \in (0, 1)$.*

Proof. In Appendix. ∎

The intuition for this Lemma is the following. Preferences and output are linear in p, and the option value of learning is strictly positive, hence the value function with the option of learning is convex. To see this, observe that since the measure of both types of firms are strictly positive, market clearing requires that workers with some p's will be employed by high type firms while workers with other p's will be employed by low type firms. This implies that some worker has to change jobs at some point and the option value of learning $\Sigma_y(p){W_y}''(p)$ is strictly positive. Hence we have $W_y''(p) > 0$, for all $p \in (0, 1)$ since $\Sigma_y(p) > 0$. On the other hand, when $p = 0$ or 1, the posterior belief will always stay at 0 or 1 by Bayes' rule such that learning never happens. It is easy to verify that $W_y''(p) = 0$ for $p = 0$ or 1.

Given the strict convexity of equilibrium value functions and the smooth pasting condition, we can immediately derive the following Lemma:

Lemma 3 *The equilibrium value functions W_y are strictly increasing.*

Proof. In Appendix. ∎

One important implication is that if we define $\mathcal{W}(p)$ as the envelope of all equilibrium value functions $W_y(p)$, then this envelope function $\mathcal{W}(p)$ is continuous, strictly increasing and strictly

[15] We slightly abuse notation hers since W_L is not defined on p_2. A more precise way of writing the equations is $W_L(p_2+) = W_H(p_2)$ and ${W_L}'(p_2+) = {W_H}'(p_2)$. In what follows, we will continue to use the expression in the text in order to economize on notation.

convex for $p \in (0,1)$. Suppose workers with $p \in [0, \underline{p})$ are employed by type y firm and workers with $p \in (\bar{p}, 1]$ are employed by type $-y$ firm. Then we should have: $W'_y(0) = \frac{\mu_{Hy} - \mu_{Ly}}{r+\delta} < W'_{-y}(1) = \frac{\mu^{H,-y} - \mu^{L,-y}}{r+\delta}$. This gives us another result:

Lemma 4 *Under supermodularity, in any equilibrium $p = 0$ workers match with L firms; $p = 1$ workers match with H firms. The opposite under strict submodularity. Moreover,*

$$\frac{\min(\Delta_H, \Delta_L)}{r+\delta} < W'(p) < \frac{\max(\Delta_H, \Delta_L)}{r+\delta},$$

where $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$.

Intuitively this result is best understood by using the standard sorting argument from Becker (1973). At $p = 0$ and $p = 1$ there is no value of learning. As a result, there the value function can be interpreted as being determined by the no-learning allocation.

The properties derived above are mainly concerned with on-equilibrium path behavior. We also need to specify what happens in the event of deviations and consider behavior off-equilibrium path. We contemplate the equivalence of a one-shot deviation in continuous time because we think of the continuum as an idealization of discrete time. This amounts to a worker playing the deviant action over an interval $[t, t + dt)$ according to the belief p at time t, and considering the limit as $dt \to 0$.[16] This is very important because it allows us to derive the value function for deviation. On the contrary, if the deviation only takes place at a single point in time t, then the value function for deviation is essentially the same as the one without deviation because no information will be extracted from just a single time point.

The next Lemma establishes that if we consider off-the-equilibrium path deviations, we actually derive one additional condition, which we call the *no-deviation* condition.

Lemma 5 *To deter possible deviations, a necessary condition is:*

$$W''_H(\underline{p}) = W''_L(\underline{p}) \quad \textit{(No-deviation condition)} \tag{6}$$

for any possible cutoff $\underline{p}$.

Proof. Given wage schedule $w_y(p)$, a worker is facing a bang-bang control problem, which is:

$$W(p) = \max_{a(p)=\{H,L\}} \mathbb{E}\left\{\int_t^\infty e^{-(r+\delta)(s-t)} w_{a(p_s)}(p_s) ds\right\}$$

such that

$$p_t = p_0 \quad \text{and} \quad dp_s = s_{a(p_s)} p_s (1 - p_s) d\bar{Z}_{y_s,s}.$$

[16]This notion is also implicitly used in Sannikov (2007, Proposition 2), and also in Cohen and Solan (2009) who consider deviations from Markovian strategies in bandit problems.

Without loss of generality, we assume that on equilibrium path, a worker with $p > \underline{p}$ accepts offers from H firms and a worker with $p < \underline{p}$ accepts offers from L firms. Consider one possible one-shot deviation: at time t, a $p > \underline{p}$ worker matches with a low type firm for dt and then switch back. The on-equilibrium-path value function is defined as before (from Hamilton-Jacobi-Bellman equation):

$$(r+\delta)W(p) = (r+\delta)W_H(p) = w_H(p) + \Sigma_H(p)W_H''(p).$$

The deviator's new value could be written as:

$$\tilde{W}_L(p) = \mathbb{E}\left\{\int_t^{t+dt} e^{-(r+\delta)(s-t)}w_L(p_s)ds + e^{-(r+\delta)dt}W(p_{t+dt})\right\}. \quad (7)$$

Potentially, p_{t+dt} can take any value between 0 and 1. We have to show that as dt becomes very small, almost surely, p_{t+dt} will be close to p such that it is in the region where the worker will still accept offers from high type firms and hence $W(p_{t+dt}) = W_H(p_{t+dt})$.

By construction, the deviator's belief updating follows a Brownian motion: $dp_t = s_L p(1-p)d\bar{Z}_{y_t,t}$. Therefore, the probability that a worker $p > \underline{p}$ will have belief $p_{t+dt} \leq \underline{p}$ is given by $\Phi\left(\frac{\underline{p}-p}{s_L p(1-p)\sqrt{dt}}\right)$, where $\Phi(\cdot)$ is the cumulative distribution function for a standard normal distribution. Apply L'Hopital's rule and it is straightforward to see that[17]

$$\lim_{dt\to 0}\frac{\Phi\left(\frac{\underline{p}-p}{s_L p(1-p)\sqrt{dt}}\right)}{dt} = 0.$$

It is also possible for p_{t+dt} to increase above another cutoff $\bar{p}$ (if it exists) such that the worker will accepts offers from low type firms. Use the same logic and it is easy to find that the probability also goes to zero as $dt \to 0$.

Notice that for any $dt > 0$,

$$\begin{aligned} W_H(p) &> \tilde{W}_L(p) > \mathbb{E}\left\{\int_t^{t+dt} e^{-(r+\delta)(s-t)}w_L(p_s)ds\right\} \\ &+ \mathbb{E}e^{-(r+\delta)dt}\left[W_H(p_{t+dt})(1-\Pr(p_{t+dt}\notin(\underline{p},\bar{p}))) + \Pr(p_{t+dt}\notin(\underline{p},\bar{p}))W(0)\right]. \end{aligned} \quad (8)$$

The first inequality comes from the fact that there should be no profitable deviation. The second inequality is true because we replace the value for $p_{t+dt} \notin (\underline{p}, \bar{p})$ with the lowest value $W(0)$ ($W(\cdot)$ is an increasing function by Lemma 3). From Ito's Lemma, we can get for the deviator:

$$\mathbb{E}W_H(p_{t+dt}) = W_H(p) + \Sigma_L(p)W_H''(p)dt + o(dt).$$

[17] Another way to prove the claim is by applying a property of Brownian motion: For a Brownian motion X_t and any $\alpha < 1/2$, if h is sufficiently small, almost surely $|X_{t+h} - X_t| < Ch^{\alpha}$.

For any $dt > 0$, the no deviation condition implicit in equation (8) implies:

$$\frac{\mathbb{E}\{\int_t^{t+dt} e^{-(r+\delta)(s-t)} w_L(p_s) ds\}}{dt}$$
$$+ \quad \frac{\mathbb{E}\left\{e^{-(r+\delta)dt}[W_H(p_{t+dt})(1-\Pr(p_{t+dt} \notin (\underline{p}, \bar{p}))) + \Pr(p_{t+dt} \notin (\underline{p}, \bar{p}))W(0)]\right\} - W_H(p)}{dt} < 0.$$

Let $dt \to 0$ and first, it follows immediately that:

$$\lim_{dt \to 0} \frac{\mathbb{E}\left\{\int_t^{t+dt} e^{-(r+\delta)(s-t)} w_L(p_s) ds\right\}}{dt} = w_L(p).$$

Second, as proved earlier,

$$\lim_{dt \to 0} \frac{\Pr(p_{t+dt} \notin (\underline{p}, \bar{p}))}{dt} = 0.$$

Finally,

$$\begin{aligned} & \lim_{dt \to 0} \frac{\mathbb{E}\left\{e^{-(r+\delta)dt} W_H(p_{t+dt})(1-\Pr(p_{t+dt} \notin (\underline{p}, \bar{p})))\right\} - W_H(p)}{dt} \\ = & \lim_{dt \to 0} \frac{(e^{-(r+\delta)dt} - 1)W_H(p) + \Sigma_L(p) W_H''(p) dt + o(dt)}{dt} = \Sigma_L(p) W_H''(p) - (r+\delta) W_H(p). \end{aligned}$$

Therefore, the necessary condition such that a $p > \underline{p}$ worker has no incentive to deviate can be written as:

$$w_L(p) + \Sigma_L(p) W_H''(p) - (r+\delta) W_H(p) = w_L(p) + \Sigma_L(p) W_H''(p) - w_H(p) - \Sigma_H(p) W_H''(p) < 0. \quad (9)$$

The above inequality must hold for any $p > \underline{p}$. Let $p \to \underline{p}$ and we have:[18]

$$\begin{aligned} & w_L(\underline{p}) - w_H(\underline{p}) + [\Sigma_L(\underline{p}) - \Sigma_H(\underline{p})] W_H''(\underline{p}) \leq 0 \\ \Rightarrow\ & w_L(\underline{p}) + \Sigma_L(\underline{p}) W_L''(\underline{p}) - (w_H(\underline{p}) + \Sigma_H(\underline{p}) W_H''(\underline{p})) + (W_H''(\underline{p}) - W_L''(\underline{p}))\Sigma_L(\underline{p}) \leq 0 \\ & \Rightarrow W_H''(\underline{p}) \leq W_L''(\underline{p}). \end{aligned} \quad (10)$$

Similarly, we can consider another possible one-shot deviation: a $p < \underline{p}$ worker matches with a high type firm for dt and then switches back. The same logic establishes that to deter such deviation, it must be the case that:

$$w_H(p) - w_L(p) + [\Sigma_H(p) - \Sigma_L(p)] W_L''(p) < 0 \quad (11)$$

[18] As p goes to $\underline{p}+$, notice that $w_L(\underline{p}-) = w_L(\underline{p}+), \Sigma_L(\underline{p}-) = \Sigma_L(\underline{p}+)$. Hence, we will have: $w_L(\underline{p}-) + \Sigma_L(\underline{p}-)W_L''(\underline{p}-) - (w_H(\underline{p}+) + \Sigma_H(\underline{p}+)W_H''(\underline{p}+)) + (W_H''(\underline{p}+) - W_L''(\underline{p}-))\Sigma_L(\underline{p}-) \leq 0$.

for any $p < \underline{p}$. As p goes to $\underline{p}$, we should have:

$$w_H(\underline{p}) - w_L(\underline{p}) + [\Sigma_H(\underline{p}) - \Sigma_L(\underline{p})]W_L''(\underline{p}) \leq 0 \Rightarrow W_H''(\underline{p}) \geq W_L''(\underline{p}). \tag{12}$$

(10) and (12) imply that $W_H''(\underline{p}) = W_L''(\underline{p})$. ■

This no-deviation condition is quite unique for the two-armed bandit problem. This condition is absent in an one-armed bandit problem. Most of the models in the literature on continuous time learning models (Jovanovic (1979) and Moscarini (2005)) and continuous time games (see amongst others, Sannikov (2009)) are essentially investigating a one-armed bandit problem. There, we can directly look at equilibria in cutoff strategies. In the one-armed bandit problems, the safe arm essentially is an absorbing state so we only need to worry about the potential deviation from the risky arm to the safe arm.[19] Then the no-deviation condition becomes $W_H''(\underline{p}) \geq W_L''(\underline{p}) = 0$ but this is already implied by the convexity property.[20]

We provide some intuition for the no-deviation condition. By assuming Sequential Rationality, i.e., the equilibrium is robust to a one-shot deviation, we basically impose that the equilibrium wage is self-enforcing. There is no commitment to future realizations of X_t and therefore of future beliefs p. Now we can interpret W'' as the marginal value of learning: W' is the marginal change of W with respect to the posterior p, and learning changes p and is therefore quantified by the change in W' which is W''. The condition states that there is no deviation if the marginal value of learning at $\underline{p}$ is the same in both firms.

Now in our two-armed bandit problem, we first need to answer the question whether there exist non-cutoff stationary equilibria, i.e., a worker with $p \in [p_1, p_2)$ accepts the offer from a high type firm, with $p \in [p_2, p_3)$ accepts the offer from a low type firm and with $p \in [p_3, p_4)$ accepts the offer from a high type firm again. Surprisingly, Lemmas 2–5 imply that all possible stationary competitive equilibria must be in cutoff strategies. The next theorem therefore establishes uniqueness and sorting under supermodularity. It does not shown existence yet, which we do in Theorem 3 below.

Theorem 1 *If an equilibrium exists, PAM is the unique stationary competitive equilibrium allocation under strict supermodularity. Likewise for NAM under strict submodularity.*

To prove this theorem, we only need to prove the following Claim:

[19] For example, in our model assume $\mu_{HL} = \mu_{LL}$ and the return in the low type firm is deterministic.

[20] In a model of option pricing by Dumas (1991), there does exist a condition on the second derivative called the "super contact" condition, which is of a very different nature. It arises as the optimal solution to the option pricing problem with proportional cost.

Claim 2 *Under strict supermodularity, it is impossible to have $p_1 < p_2$ and equilibrium value functions W_H (for $p \in [p_1, p_2]$), W_{L1} (for $p < p_1$), W_{L2} (for $p > p_2$) such that:*

$$W_H(p_1) = W_{L1}(p_1) \quad \text{and} \quad W''_H(p_1) = W''_{L1}(p_1)$$

$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W''_H(p_2) = W''_{L2}(p_2)$$

are satisfied simultaneously.

Under strict submodularity, it is impossible to have $p_1 < p_2$ and equilibrium value functions W_L (for $p \in [p_1, p_2]$), W_{H1} (for $p < p_1$), W_{H2} (for $p > p_2$) such that:

$$W_L(p_1) = W_{H1}(p_1) \quad \text{and} \quad W''_L(p_1) = W''_{H1}(p_1)$$

$$W_L(p_1) = W_{H2}(p_2) \quad \text{and} \quad W''_L(p_2) = W''_{H2}(p_2)$$

are satisfied simultaneously.

Proof. In Appendix. ■

This result states that it is not benefial for a worker of type p to learn in the high type firm H in the middle as long as there there are still types p on both sides who work in the low type firms. Given the above claim, it is easy to prove the theorem:

Proof. Under supermodularity, by Lemma 5, workers with sufficiently low p's will accept a low type firm's wage offer and workers with sufficiently high p's will accept a high type firm's offer. But Claim 2 implies it is impossible to have worker first accept low type firm's offer, then accept high type firm's offer and finally accept low type firm's offer again. Hence, we must have some cutoff $\underline{p}$ such that $p < \underline{p}$ will accept low type firm's offer and $p > \underline{p}$ will accept high type firm's offer. This is exactly a PAM allocation. Use the same logic, NAM is the only possible stationary competitive equilibrium allocation under strict submodularity. ■

Before we turn to the equilibrium distribution, we show that the no-deviation condition in Lemma 5 is not just necessary but also sufficient under strict supermodularity:

Lemma 6 *Under strict supermodularity, $W''_H(\underline{p}) = W''_L(\underline{p})$ implies that no deviation will happen for the PAM equilibrium allocation.*

Proof. In Appendix. ■

4.2 The Equilibrium Distribution

The previous section shows that under strict supermodularity (submodularity), PAM (NAM) is the unique candidate stationary competitive equilibrium allocation. Note that this doesn't necessarily mean the equilibrium exists. We still need to construct such an equilibrium. To do that, we assume strict supermodularity and worker and firm monotonicity: ($\mu_{HH} > \mu_{HL}$ and $\mu_{LH} > \mu_{LL}$).[21] Now consider a strictly positive assortative matching equilibrium such that workers with beliefs less than $\underline{p}$ will choose L firms and workers with beliefs higher than $\underline{p}$ will choose H firms. From equation (4) we hence have $k_{L1} = 0$ and $k_{L2} > 0$ for $y = L$ and $k_{H2} = 0$ and $k_{H1} > 0$ for $y = H$. Let $k_L = k_{L2}$, $k_H = k_{H1}$ and worker's value functions become:

$$W_L(p) = \frac{w_L(p)}{r+\delta} + k_L p^{\alpha_L}(1-p)^{1-\alpha_L} \tag{13}$$

and

$$W_H(p) = \frac{w_H(p)}{r+\delta} + k_H p^{1-\alpha_H}(1-p)^{\alpha_H}, \tag{14}$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

To discuss market clearing conditions, we need to consider the ergodic distribution of p's. From the Fokker-Planck (Kolmogorov forward) equation, the stationary and ergodic density f_y should satisfy the following differential equation:

$$0 = \frac{df_y(p)}{dt} = \frac{d^2}{dp^2}[\Sigma_y(p) f_y(p)] - \delta f_y(p). \tag{15}$$

The general solution to this differential equation is (see also Moscarini (2005)):[22]

$$f_y(p) = [f_{y0} p^{\gamma_{y1}}(1-p)^{\gamma_{y2}} + f_{y1}(1-p)^{\gamma_{y1}} p^{\gamma_{y2}}] \tag{16}$$

where

$$\gamma_{y1} = -\frac{3}{2} + \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > -1$$

and

$$\gamma_{y2} = -\frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} < -2.$$

First, the integrability of f_y requires that $f_{y1} = 0$ if 0 is included in the domain and $f_{y0} = 0$ if 1

[21] Monotonicity is just to help us find one particular way to divide the surplus. The whole construction of equilibrium also goes through if we do not make this assumption.

[22] Here the assumption that there is no heterogeneity in the prior p_0 substantially simplifies the solution to this differential equation. While there is no solution for a general distribution of priors, we have been able to solve the stationary distribution if the priors are drawn from a beta distribution. See also Papageorgiou (2009).

is included in the domain. Second, the Fokker-Planck (Kolmogorov forward) equation is only valid for $p \neq p_0$. Since there is a flow in of new workers, for $p = p_0$ we should have a kink in the density function. This also raises the issue of the relative position between p_0 and $\underline{p}$. We first consider the case where $\underline{p} < p_0$. We then derive in abbreviated format the result when $\underline{p} > p_0$.

Given any $p_0 \in (0,1)$, if $\underline{p} < p_0$, then the density functions are:

$$f_H(p) = [f_{H0} p^{\gamma_{H1}} (1-p)^{\gamma_{H2}} + f_{H1}(1-p)^{\gamma_{H1}} p^{\gamma_{H2}}] \mathbb{I}(\underline{p} < p \leq p_0) + f_{H2}(1-p)^{\gamma_{H1}} p^{\gamma_{H2}} \mathbb{I}(p > p_0) \quad (17)$$

and

$$f_L(p) = f_{L0} p^{\gamma_{L1}} (1-p)^{\gamma_{L2}}. \quad (18)$$

The density functions are subject to the following boundary conditions. The derivations of these boundary conditions are shown in the appendix. First, once the posterior belief reaches the equilibrium separation point $\underline{p}$, we should have the cutoff condition:

$$\Sigma_H(\underline{p}+) f_H(\underline{p}+) = \Sigma_L(\underline{p}-) f_L(\underline{p}-). \quad (19)$$

This condition guarantees that the flow speed of agents who cross $\underline{p}$ from below is equal to the flow speed of agents who cross from above. The implication is that since the speed from above Σ_H is larger than Σ_L, the densities are not continuous: $f_H(\underline{p}+) < f_L(\underline{p}-)$. It is worth comparing this condition to the standard condition when there is an absorbing state (Cox-Miller (1965), Dixit (1993), and Moscarini (2005)). In the case with only one brownian motion and an absorbing state, what is required is $\Sigma(\underline{p}+) f(\underline{p}+) = 0$ because the probability of absorption in a time interval dt must equal the flow-in speed of the Brownian motion which is proportional to $\sqrt{dt}$ (see Cox and Miller (1965, p.220)).

Second, total flows in and out of the high type firms must balance:

$$\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \delta\pi + \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+}.$$

The left-hand side of the above equation is the total inflow into high type firms, which are new workers who enter into this economy. The right-hand side of the above equation is the total outflows from the high type firms, which include workers who reach $\underline{p}$ and transfer to low type firms and workers who are hit by the death shock. We manage to show that this equation will further imply:

$$\frac{d}{dp}[\Sigma_L(p) f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+}$$

Third, the density function has to be continuous at p_0:

$$f_H(p_0-) = f_H(p_0+).$$

It is customary to impose this condition as it approximates entry from a non-degenerate distribution instead of entry of identical types p_0.

Finally, usual market clearing conditions apply:

$$\int_{\underline{p}}^{1} f_H(p)dp = \pi \quad \text{and} \quad \int_0^{\underline{p}} f_L(p)dp = 1 - \pi.$$

In summary, when $\underline{p} < p_0$, the equilibrium is characterized by a system of eight equations with nine unknowns $(V_L, V_H, k_L, k_H, \underline{p}, f_{H0}, f_{H1}, f_{H2}, f_{L0})$:[23]

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value-matching condition)} \qquad (20)$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting condition)} \qquad (21)$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation condition)} \qquad (22)$$

$$\Sigma_H(\underline{p}+) f_H(\underline{p}+) = \Sigma_L(\underline{p}-) f_L(\underline{p}-) \qquad \text{(Boundary condition)} \qquad (23)$$

$$\int_{\underline{p}}^{1} f_H(p)dp = \pi \qquad \text{(Market clearing } H\text{)} \qquad (24)$$

$$\int_0^{\underline{p}} f_L(p)dp = 1 - \pi \qquad \text{(Market clearing } L\text{)} \qquad (25)$$

$$\frac{d}{dp}[\Sigma_L(p) f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+} \qquad \text{(Flow equation at } \underline{p}\text{)} \qquad (26)$$

$$f_H(p_0-) = f_H(p_0+) \qquad \text{(Continuous density at } p_0\text{)} \qquad (27)$$

Fortunately, Equations (23)–(27) can be solved separately from Equations (20)–(22). In other words, the procedure of solving this system of equation could be: first we solve $\underline{p}$ jointly with $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ from Equations (23)–(27) and then we plug $\underline{p}$ into Equations (20)–(22) to pin down other unknowns.

Proposition 1 *Equations (23)-(27) imply* $\underline{p} < p_0$ *if and only if:*

$$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp} < \frac{\pi}{1-\pi}. \qquad (28)$$

Moreover, if such $\underline{p}$ *exists, it must be unique.*

Proof. In Appendix. ■

[23] Observe that with more unknowns than variables, the solution to our system is indeterminate. In fact, there are potentially a continuum of wages that can be supported in equilibrium, though the allocation will be unique. This indeterminacy is as in Becker: the allocation is unique, but there may be multiple ways to split the surplus. In all that follows, when we use the term uniqueness of equilibrium, we refer to the allocation, not to the wages.

The proof of Proposition 1 is quite straightforward. The idea of the proof is the following: since we have 5 equations with five unknowns, we can first express $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ as functions of $\underline{p}$ and then use the last equation to pin down $\underline{p}$.

The existence and uniqueness of the solution to the system require that $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ change monotonically with $\underline{p}$. Fortunately, this is the case as shown in the appendix. The monotonicity guarantees that if a solution exists, it must be unique. Furthermore, it enables us to only check the boundaries when determining whether a solution exists. Equation (28) given in the Proposition is thus derived.

In the second case, $\underline{p} \geq p_0$. Given any $p_0 \in (0,1)$, if $\underline{p} \geq p_0$, then the density functions are:

$$f_L(p) = f_{L0}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\mathbb{I}(p < p_0) + [f_{L1}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} + f_{L2}(1-p)^{\gamma_{L1}}p^{\gamma_{L2}}]\mathbb{I}(p_0 \leq p \leq \underline{p}) \quad (29)$$

and

$$f_H(p) = f_{H0}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}. \quad (30)$$

Then the system of equations to determine the equilibrium is:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value-matching)} \quad (31)$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting)} \quad (32)$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation)} \quad (33)$$

$$\Sigma_H(\underline{p}+)f_H(\underline{p}+) = \Sigma_L(\underline{p}-)f_L(\underline{p}-) \qquad \text{(Boundary condition)} \quad (34)$$

$$\int_{\underline{p}}^1 f_H(p)dp = \pi \qquad \text{(Market clearing } H\text{)} \quad (35)$$

$$\int_0^{\underline{p}} f_L(p)dp = 1-\pi \qquad \text{(Market clearing } L\text{)} \quad (36)$$

$$\frac{d}{dp}[\Sigma_L(p)f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p)f_H(p)]|_{\underline{p}+} \qquad \text{(Flow equation at } \underline{p}\text{)} \quad (37)$$

$$f_L(p_0-) = f_L(p_0+) \qquad \text{(Continuous density at } p_0\text{)} \quad (38)$$

Based on the above equations, we can prove the following Proposition, the counterpart to Proposition 1, in a similar fashion:

Proposition 2 *Equations (34)-(38) imply $\underline{p} \geq p_0$ if and only if:*

$$(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp} \geq \frac{\pi}{1-\pi}. \quad (39)$$

Moreover, if such $\underline{p}$ exists, it must be unique.

The idea for the proof of Proposition 2 is exactly the same as that for the proof of Proposition

1 and the proof is also shown in the appendix. Propositions 1 and 2 together provide the following existence and uniqueness result:

Theorem 2 *Under strict supermodularity, for any pair* $(p_0, \pi) \in (0,1)^2$*, there exists a unique PAM cutoff* $\underline{p}$*. Moreover,* $\underline{p} < p_0$ *if and only if:*

$$(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp} < \frac{\pi}{1-\pi}. \tag{40}$$

One of the nice properties about Equation (40) is that the whole equation only depends on p_0, π, δ/s_H^2 and δ/s_L^2. This provides a feasible way to compute $\underline{p}$. Given p_0, π, δ/s_H^2 and δ/s_L^2, we first need to decide the sign of

$$(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp} - \frac{\pi}{1-\pi}.$$

If this sign is negative, then we know that $\underline{p}$ is smaller than p_0 and we can use the system of equations in the first case to figure out $\underline{p}$. On the contrary, if this sign is not negative, then we know that $\underline{p}$ is larger than p_0 and we can use the system of equations in the second case to compute $\underline{p}$. This turns out to be a convenient way to determine the equilibrium cutoff numerically.

Before presenting the numerical results, we have a simple theoretical comparative static result:

Corollary 1 $\underline{p}$ *is strictly increasing in* p_0 *and decreasing in* π*.*

This corollary is proved in the appendix. But the intuition is quite straightforward: decreasing in π means there are more low type firms in the economy and hence $\underline{p}$ has to become larger such that more workers are matched with low type firms; increasing in p_0 means the overall quality of the workers is becoming better in the economy and $\underline{p}$ has to go up to make sure that low type firms are also matched with better workers.

Mathematically, it is not easy to derive comparative statics between $\underline{p}$ and δ/s_H^2 or δ/s_L^2. But intuitively speaking, as s_L increases, the degree of supermodularity will be reduced while the speed of learning in low type firms will increase. Both of these factors make the low type firms more attractive and hence $\underline{p}$ should increase in s_L. On the other hand, as s_H becomes higher, both the degree of supermodularity and the speed of learning in high type firms will go up, which will lead to a reduction in $\underline{p}$.

Figure 1 plots the stationary distribution of beliefs p, for the case of PAM and with parameter values: $s_H = 0.15, s_L = 0.05, p_0 = 0.5, \pi = 0.5, \delta = 0.01$.

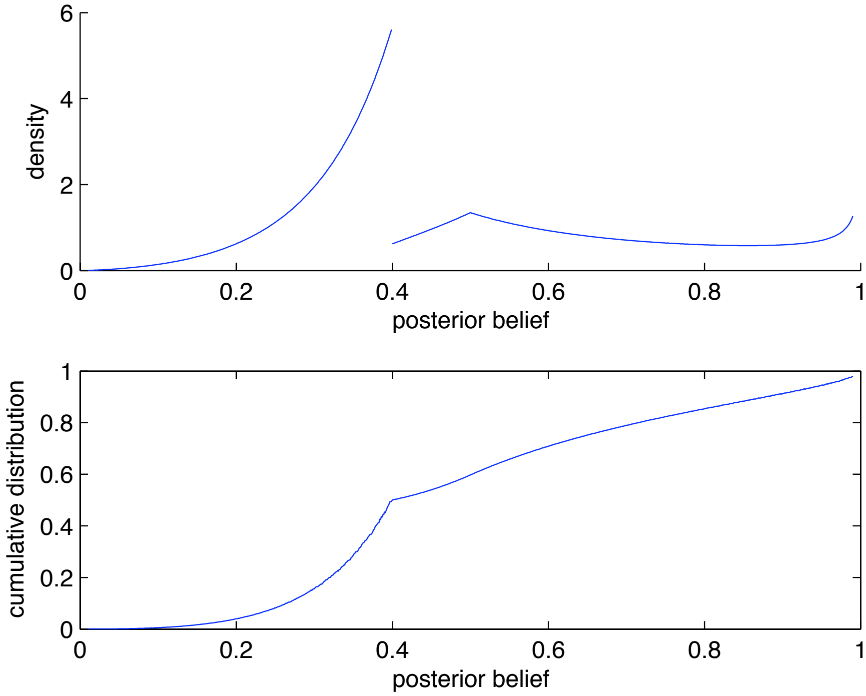

Figure 1: Equilibrium Distribution of Posterior beliefs.

4.3 Equilibrium Analysis: Value Functions

Theorem 2 implies that under strict supermodularity, the PAM cutoff $\underline{p}$ can be uniquely determined. But given this $\underline{p}$, we still have the following conditions to satisfy:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value-matching condition)} \tag{41}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting condition)} \tag{42}$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation condition)} \tag{43}$$

Equations (41)-(43) are three equations for four unknowns. The equilibrium is indeterminate in the sense that although the allocation $\underline{p}$ is unique, there could be multiple ways to divide the surplus. To make the system determinate, we assume firm monotonicity and set $\mu_{LL} = 0$. Then limited liability requires that $w_L(0)$ has to be zero and hence $V_L = 0$. Equations (41)-(43) thus could be written as:

$$\frac{\mu_L(\underline{p})}{r+\delta} + k_L \underline{p}^{\alpha_L}(1-\underline{p})^{1-\alpha_L} = \frac{\mu_H(\underline{p}) - rV_H}{r+\delta} + k_H \underline{p}^{1-\alpha_H}(1-\underline{p})^{\alpha_H}$$

$$\frac{\mu_{HL} - \mu_{LL}}{r+\delta} + k_L \underline{p}^{\alpha_L}(1-\underline{p})^{1-\alpha_L}\left(\frac{\alpha_L - \underline{p}}{\underline{p}(1-\underline{p})}\right) = \frac{\mu_{HH} - \mu_{LH}}{r+\delta} + k_H \underline{p}^{1-\alpha_H}(1-\underline{p})^{\alpha_H}\left(\frac{1-\alpha_H - \underline{p}}{\underline{p}(1-\underline{p})}\right)$$

$$k_L \underline{p}^{\alpha_L - 2}(1-\underline{p})^{-1-\alpha_L}\alpha_L(\alpha_L - 1) = k_H \underline{p}^{-1-\alpha_H}(1-\underline{p})^{\alpha_H - 2}\alpha_H(\alpha_H - 1)$$

This system of equations will give us a unique formula for V_H:

$$rV_H = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1 - \underline{p})(\alpha_L - \alpha_H)}. \tag{44}$$

As usual, $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$. Furthermore, it is easy to check that both k_H and k_L are strictly larger than zero such that the option value of learning is strictly positive.

Therefore, we finally reach our main result:

Theorem 3 *Under strict supermodularity, the stationary competitive equilibrium is unique in the sense that all equilibria are PAM and the allocation is uniquely determined by Theorem 2. Moreover, assume firm monotonicity and normalize $V_L = 0$, we can get a unique formula for V_H given by equation (44).*

4.4 Wage Gap at $\underline{p}$

The analysis of the value functions allows us to determine equilibrium wages. We start with an interesting observation:

$$\begin{aligned} w_H(\underline{p}) = \mu_H(\underline{p}) - rV_H &= \Delta_H\underline{p} + \mu_{LL} - \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1 - \underline{p})(\alpha_L - \alpha_H)} \\ &< \Delta_L\underline{p} + \mu_{LL} = w_L(\underline{p}). \end{aligned}$$

This implies that the worker with posterior belief slightly higher than $\underline{p}$ will accept the high firm's offer even though the wage provided is lower than the wage at the low firm. This obviously comes from the fact that the learning speed in the high firm is higher and this would compensate the loss in the flow wages.

On the other hand, we can see that the difference in expected productivity at $\underline{p}$ is

$$\mu_H(\underline{p}) - \mu_L(\underline{p}) = (\mu_{HL} - \mu_{LL}) + (\Delta_H - \Delta_L)\underline{p} < rV_H.$$

This implies the high firm can enjoy a strictly positive rent from a higher learning speed. This above result actually does not depend on the assumption $V_L = 0$ and it can be generalized for any possible division of surplus.[24] This is illustrated by Figure 2:

Lemma 7 *Under strict supermodularity, we have: $w_H(\underline{p}) < w_L(\underline{p})$ and $rV_H - rV_L > \mu_H(\underline{p}) - \mu_L(\underline{p})$.*

[24] Generally, value matching and no-deviation conditions imply that

$$(r + \delta)W_H(\underline{p}) = w_H(\underline{p}) + \Sigma_H(\underline{p})W_H''(\underline{p}) = (r + \delta)W_L(\underline{p}) = w_L(\underline{p}) + \Sigma_L(\underline{p})W_L''(\underline{p})$$

and

$$W_H''(\underline{p}) = W_L''(\underline{p}).$$

These immediately mean that $w_H(\underline{p}) < w_L(\underline{p})$ and $rV_H - rV_L > \mu_H(\underline{p}) - \mu_L(\underline{p})$.

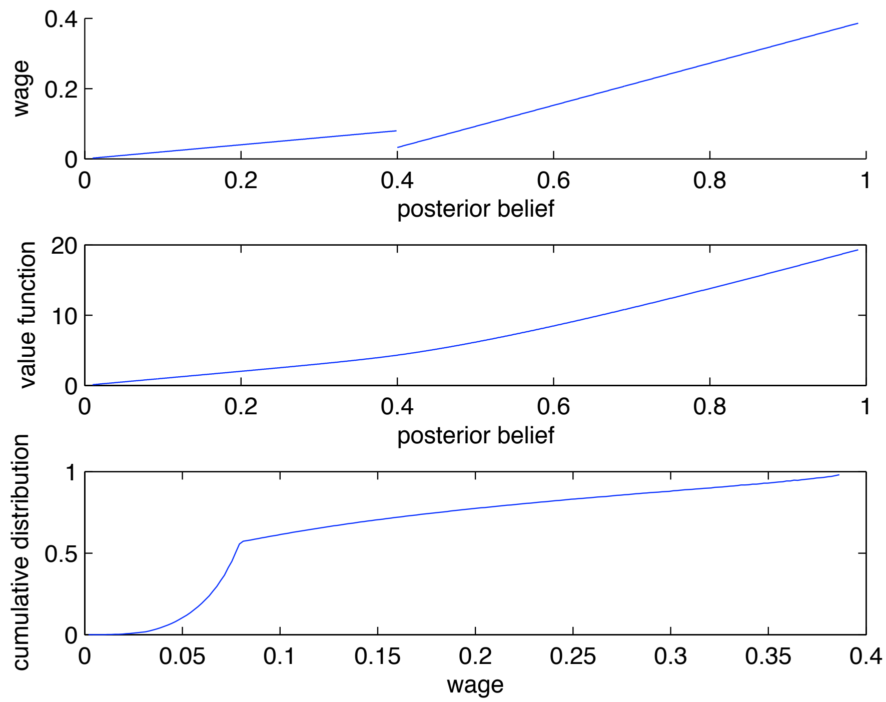

Figure 2: Equilibrium wage function and value function in terms of beliefs p; Stationary wage distribution.

5 Firm-dependent Volatility: σ_y

A valid criticism of our approach is that we give the H firms too much of an edge under supermodularity (likewise for the L firms under submodularity). Not only are they superior in the production of output, by assuming that the volatility σ is common to both types of firms, effectively the signal-to-noise ratio is higher in H firms:

$$s_H = \frac{\mu_{HH} - \mu_{LH}}{\sigma} > \frac{\mu_{HL} - \mu_{LL}}{\sigma} = s_L,$$

from supermodularity. With firm-dependent volatility, that need not be the case. In particular, for σ_H sufficiently high, it may well be the case that $s_H < s_L$.

Mere observation of the value function in Equation (3), $rW_y(p) = \mu_y(p) - V_y + \Sigma_y(p)W_y''(p) - \delta W_y(p)$, reveals that firm-dependent volatility will play a crucial role here. Since $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$, for sufficiently high σ_H and therefore low s_H, it appears intuitive that the value W_H can be smaller than the value of W_L for high p. It turns out that this intuition is wrong. First, in this competitive equilibrium, wages are endogenous and therefore as the value of learning changes, so does $\mu_y(p) - V_y$. Second, the no-deviation condition requires that at the marginal type $\underline{p}$, $W_H'' = W_L''$. It turns out that as a result these two features, in equilibrium the learning effect is the same in both firms, no matter what the volatility σ_y is.

To make this argument formal, when $\sigma_H \neq \sigma_L$, we generally define $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma_y,\ y = H, L$. It is trivial to show that belief updating also satisfies the formula:

$$dp_t = p_t(1-p_t)s_y d\bar{Z}_{y,t}.$$

Furthermore, Lemmas 2–5 still hold because none of these results depend explicitly on σ_y. As shown in the appendix, the statement in Claim 2 is generalized to *any* combination of (σ_H, σ_L).[25]

With the proof of Claim 2 in hand, the result of Theorem 1 immediately extends: PAM (NAM) is the unique candidate stationary competitive equilibrium allocation under strict supermodularity (submodularity) thus holds for *any* combination of (σ_H, σ_L). Surprisingly, this implies that under strict supermodularity, even if we have an extremely high σ_H such that the learning rate in high type firms is smaller than that in low type firms, we still have PAM. It is equivalent to assert that the direct productivity consideration dominates the learning in our model. The reason comes from the fact that the equilibrium wage schedules adjust to offset the impact of change in learning rate. The key insight here is the no-deviation condition. At $\underline{p}$, the no-deviation condition requires that the second-order effect on the value function is the same in both firms. This second-order effect W_y'' exactly captures the effect of learning through $\Sigma_y(\underline{p})W_y''(\underline{p})$ where $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$. Because equilibrium wages adjust to satisfy the no-deviation condition at the cutoff, the impact of differential learning rates is completely offset by the change of wage schedule, and the equilibrium allocation is solely determined by the productivity consideration.

6 The Planner's Problem

A priori, we might expect the competitive equilibrium not to decentralize the planner's problem. Wage contracts cannot condition on future realizations or actions and are assumed to be self-enforcing. As a result of this lack of commitment, there is a missing market. With incomplete markets, the competitive equilibrium in general does not necessarily decentralize the planner's problem. It turns out however as we show below that this market incompleteness does not preclude the efficiency of the decentralized equilibrium. As will become apparent, this efficiency result is driven by the martingale property present in all models of learning.

We consider a planner's problem under stationarity, i.e., in the presence of an ergodic distribution. The planner chooses an allocation rule and as a consequence of the Kolmogorov forward equation, the ergodic distribution associated with this allocation rule. The objective is to maximize the aggregate flow of output. Given stationarity of the problem, the focus on output maximization yields the same outcome as maximization of aggregate values.

[25] The sufficiency of the no-deviation condition is also extended to include all of the combinations of (σ_H, σ_L) by proving a generalized version of Claim 2 and Lemma 6 in the appendix.

Before we state and prove the efficiency result, we need to derive the stationary distribution under multiple cutoffs. Consider any allocation with multiple cutoffs:

$$0 < \underline{p}_N < \cdots < \underline{p}_1 < 1, \quad N \text{ odd}.$$

Without loss of generality, we assume workers with $p \in (p_1, 1]$ are allocated to the high type firms while workers with $p \in [0, p_N)$ are allocated to the low type firms since for workers with $p = 0$ or 1, there is no need for learning and it is optimal to allocate them according to instantaneous production efficiency (PAM).[26] This also implies that generically N is odd. Denote by Ω_y the set of p's that match with firms of type y.

Formally, the planner will choose Ω_y to solve the problem:

$$\begin{aligned} \max_{\Omega_y} S = \int_{\Omega_H} \mu_H(p) f_H(p) dp &+ \int_{\Omega_L} \mu_L(p) f_L(p) dp \\ \text{s.t. } \frac{d^2}{dp^2}[\Sigma_y(p) f_y(p)] - \delta f_y(p) = \frac{df_y(p)}{dt} &= 0 \qquad \text{Kolmogorov forward equation} \\ \int_{\Omega_H} p f_H(p) dp + \int_{\Omega_L} p f_L(p) dp &= p_0 \qquad \text{Martingale property} \\ \int_{\Omega_L} f_L(p) dp = 1 - \pi, \quad \int_{\Omega_H} f_H(p) dp &= \pi. \qquad \text{Market clearing} \end{aligned}$$

It turns out that the martingale property enables an easier way to compare different allocations, hence the following Lemma:

Lemma 8 *Consider two possible allocations with ergodic density functions $f_H(p)$, $f_L(p)$ (allocation 1) and $\tilde{f}_H(p)$, $\tilde{f}_L(p)$ (allocation 2) respectively. Then allocation 1 generates higher aggregate output than the allocation 2 if and only if $\int_{\Omega_H} p f_H(p) dp > \int_{\tilde{\Omega}_H} p \tilde{f}_H(p) dp$ or alternatively, $\int_{\Omega_L} p f_L(p) dp < \int_{\tilde{\Omega}_L} p \tilde{f}_L(p) dp$.*

Proof. In Appendix. ■

To prove that the competitive equilibrium decentralizes the planner's stationary solution under supermodularity, it suffices to show that the PAM allocation is better than *any* allocation with multiple cutoffs because from Theorem 2, we know that PAM allocation is unique and will be the same as the competitive equilibrium allocation for any combination of (s_H, s_L). The key technical issue is that the ergodic distribution is endogenously determined by the allocation rule. It is infeasible to compute the ergodic density functions for each possible allocation. Our strategy of proof is therefore to use a variational argument to circumvent this difficulty.

[26]This property is also established in the one-sided model of Anderson and Smith (2010). Our results shows that not only at the extremes but also at the interior the planner's (and the equilibrium) allocation exhibit PAM.

The proof heavily uses the martingale property and works as follows. First we consider a candidate allocation with 3 cutoffs. Under this candidate allocation, there will be an interior interval of p's that are matched to L type firms associated with some ergodic distribution. We move the bounds of that interval slightly to the left, thus generating a new density in this interval while keeping all other cutoffs and distributions unchanged. The new interval is chosen by imposing market clearing conditions. Lemma 8 then shows that under supermodularity this experiment strictly increases aggregate output. This holds until cutoffs coincide such that the interior rang of p's matched with L firms disappears, thus reducing the number of cutoffs to $N = 1$. We use a similar argument to establish that output increases when moving from N to $N - 2$ cutoffs. The result then follows by induction. We derive the result under supermodularity. The same logic applies under submodularity.

Theorem 4 *The competitive equilibrium decentralizes the planner's stationary solution that maximizes the aggregate flow of output.*

Proof. In Appendix. ■

7 On-the-job Human Capital Accumulation

On the job, workers and firms not only learn about their unknown innate skills, they also accumulate human capital. In reality, human capital accumulation is an ongoing, continuous process. The longer the tenure of a worker, the higher her productivity. This monotonically increasing relation between tenure and human capital experience is likely also to be concave. For modeling purposes, here we consider a very simple form that captures this relation. With probability λ, a worker transitions from being unexperienced to being experienced.[27] Once a worker is experienced, her productivity increases to $\mu_{xy} + \xi_x$ and the status of experience is complete information.[28] Now there are the same value functions for experienced workers as before W_y^e.

$$rW_y^e(p) \quad = \quad \mu_y(p) + \xi(p) - rV_y + \Sigma_y^e(p)W_y^{e''}(p) - \delta W_y^e(p)$$

where $\xi(p) = p\xi_H + (1-p)\xi_L$ is the expected experience.[29] For the unexperienced worker there is now one additional value function. As before, there are unexperienced workers who are matched with L firms, and who continue to match with an L firms; and there are those who match with H

[27] Having a continuous relation between tenure and human capital renders the system of differential equations into a system of partial differential equations. Typically there is no solution. In the current setup, there is an additional state (experienced versus unexperienced) and the model remains tractable.

[28] Observe that experience is worker dependent, but not firm dependent. While it is likely a realistic feature to have experience dependent on the job type, the reason is that we would have a different level of experience for different histories which makes the problem non-tractible.

[29] In this section we maintain the earlier assumption that $\sigma_H = \sigma_L = \sigma$.

firms both when unexperienced as well as when experienced. We denote those values by W^u_{LL}, W^u_{HH}. There are now also some types p who match with an L firm when unexperienced and who switch to an H firm when they become experienced, the value of which is denoted by W^u_{LH}. This requires that the reservation type of an experienced worker ($\underline{p}^e$) is lower than that of the unexperienced worker ($\underline{p}^u$). We start from this premise and later verify that this is indeed the case. The value functions then are:

$$\begin{aligned} rW^u_{yy}(p) &= \mu_y(p) - rV_y + \Sigma^u_y(p)W^{u\prime\prime}_{yy}(p) + \lambda W^e_y(p) - (\delta+\lambda)W^u_{yy}(p) \\ rW^u_{LH}(p) &= \mu_L(p) - rV_L + \Sigma^u_L(p)W^{u\prime\prime}_{LH}(p) + \lambda W^e_H(p) - (\delta+\lambda)W^u_{LH}(p) \end{aligned}$$

Observe that even though experience is completely observable, it does affect the inference from learning in the sense that the signal-to-noise ratio changes to $[(\mu_{Hy} + \xi_H - \mu_{Ly} - \xi_L)]/\sigma^2$. As a result, Σ_y depends on experience u, e.

$$\begin{aligned} W^u_{yy}(p) &= \frac{\mu_y(p) - rV_y}{r+\delta+\lambda} + k^u_{y1}p^{1-\alpha^u_y}(1-p)^{\alpha^u_y} + k^u_{y2}p^{\alpha^u_y}(1-p)^{1-\alpha^u_y} \\ &+ \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}[\mu_y(p) + \xi(p) - rV_y] \\ &+ \frac{\lambda}{(\lambda+\delta+r) - \frac{(s^u_y)^2}{(s^e_y)^2}(r+\delta)}[k^e_{y1}p^{1-\alpha^e_y}(1-p)^{\alpha^e_y} + k^e_{y2}p^{\alpha^e_y}(1-p)^{1-\alpha^e_y}] \\ W^u_{LH}(p) &= \frac{\mu_L(p) - rV_L}{r+\delta+\lambda} + k^u_{L1}p^{1-\alpha^u_L}(1-p)^{\alpha^u_L} + k^u_{L2}p^{\alpha^u_L}(1-p)^{1-\alpha^u_L} \\ &+ \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}[\mu_H(p) + \xi(p) - rV_H] \\ &+ \frac{\lambda}{(\lambda+\delta+r) - \frac{(s^u_L)^2}{(s^e_H)^2}(r+\delta)}[k^e_{H1}p^{1-\alpha^e_H}(1-p)^{\alpha^e_H} + k^e_{H2}p^{\alpha^e_H}(1-p)^{1-\alpha^e_H}] \\ W^e_y(p) &= \frac{\mu_y(p) + \xi(p) - rV_y}{r+\delta} + k^e_{y1}p^{1-\alpha^e_y}(1-p)^{\alpha^e_y} + k^e_{y2}p^{\alpha^e_y}(1-p)^{1-\alpha^e_y} \end{aligned}$$

where

$$\alpha^u_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta+\lambda)}{(s^u_y)^2}} \geq 1$$

$$\alpha^e_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{(s^e_y)^2}} \geq 1$$

There are now two cut-offs $\underline{p}^u, \underline{p}^e$. Since we just want to compare $\underline{p}^u$ and $\underline{p}^e$, we can consider the following thought experiment. First, we assume that $\underline{p}^u = \underline{p}^e = \underline{p}$. Then we can get two systems of equations: one system is the set of value-matching, smooth-pasting and no-deviation conditions for the unexperienced workers and the other one is for the experienced workers. Second, we can solve

$\Delta V = V_H - V_L$ the way we did previously but now we can get two possible values for ΔV. Denote them to be ΔV^e and ΔV^u. Notice that ΔV^e and ΔV^u are both increasing in the cutoff $\underline{p}$. Finally, we compare ΔV^e and ΔV^u under the assumption that $\underline{p}^u = \underline{p}^e = \underline{p}$. If $\Delta V^e > \Delta V^u$, this means that we should decrease $\underline{p}^e$ or increase $\underline{p}^u$ and hence $\underline{p}^u > \underline{p}^e$; on the contrary, if $\Delta V^e < \Delta V^u$, this means that we should decrease $\underline{p}^u$ or increase $\underline{p}^e$ and hence $\underline{p}^u < \underline{p}^e$. We derive this in the Appendix and can show this to hold when human capital accumulation is not too different for H and L types.

Proposition 3 *Assume supermodularity and $\xi_H \simeq \xi_L$. Then $\underline{p}^e < \underline{p}^u$.*

Proof. In Appendix. ■

With human capital accumulation, we can now characterize the entire equilibrium, including wage schedules and the ergodic distribution of types. Even though there are types who gradually learn they are of low productivity, wages need not decrease over the life cycle as they accumulate human capital.

Turnover and Tenure. We express the expected future duration of a match by tenure $\tau_y(p)$. Tenure relates inversely to turnover. For $p < \underline{p}^e$ and $p > \underline{p}^u$, $\tau_y(p)$ satisfies the following differential equation (see also Moscarini 2005):

$$\Sigma_y(p)\tau_y''(p) - \delta\tau_y(p) = -1,$$

with solutions:

$$\tau_H^u(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^u}\right)^{1/2-\sqrt{1/4+2\delta/(s_H^u)^2}}\left(\frac{1-p}{1-\underline{p}^u}\right)^{1/2-\sqrt{1/4-2\delta/(s_H^u)^2}}\right\};$$

$$\tau_L^u(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^u}\right)^{1/2-\sqrt{1/4-2\delta/(s_L^u)^2}}\left(\frac{1-p}{1-\underline{p}^u}\right)^{1/2-\sqrt{1/4+2\delta/(s_L^u)^2}}\right\};$$

$$\tau_H^e(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^e}\right)^{1/2-\sqrt{1/4+2\delta/(s_H^e)^2}}\left(\frac{1-p}{1-\underline{p}^e}\right)^{1/2-\sqrt{1/4-2\delta/(s_H^e)^2}}\right\};$$

$$\tau_L^e(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^e}\right)^{1/2-\sqrt{1/4-2\delta/(s_L^e)^2}}\left(\frac{1-p}{1-\underline{p}^e}\right)^{1/2-\sqrt{1/4+2\delta/(s_L^e)^2}}\right\}.$$

If $p \in (\underline{p}^e, \underline{p}^u)$, the only difference is that

$$\Sigma_y(p)\tau_L^{u\prime\prime}(p) - (\delta + \lambda)\tau_L^u(p) = -1,$$

since unexperienced workers will switch jobs once they become experienced. An immediate implication of the Proposition above is the following:

Proposition 4 *(Tenure) Assume supermodularity and* $\xi_H \simeq \xi_L$. *Then,* $\tau^u_L(p) > \tau^e_L(p)$ *for* $p < \underline{p}^e$ *and* $\tau^u_H(p) < \tau^e_H(p)$ *for* $p > \underline{p}^u$. *For* $p \in (\underline{p}^e, \underline{p}^u)$, *there is a cutoff such that* $\tau^u_L(p) < \tau^e_H(p)$ *for* p *higher than this cutoff and* $\tau^u_L(p) > \tau^e_H(p)$ *for* p *smaller than this cutoff.*

For the lowest types p, tenure for the unexperienced worker is longer as the experienced workers are more likely to be hired by an H firm given positive information revelation. The opposite is true for the highest p: the unexperienced types face a higher cut-off type and will therefore upon bad information be more likely to switch to an L firm. In the intermediate range, tenure depends on how close p is to either of the cut-offs.

8 Robustness

8.1 Generalized Lévy Processes

One may suspect that our results are exclusively driven by the specific assumptions of the Brownian motion. In the section, we illustrate that this is not the case by considering a generalized Lévy process, i.e., a compound Poisson process. Let λ_{xy} denote the expected arrival rate of jumps for a type x worker in a type y firm. Following Cohen and Solan (2009), the worker's value function can be written as:

$$W_y(p) = w_y(p)dt + (1 - rdt - \delta dt)\{[p\lambda_{Hy} + (1-p)\lambda_{Ly}]dtW_{y'}(p_h) + (1 - [p\lambda_{Hy} + (1-p)\lambda_{Ly}]dt)W_y(p + dp)$$

where $p_h = \frac{p\lambda_{Hy}}{p\lambda_{Hy}+(1-p)\lambda_{Ly}}$ and y' is the firm type which matches with worker p_h. If no jump occurs, the updating of the posterior belief in firm y follows:

$$dp = -p(1-p)(\lambda_{Hy} - \lambda_{Ly})dt + p(1-p)s_y d\bar{Z}.$$

As usual, the value function could be rewritten as a differential equation:

$$(r+\delta+[p\lambda_{Hy}+(1-p)\lambda_{Ly}])W_y(p) = w_y(p)+[p\lambda_{Hy}+(1-p)\lambda_{Ly}]W_{y'}(p_h)-p(1-p)(\lambda_{Hy}-\lambda_{Ly})W'_y(p)+\Sigma_y(p)W''_y(p).$$

The no-deviation condition derived earlier still holds in this situation. The proof is similar and is omitted here.

Lemma 9 *To deter possible deviations, a necessary condition is:*

$$W''_H(\underline{p}) = W''_L(\underline{p}) \quad \textit{(No-deviation condition-Lévy)} \tag{45}$$

for any possible cutoff $\underline{p}$.

Consider the simplifying assumption that $\lambda_{Ly} = 0$ and denote λ_{Hy} by λ_y. Then p_h is always 1 and the value function becomes:

$$(r + \delta + p\lambda_y)W_y(p) = w_y(p) + p\lambda_y W_{y'}(1) - p(1-p)\lambda_y W_y'(p) + \Sigma_y(p)W_y''(p).$$

The differential equation could be solved explicitly by guess and verify:

$$W_y(p) = A_y + B_y p + k_{y1}p^{\alpha_{y1}}(1-p)^{1-\alpha_{y1}} + k_{y2}p^{\alpha_{y2}}(1-p)^{1-\alpha_{y2}}$$

where $A_y = \frac{\mu_{Ly} - rV_y}{r+\delta}$, $B_y = \frac{\Delta_y + \lambda_y(W_{y'}(1) - A_y)}{r+\delta+\lambda_y}$ and

$$\begin{aligned}\alpha_{y1} &= \frac{1}{2} + \frac{\lambda_y}{s_y^2} + \sqrt{(\frac{1}{2} + \frac{\lambda_y}{s_y^2})^2 + \frac{2(r+\delta)}{s_y^2}} > 1 + 2\frac{\lambda_y}{s_y^2} \\ \alpha_{y2} &= \frac{1}{2} + \frac{\lambda_y}{s_y^2} - \sqrt{(\frac{1}{2} + \frac{\lambda_y}{s_y^2})^2 + \frac{2(r+\delta)}{s_y^2}} < 0.\end{aligned}$$

Obviously, the envelope of W_y is a strictly increasing and strictly convex function for $p \in (0,1)$. First, we would like to argue that for $p = 1$, $y' = H$. Since the function is strictly convex, it must be the case that 0 and 1 workers are matched with different types of firms. Now suppose $y' = L$. Then since 0 workers are matched with H firms, $A_H > A_L$ and hence $W_L(1) = \frac{\Delta_L}{r+\delta} + A_L < \frac{\Delta_H}{r+\delta} + A_H = W_H(1)$. A contradiction.

Therefore, the value function could be rewritten as:

$$(r + \delta + p\lambda_y)W_y(p) = w_y(p) + p\lambda_y W_1(1) - p(1-p)\lambda_y W_y'(p) + \Sigma_y(p)W_y''(p). \tag{46}$$

with general solution:

$$W_y(p) = A_y + B_y p + k_{y1}p^{\alpha_{y1}}(1-p)^{1-\alpha_{y1}} + k_{y2}p^{\alpha_{y2}}(1-p)^{1-\alpha_{y2}}. \tag{47}$$

Notice that the equilibrium payoffs are such that $A_L > A_H$, $B_L < B_H$ and $A_L + B_L < A_H + B_H$. At any cutoff $\underline{p}$, the following three equations should hold simultaneously:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value-matching condition)} \tag{48}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting condition)} \tag{49}$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation condition)} \tag{50}$$

Then from Equation (46), it is immediate to get at $\underline{p}$,

$$(\lambda_H-\lambda_L)\underline{p}W_H(\underline{p}) = w_H(p)-w_L(p)+(\lambda_H-\lambda_L)\underline{p}W_H(1)-(\lambda_H-\lambda_L)\underline{p}(1-\underline{p})W'_H(\underline{p})+(\Sigma_H(p)-\Sigma_L(p))W''_H(p).$$

Apply Equation (47) and the above equation could be simplified as:

$$0 = w_H(\underline{p}) - w_L(\underline{p}) + (r+\delta+\lambda_L)[A_L - A_H + (B_L - B_H)\underline{p}].$$

The RHS of the above equation is linear in $\underline{p}$. Therefore, if we can prove the slope is not zero then there cannot exist two $\underline{p}$'s satisfying the equation simultaneously. Fortunately, this is the case. The slope is

$$\Delta_H - \Delta_L + (r+\delta+\lambda_L)(B_L - B_H).$$

Notice that $B_H = \frac{\Delta_H}{r+\delta}$ and $(r+\delta+\lambda_L)B_L = \Delta_L + \lambda_L(W_H(1) - A_L)$. Hence,

$$\Delta_H - \Delta_L + (r+\delta+\lambda_L)(B_L - B_H) = \lambda_L(A_L - A_H) > 0.$$

The following result summarizes the findings above and corresponds to Theorem 1 in the Brownian motion case:

Proposition 5 *Given the Lévy process and provided an equilibrium exists, PAM is the unique stationary competitive equilibrium allocation under strict supermodularity.*

Under PAM, $k_{L1} > 0$, $k_{L2} = 0$ and $k_{H1} = 0$, $k_{H2} > 0$. We can use the procedure introduced in the previous sections to pin down the equilibrium cutoff $\underline{p}$ and derive value functions based on $\underline{p}$.

Notice also that under the Lévy process, beliefs are formed through Bayesian updating. We conjecture that PAM will always be the competitive equilibrium allocation under strict supermodularity for any stochastic process as long as there is Bayesian updating. This is because under Bayesian learning, the belief updating process is always a martingale. Of course, establishing this result for general information processes is impossible because it requires the explicit solution of the differential equations for the value function, which generally does not exist.

8.2 Non-Bayesian Updating

Suppose instead that the belief updating is not a martingale. Then it must be generated by some non-Bayesian learning process. We will now show for an example that the competitive equilibrium can be non-PAM even if there is supermodularity.

Suppose the belief updating process in firm y is given by: $dp = \lambda_y p dt$ for $p < 1$, with λ_y a constant, and once p reaches 1, $dp = 0$. We may think p as a special human capital with 1 as an upper bound on the accumulation. The value function of a worker is given by:[30]

$$(r+\delta)W_y(p) = w_y(p) + \lambda_y p W_y'(p)$$

with solution:

$$W_y(p) = C_y p^{\frac{r+\delta}{\lambda_y}} + \frac{\Delta_y}{r+\delta-\lambda_y}p + \frac{\mu_{Ly} - rV_y}{r+\delta}.$$

Suppose PAM is the equilibrium allocation, then

$$\lim_{p\to 1} W_H(p) = W_H(1) = \frac{\Delta_H}{r+\delta}p + \frac{\mu_{LH} - rV_H}{r+\delta},$$

which implies that:

$$C_H = -\frac{\lambda_H \Delta_H}{(r+\delta)(r+\delta-\lambda_H)}.$$

At the cutoff $\underline{p}$ we have:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value-matching condition)} \tag{51}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}), \qquad \text{(Smooth-pasting \& No-deviation condition)} \tag{52}$$

where it turns out that for this belief-updating process, the no-deviation condition coincides with the smooth-pasting condition. We derive the no-deviation condition in the Appendix.

This is a system of equations in C_L and $\underline{p}$. Substitute C_L and $\underline{p}$ could be expressed as:

$$\frac{\Delta_L}{r+\delta}\underline{p} + \frac{\mu_{LL} - rV_L}{r+\delta} = \frac{\lambda_L - \lambda_H}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}(\underline{p})^{\frac{r+\delta}{\lambda_H}} + (1 - \frac{\lambda_L}{r+\delta})\frac{\Delta_H}{r+\delta-\lambda_H}\underline{p} + \frac{\mu_{LH} - rV_H}{r+\delta}$$

or

$$\frac{\Delta_L - \Delta_H}{r+\delta}\underline{p} + \frac{\mu_{LL} - rV_L}{r+\delta} = \frac{\lambda_H - \lambda_L}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}] + \frac{\mu_{LH} - rV_H}{r+\delta}. \tag{53}$$

Notice that PAM requires that the $p = 0$ worker has incentive to be matched with L firms. Hence,

$$\frac{\mu_{LL} - rV_L}{r+\delta} > \frac{\mu_{LH} - rV_H}{r+\delta}$$

Also notice that

$$\frac{\lambda_H - \lambda_L}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}] < 0$$

if $\lambda_L > \lambda_H$ and $r+\delta > \lambda_H$.

[30] We can write the value of a worker of type p in firm y as $W_y(p) = w_y(p)dt + (1-(r+\delta)dt)W_y(p+dp)$. Using a Taylor expansion $W_y(p+dp) = W_y(p) + W_y'(p)dp + o(dt)$ and the fact that $dp = \lambda_y p dt$, we obtain the expression for $W_y(p)$.

If we can show that

$$\frac{\Delta_H - \Delta_L}{r+\delta}\underline{p} < \frac{\lambda_L - \lambda_H}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}],$$

then Equation (53) cannot hold as equality, which is the result we are looking for. First notice that the LHS of the inequality goes to zero as $\Delta_H - \Delta_L$ decreases to zero. Meanwhile, the belief updating process implies the ergodic distribution only depends on λ's and will not depend on Δ's. From previous sections, if PAM is indeed the equilibrium allocation, then $\underline{p}$ should not depend on Δ's. Therefore, fix any $\lambda_L > \lambda_H$ and $r + \delta > \lambda_H$ and we can derive some corresponding $\underline{p} \in (0, 1)$. Then, let $\Delta_H - \Delta_L$ decreases to zero and it is immediate to see that eventually we will have:

$$\frac{\Delta_H - \Delta_L}{r+\delta}\underline{p} < \frac{\lambda_L - \lambda_H}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}].$$

This implies that PAM cannot be an equilibrium if $\lambda_L > \lambda_H$ and the degree of supermodularity is sufficiently small.

9 Concluding Remarks

In this paper, we have proposed a competitive equilibrium model of the labor market that unifies frictionless sorting and a learning-based theory of turnover. In equilibrium under supermodularity, workers with better posteriors about their ability tend to sort into more productive jobs. The main technical contribution of this paper is that we find a new constraint on the worker's value function as a result of sequential rationality in the presence of competitively determined payoffs. At the cutoff type, the second derivative of the workers' value function must equate. In addition to the standard conditions of value-matching (zero-th derivative) and smooth-pasting (first derivative), we now also have the no-deviation condition (second derivative).

What is possibly most surprising is that the result of positive sorting under supermodularity is not determined by the speed of learning. In the trade-off between the learning speed and instantaneous productive efficiency, productive efficiency always takes the upper hand. As such, the equilibrium allocation does not depend on the signal-to-noise ratio (the ratio of the average payoff gain, which measures the efficiency, over the noise term). This seems to indicate in this competitive environment the sorting aspect dominates the learning. Quite surprisingly, this sorting result does not hinge on the particular information structure and is robust to general Bayesian learning processes.

Our analysis has certain limitations and several issues remain unanswered. First, like most experimentation models, payoffs are linear and agents are risk neutral. Non-linearity is desirable for the economic interpretation. However, it renders the solution to the differential equation of the value function much harder to solve.

Second, ideally we would like to extend the analysis to general distributions of worker and firm types. Like in much of the experimentation literature the realized type is either high or low on a risky arm. Here, in addition we have *two* risky arms that are correlated, since there is learning in both types of firms. The focus on the two firm-type case (two arms) keeps down the dimensionality of the continuous time problem. With more than two firm types, analyzing the Brownian motion process is mathematically substantially more demanding.

Finally, our result that PAM obtains under supermodularity and that the planner's problem can be decentralized, is established for a stationary equilibrium. While a solution of a general non-stationary equilibrium is too complex, one can easily construct a two-period counterexample in which PAM will not necessarily obtain in a non-stationary environment.

Appendix

Proof of Lemma 2

Proof. The worker $p \in (0,1)$ always has the choice that stays in one firm y forever. Then the value is $\frac{\mu_y(p) - rV_y}{r+\delta}$. But obviously, this is not an optimal choice (Suppose not, then all of the workers will stay in one type of firms and the market is not cleared). So we have that the equilibrium value function $W_y(p)$ must satisfy: $W_y(p) > \frac{\mu_y(p) - rV_y}{r+\delta}$. This immediately implies:

$$\Sigma_y(p) W_y{}''(p) = (r+\delta) W_y(p) - (\mu^i(p) - rV^i) > 0.$$

So the equilibrium value functions W_y convex for $p \in (0,1)$. ■

Proof of Lemma 3

Proof. Suppose workers with $p \in [0, \underline{p})$ are employed by type y firm. This implies that $W_y(p) = \frac{\mu_y(p) - rV_y}{r+\delta} + k_{y2} p^{\alpha_y} (1-p)^{1-\alpha_y}$ since 0 is included in the domain. It is easy to see that $W_y{}'(0) = \frac{\mu_{Hy} - \mu_{Ly}}{r+\delta} > 0$ and since W_y is strictly convex, $W_y'(p) > 0$ for all $p \in [0, \underline{p})$. At $\underline{p}$, worker will transfer to type $-y$ firm but smooth pasting condition implies $W'_{-y}(\underline{p}) = W_y'(\underline{p}) > 0$. Strict convexity implies $W_{y'}'(p) > 0$ so on and so forth. Therefore, we must have the equilibrium value functions W_y are strictly increasing. ■

Proof of Claim 2

Proof. We will actually prove a more general claim, i.e., that the result holds for any combination (s_H, s_L), including $s_H < s_L$. This makes the proof also applicable to the case of $\sigma_H \neq \sigma_L$. Under strict supermodularity, for any combination of (s_H, s_L), it is impossible to have $p_1 < p_2$ and equilibrium value functions W_H (for $p \in [p_1, p_2]$), W_{L1} (for $p < p_1$), W_{L2} (for $p > p_2$) such that:

$$W_H(p_1) = W_{L1}(p_1) \quad \text{and} \quad W_H''(p_1) = W_{L1}''(p_1)$$

$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W_H''(p_2) = W_{L2}''(p_2)$$

are satisfied simultaneously.

Suppose on the contrary the equations described above hold simultaneously. Then from Equation (3), we should get:

$$w_H(p_1) + \Sigma_H(p_1) W_H''(p_1) = w_L(p_1) + \Sigma_L(p_1) W_{L1}''(p_1)$$

and

$$w_H(p_2) + \Sigma_H(p_2) W_H''(p_2) = w_L(p_2) + \Sigma_L(p_2) W_{L2}''(p_2)$$

since

$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W_H(p_1) = W_{L1}(p_1).$$

Notice that

$$W_H''(p_2) = W_{L2}''(p_2) \quad \text{and} \quad W_H''(p_1) = W_{L1}''(p_1),$$

by Lemma 5 and hence:

$$\frac{\Sigma_H(p_1) - \Sigma_L(p_1)}{\Sigma_H(p_1)} (r+\delta) W_H(p_1) = w_L(p_1) - \frac{\Sigma_L(p_1)}{\Sigma_H(p_1)} w_H(p_1) \tag{54}$$

and

$$\frac{\Sigma_H(p_2)-\Sigma_L(p_2)}{\Sigma_H(p_2)}(r+\delta)W_H(p_2)=w_L(p_2)-\frac{\Sigma_L(p_2)}{\Sigma_H(p_2)}w_H(p_2). \tag{55}$$

By definition,

$$\frac{\Sigma_H(p_1)-\Sigma_L(p_1)}{\Sigma_H(p_1)}=\frac{\Sigma_H(p_2)-\Sigma_L(p_2)}{\Sigma_H(p_2)}=\frac{s_H^2-s_L^2}{s_H^2}.$$

First, if $s_H^2=s_L^2$, Equations (54) and (55) imply that: $w_H(p_1)-w_L(p_1)=w_H(p_2)-w_L(p_2)=0$ which cannot hold simultaneously for $p_1\neq p_2$ since $w_H(\cdot)$ and $w_L(\cdot)$ are linear functions with different slopes Δ_H and Δ_L.

Second, if $s_H^2>s_L^2$, then Equations (54) and (55) could be simplified as:

$$\frac{s_H^2-s_L^2}{s_H^2}(r+\delta)(W_H(p_2)-W_H(p_1))=w_L(p_2)-w_L(p_1)-\frac{\Sigma_L(p_2)}{\Sigma_H(p_2)}(w_H(p_2)-w_H(p_1)).$$

Under strict supermodularity, the LHS of the above equation is strictly larger than $\frac{s_H^2-s_L^2}{s_H^2}(r+\delta)W_H'(p_1)(p_2-p_1)$ by the convexity of the value function. And $\frac{s_H^2-s_L^2}{s_H^2}(r+\delta)W_H'(p_1)(p_2-p_1)$ is larger than $\frac{s_H^2-s_L^2}{s_H^2}\Delta_L(p_2-p_1)$ by Lemma 4. Meanwhile, the RHS of the above equation is strictly smaller than

$$\Delta_L(p_2-p_1)-\frac{\Sigma_L(p_2)}{\Sigma_H(p_2)}\Delta_H(p_2-p_1))=\frac{s_H^2-s_L^2}{s_H^2}\Delta_L(p_2-p_1)$$

which contradicts the fact that LHS is the same as RHS. The impossibility in $s_H^2<s_L^2$ case could be proved similarly and is thus omitted. By contradiction, we immediately know the claim at the beginning of the proof is correct.

For the strict submodularity case, it suffices to relabel 'H' by 'L' and 'L' by 'H'. The claim is obviously correct given we have already proved the strict supermodularity result. ■

Proof of Lemma 6

Proof. We will actually prove a more general Lemma, i.e., that the result holds for any combination (s_H,s_L), including $s_H<s_L$. This makes the proof also applicable to the case of $\sigma_H\neq\sigma_L$. First of all, we want to show all of the one-shot deviations are ruled out by our no-deviation condition as $dt\to 0$.

Under strict supermodularity, PAM is the only candidate equilibrium allocation by Theorem 1. The value functions thus are given by:

$$W_L(p)=\frac{w_L(p)}{r+\delta}+k_Lp^{\alpha_L}(1-p)^{1-\alpha_L}$$

and

$$W_H(p)=\frac{w_H(p)}{r+\delta}+k_Hp^{1-\alpha_H}(1-p)^{\alpha_H}.$$

Let

$$\mathcal{G}_L(p)=k_Lp^{\alpha_L}(1-p)^{1-\alpha_L}(\frac{\alpha_L-p}{p(1-p)})>0$$

and

$$\mathcal{G}_H(p) = k_H p^{1-\alpha_H}(1-p)^{\alpha_H}(\frac{1-\alpha_H-p}{p(1-p)}) < 0$$

be the first derivatives for the non-linear parts of the value functions. Smooth pasting at $\underline{p}$ implies:

$$\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p}) = \frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p}).$$

From the proof of Lemma 5, it suffices to show that inequality (11) holds for $p < \underline{p}$ and inequality (9) holds for $p > \underline{p}$.

For $p < \underline{p}$, define:

$$Z_L(p) = w_H(p) - w_L(p) + \frac{s_H^2 - s_L^2}{s_L^2}((r+\delta)W_L(p) - w_L(p)). \quad (56)$$

Obviously, we have $\lim_{p\nearrow\underline{p}} Z_L(p) = 0$ from Lemma 5. If we can show that $Z_L(p)$ is increasing in p as p increases from 0 to $\underline{p}$, then we are done since $Z_L(p) < Z_L(\underline{p}) = 0$. Notice that

$$Z_L'(p) = \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)W_L'(p)$$

and $W_L'(p)$ lies between $\frac{\Delta_L}{r+\delta}$ and $\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})$ for $p \in [0, \underline{p}]$.[31]

If $s_H^2 \geq s_L^2$, then

$$Z_L'(p) \geq \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)\frac{\Delta_L}{r+\delta} = \Delta_H - \Delta_L > 0;$$

if $s_H^2 < s_L^2$, then

$$\begin{aligned} Z_L'(p) &\geq \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)[\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})] \\ &= \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)[\frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p})] \\ &= \frac{s_H^2}{s_L^2}(\Delta_H - \Delta_L) + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)\mathcal{G}_H(\underline{p}) > 0. \end{aligned}$$

Therefore, we conclude that $Z_L'(p) > 0$ for both $s_H \geq s_L$ and $s_H < s_L$ cases, which implies that $Z_L(p) < 0$ for all $p < \underline{p}$ and hence there is no profitable one-shot deviation as dt is sufficiently small.

For $p > \underline{p}$, similarly define:

$$Z_H(p) = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W_H''(p). \quad (57)$$

Under PAM equilibrium, we have $Z_H(\underline{p}+) = 0$ from Lemma 5. Notice that

$$Z_H(p) = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W_H''(p) = w_L(p) - w_H(p) + \frac{s_L^2 - s_H^2}{s_H^2}((r+\delta)W_H(p) - w_H(p)),$$

[31]This comes from the fact that $W_L(\cdot)$ is a strictly convex function.

with $W'_H(p)$ lies between $\frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p})$ and $\frac{\Delta_H}{r+\delta}$ for $p \in [\underline{p}, 1]$. Similar to the proof for $p < \underline{p}$ case, if $s_L^2 > s_H^2$

$$Z'_H(p) \leq \Delta_L - \Delta_H < 0;$$

and if $s_L^2 \leq s_H^2$

$$Z'_H(p) \leq \Delta_L - \frac{s_L^2}{s_H^2}\Delta_H + \frac{s_L^2 - s_H^2}{s_H^2}(r+\delta)(\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})) < 0.$$

Therefore, $Z'_H(p) < 0$ for both $s_H \geq s_L$ and $s_H < s_L$ cases and hence $Z_H(p) < 0$ for all $p > \underline{p}$.

Second, since there is no one-shot deviation for any p, obviously there will be no any other deviation for any p. Consider any deviation starting at p. Then the above result says it is better not to deviate for at least dt time. Suppose after dt, we achieve a new p'. Similarly, there should be no profitable deviation for at least dt' time. Keep using the same logic and we can conclude that any deviation is not profitable. ■

Derivation of the Boundary Conditions

Here, we just investigate the boundary conditions for the first case: $\underline{p} < p_0$. The derivation is similar for the second case.

In a stationary equilibrium, both the total measure $\int_0^1 f_y(p,t)dp$ and the expectations $\int_0^1 pf_y(p,t)dp$ are constant over time. Hence, it must be the case that $\int_0^1 \frac{\partial f_y(p,t)}{\partial t}dp = 0$ and $\int_0^1 p\frac{\partial f_y(p,t)}{\partial t}dp = 0$

From

$$\frac{\partial f_y(p,t)}{\partial t} = \frac{d^2}{dp^2}[\Sigma_y(p)f_y(p,t)] - \delta f_y(p,t),$$

we should have:

$$\int_0^{\underline{p}} \{\frac{d^2}{dp^2}[\Sigma_L(p)f_L(p)] - \delta f_L(p)\}dp = 0$$

and

$$\int_{\underline{p}}^{p_0} \{\frac{d^2}{dp^2}[\Sigma_H(p)f_H(p)] - \delta f_H(p)\}dp + \int_{p_0}^{1} \{\frac{d^2}{dp^2}[\Sigma_H(p)f_H(p)] - \delta f_H(p)\}dp = 0.$$

The above two equations give us:

$$\frac{d}{dp}[\Sigma_L(p)f_L(p)]|_{\underline{p}-} = \delta(1-\pi)$$

and

$$\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \frac{d}{dp}[\Sigma_H(p)f_H(p)]|_{\underline{p}+} + \delta\pi$$

since the market clearing conditions imply:

$$\int_0^{\underline{p}} f_L(p)dp = 1-\pi$$

$$\int_{\underline{p}}^1 f_H(p)dp = \pi$$

and there is continuity at p_0:

$$f_H(p_0-) = f_H(p_0+).$$

Meanwhile, notice that inflow at p_0 must be the same as δ, which implies that $\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \delta$. This immediately gives us the flow equation at $\underline{p}$:

$$\frac{d}{dp}[\Sigma_L(p)f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p)f_H(p)]|_{\underline{p}+}.$$

Now apply similar logic and we can get:

$$\int_0^{\underline{p}} \left\{ p\frac{d^2}{dp^2}[\Sigma_L(p)f_L(p)] - p\delta f_L(p) \right\} dp + \int_{\underline{p}}^1 \left\{ p\frac{d^2}{dp^2}[\Sigma_H(p)f_H(p)] - p\delta f_H(p) \right\} dp = 0.$$

Notice that

$$\int_0^{\underline{p}} p\delta f_L(p)dp + \int_{\underline{p}}^1 p\delta f_H(p)dp = \delta p_0$$

by the martingale property. Meanwhile, we still have: $\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \delta$. Hence,after some tedious algebra, we can get:

$$\left\{ p\frac{d}{dp}[\Sigma_L(p)f_L(p)] + \Sigma_L(p)f_L(p) \right\} |_{\underline{p}-} = \left\{ p\frac{d}{dp}[\Sigma_H(p)f_H(p)] + \Sigma_H(p)f_H(p) \right\} |_{\underline{p}+}$$

which gives us the boundary condition at $\underline{p}$:

$$\Sigma_H(\underline{p}+)f_H(\underline{p}+) = \Sigma_L(\underline{p}-)f_L(\underline{p}-).$$

Proof of Proposition 1

Proof. First, we can express $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ as functions of $\underline{p}$. Equations (25) and (27) imply:

$$f_{L0} = \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}.$$

and

$$f_{H2} = f_{H0}(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{H2}} + f_{H1}$$

From Equations (23) and (26), f_{H0} and f_{H1} as could be written as:

$$f_{H0} = \frac{\eta_H + \eta_L}{2\eta_H}\frac{s_L^2}{s_H^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L - \eta_H} f_{L0}$$

and

$$f_{H1} = -\frac{\eta_L - \eta_H}{2\eta_H}\frac{s_L^2}{s_H^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L + \eta_H} f_{L0}.$$

Here,

$$\eta_L = \sqrt{\frac{1}{4} + \frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_H^2}} > 1/2.$$

Next, we want to show that both f_{H0} and f_{H1} are decreasing in $\underline{p}$.

Rewrite f_{H0} as:

$$f_{H0} = \frac{\eta_H + \eta_L}{2\eta_H} \frac{s_L^2}{s_H^2} (\frac{\underline{p}}{1-\underline{p}})^{\eta_L - \eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp}.$$

and it suffices to show that $(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}$ is decreasing in $\underline{p}$. Notice that

$$(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H} = \int_0^{\underline{p}} [(\frac{p}{1-p})^{\eta_L-\eta_H}]' dp = \int_0^{\underline{p}} (\eta_L - \eta_H)(\frac{p}{1-p})^{\eta_L-\eta_H-1}(\frac{1}{1-p})^2 dp.$$

Let $G_1(p) = p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}$ and $G_2(p) = (\frac{p}{1-p})^{\eta_L-\eta_H-1}(\frac{1}{1-p})^2$ such that:

$$\frac{G_1(p)}{G_2(p)} = p^{-\frac{1}{2}+\eta_H}(1-p)^{-\frac{1}{2}-\eta_H}$$

is increasing in p. Therefore, we could derive:

$$(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp}$$

is decreasing in $\underline{p}$[32] and hence f_{H0} is decreasing in $\underline{p}$ as well.

Similarly, we can rewrite f_{H1} as:

$$f_{H1} = -\frac{\eta_L - \eta_H}{2\eta_H} \frac{s_L^2}{s_H^2} (\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp}.$$

Similarly,

$$(\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H} = \int_0^{\underline{p}} (\eta_L + \eta_H)(\frac{p}{1-p})^{\eta_L+\eta_H-1}(\frac{1}{1-p})^2 dp.$$

Let $G_3(p) = (\frac{p}{1-p})^{\eta_L+\eta_H-1}(\frac{1}{1-p})^2$ and we have:

$$\frac{G_1(p)}{G_3(p)} = p^{-\frac{1}{2}-\eta_H}(1-p)^{-\frac{1}{2}+\eta_H}$$

is decreasing in p. Therefore, it must be the case that

$$-(\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp}$$

is decreasing in $\underline{p}$ and hence f_{H1} is also decreasing in $\underline{p}$.

Finally, it is immediate that

$$f_{H2} = f_{H0}(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{H2}} + f_{H1}$$

is also decreasing in $\underline{p}$. Therefore, we can expressing f_{H0}, f_{H1} and f_{H2} as $\xi_0(\underline{p})$, $\xi_1(\underline{p})$ and $\xi_2(\underline{p})$ respectively such that $\xi_0{}' < 0$, $\xi_1{}' < 0$ and $\xi_2{}' < 0$.

[32] Actually, we are using the result that if $\frac{G_2(p)}{G_1(p)}$ is decreasing in p, then $\frac{\int_0^{\underline{p}} G_2(p)dp}{\int_0^{\underline{p}} G_1(p)dp}$ will also be decreasing in $\underline{p}$. This is true because by the definition of Riemann integral, $\int_0^{\underline{p}} G_1(p)dp$ and $\int_0^{\underline{p}} G_2(p)dp$ could be written as the limit of Riemann sum. The ratio of two Riemann sums is always decreasing in $\underline{p}$ since $\frac{G_2(p)}{G_1(p)}$ is decreasing in p.

Hence, the market clearing condition (24) implies:

$$H(\underline{p}) = \int_{\underline{p}}^{p_0} [\xi_0(\underline{p}) p^{\gamma_{H1}} (1-p)^{\gamma_{H2}} + \xi_1(\underline{p}) p^{\gamma_{H2}} (1-p)^{\gamma_{H1}}] dp + \int_{p_0}^{1} \xi_2(\underline{p}) p^{\gamma_{H2}} (1-p)^{\gamma_{H1}} dp = \pi.$$

It is easy to check that $H' < 0$ since $\xi_0{}' < 0$, $\xi_1{}' < 0$ and $\xi_2{}' < 0$. There exists $\underline{p} \in (0, p_0)$ such that $H(\underline{p}) = \pi$ if and only if $\lim_{p \to 0} H(p) > \pi$ and $\lim_{p \to p_0} H(p) < \pi$.

As $\underline{p} \to 0$, $f_{H0} = \xi_0(\underline{p}) \to \infty$ and $f_{H1} = \xi_1(\underline{p}) \to 0$, which imply:

$$\lim_{p \to 0} H(p) \to \infty > \pi.$$

Meanwhile, when $\underline{p} \to p_0$, it is obvious that $H(\underline{p}) \to \int_{p_0}^{1} f_{H2} p^{\gamma_{H2}} (1-p)^{\gamma_{H1}} dp$. Notice that

$$f_{H2} = f_{H0} (\frac{p_0}{1-p_0})^{\gamma_{H1} - \gamma_{H2}} + f_{H1} \to \frac{s_L^2}{s_H^2} (\frac{p_0}{1-p_0})^{\eta_L + \eta_H} \frac{1-\pi}{\int_0^{p_0} p^{\gamma_{L1}} (1-p)^{\gamma_{L2}} dp}$$

as $\underline{p} \to p_0$.

As a result, $\lim_{p \to p_0} H(p) < \pi$ if and only if:

$$\frac{s_L^2}{s_H^2} (\frac{p_0}{1-p_0})^{\eta_L + \eta_H} \frac{1-\pi}{\int_0^{p_0} p^{\gamma_{L1}} (1-p)^{\gamma_{L2}} dp} \int_{p_0}^{1} p^{\gamma_{H2}} (1-p)^{\gamma_{H1}} dp < \pi,$$

which establishes Equation 28 in the proposition. Moreover, since $H(\cdot)$ is strictly decreasing, the solution to $H(p) = \pi$ must be at most one. This completes our proof of Proposition 1. ■

Proof of Corollary 1

Proof. To make the proof, we have to redefine the $H(\cdot)$ function in the proof of Proposition 1 as $H(p; \pi, p_0)$ with equilibrium cutoff $\underline{p}$ satisfying $H(\underline{p}; \pi, p_0) = \pi$. It is obviously to verify that H is linear in $(1-\pi)$. So as π increases, $\pi/(1-\pi)$ increases and we have to decrease $\underline{p}$ to balance the equation. On the other hand,

$$\begin{aligned} \frac{\partial H}{\partial p_0} &= \xi_0(\underline{p}) p_0^{\gamma_1^H} (1-p_0)^{\gamma_2^H} + \xi_1(\underline{p}) p_0^{\gamma_2^H} (1-p_0)^{\gamma_1^H} - \xi_2(\underline{p}) p_0^{\gamma_2^H} (1-p_0)^{\gamma_1^H} \\ &+ \int_{p_0}^{1} \frac{\partial \xi_2(\underline{p})}{\partial p_0} p^{\gamma_2^H} (1-p)^{\gamma_1^H} dp. \end{aligned}$$

It is easy to verify that the first line on the RHS is zero while the second line is strictly positive. Hence $H(\underline{p}; \pi, p_0)$ is increasing in p_0 and we have to increase $\underline{p}$ to keep the equation as p_0 increases.

The proof for the comparative statics for $\underline{p} > p_0$ case is similar and hence is omitted. ■

Proof of Proposition 2

Proof. First, from equation (35), we have:

$$f_{H0} = \frac{\pi}{\int_{\underline{p}}^{1} p^{\gamma_{H2}} (1-p)^{\gamma_{H1}} dp}.$$

Second, Equations (34) and (37) imply:

$$f_{L1} = \frac{\eta_L - \eta_H}{2\eta_L}\frac{s_H^2}{s_L^2}(\frac{\underline{p}}{1-\underline{p}})^{-\eta_L-\eta_H} f_{H0}$$

and

$$f_{L2} = \frac{\eta_L + \eta_H}{2\eta_L}\frac{s_H^2}{s_L^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H} f_{H0}.$$

Here,

$$\eta_L = \sqrt{\frac{1}{4}+\frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4}+\frac{2\delta}{s_H^2}} > 1/2.$$

It is easy to verify that f_{H0}, f_{L1}, f_{L2} are increasing in $\underline{p}$ and hence $f_{L0} = f_{L1} + f_{L2}(\frac{p_0}{1-p_0})^{-2\eta_L}$ is also increasing in $\underline{p}$ by Equation (38).

Hence, we can express f_{L0}, f_{L1}, f_{L2} as $\xi_0(\underline{p})$, $\xi_1(\underline{p})$ and $\xi_2(\underline{p})$ respectively such that ${\xi_0}' > 0$, ${\xi_1}' > 0$ and ${\xi_2}' > 0$.

Finally, the market clearing condition (36) implies:

$$H(\underline{p}) = \int_0^{p_0} \xi_0(\underline{p}) p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp + \int_{p_0}^{\overline{p}} [\xi_1(\underline{p}) p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} + \xi_2(\underline{p}) p^{\gamma_{L2}}(1-p)^{\gamma_{L1}}] dp = 1-\pi.$$

Obviously, $H(\cdot)$ is strictly increasing, which guarantees the solution is unique if it exists and $\lim_{p\to p_0} H(p) \le 1-\pi$ will give us Equation (39) in Proposition 2. ■

Proof of Lemma 8

Proof. By substituting $\mu_H(p)$ and $\mu_L(p)$, the total expected surplus for allocation 1 could be written as:

$$S_1 = \int_{\Omega_H} (\Delta_H p + \mu_{LH}) f_H(p) dp + \int_{\Omega_L} (\Delta_L p + \mu_{LL}) f_L(p) dp.$$

From market clearing and martingale property conditions, we can furthermore rewrite S_1 as:

$$S_1 = (\Delta_H - \Delta_L)\int_{\Omega_H} p f_H(p) dp + \Delta_L p_0 + \pi\mu_{LH} + (1-\pi)\mu_{LL}.$$

And similarly,

$$S_2 = (\Delta_H - \Delta_L)\int_{\tilde{\Omega}_H} p f_H(p) dp + \Delta_L p_0 + \pi\mu_{LH} + (1-\pi)\mu_{LL}.$$

Therefore, $S_1 > S_2$ if and only if $\int_{\Omega_H} p f_H(p) dp > \int_{\tilde{\Omega}_H} p \tilde{f}_H(p) dp$ or alternatively, $\int_{\Omega_L} p f_H(p) dp < \int_{\tilde{\Omega}_L} p \tilde{f}_L(p) dp$. ■

Proof of Theorem 4

Proof.

We establish the proof of Theorem 4 under supermodularity. The same logic goes through for submodularity. The proof is constructed in the following three steps: 1. for $N = 3$ we show that the planner can increase output when changing the cutoffs; 2. for $N = 3$ no allocation dominates PAM; 3. For any N, the allocation with $N - 2$ cutoffs dominates that with N cutoffs.

1. For $N = 3$, output increases from changing the cutoffs

Consider any allocation with three cutoffs $0 < \underline{p}_3 < \underline{p}_2 < \underline{p}_1 < 1$ such that workers with $p \in (\underline{p}_1, 1]$ and $p \in (\underline{p}_3, \underline{p}_2)$ are allocated to the high type firms while workers with $p \in [0, \underline{p}_3)$ and $p \in (\underline{p}_2, \underline{p}_1)$ are allocated to the low type firms. Furthermore, denote the ergodic density function for this allocation to be f_y and for p close to 0, let the density function be $f_L(p) = \tilde{f}_{L0} p^{\gamma_L} (1-p)^{1-\gamma_L}$ while the ergodic density function for p close to 1 is denoted by $f_H(p) = \tilde{f}_{H0} p^{1-\gamma_H}(1-p)^{\gamma_H}$ where $\tilde{f}_{L0}$ and $\tilde{f}_{H0}$ are constants. Correspondingly, denote the ergodic density under the PAM allocation to be f_y^* with the unique cutoff $\underline{p}$.

1. Suppose the planner changes the allocation by moving the interval to the left: $(\underline{p}_2, \underline{p}_1) \rightarrow (\underline{p}'_2, \underline{p}'_1)$ where $(\underline{p}'_2, \underline{p}'_1) = (\underline{p}_2 - \epsilon_2, \underline{p}_1 - \epsilon_1)$. Choose ϵ_1, ϵ_2 such that market clearing is satisfied:

$$\int_{\underline{p}'_1}^{\underline{p}_1} f_H(p)dp = \int_{\underline{p}'_2}^{\underline{p}_2} f_H(p)dp.$$

2. Given the new cutoffs, the Kolmogorov forward equation will pin down a new density $\hat{f}_L$ in the interval $(\underline{p}'_2, \underline{p}'_1)$. Globally, we need to satisfy market clearing and the martingale property conditions. The market clearing condition for the H types is satisfied by the construction. For the L type firms it requires that:

$$\int_{\underline{p}'_2}^{\underline{p}'_1} \hat{f}_L(p)dp = \int_{\underline{p}_2}^{\underline{p}_1} f_L(p)dp.$$

The martingale property condition requires that $\mathbb{E}_{\Omega'_H} p + \mathbb{E}_{\Omega'_L} p = p_0$ or:

$$\int_0^{p_3} p f_L(p)dp + \int_{p_3}^{p'_2} p f_H(p)dp + \int_{p'_2}^{p'_1} p \hat{f}_L(p)dp + \int_{p'_1}^{1} p f_H(p)dp = p_0.$$

Above are a system of two linear equations about the distributional parameters for $\hat{f}_L$ and $\hat{f}_L$ could be solved as a result.[33]

3. Then comparing the original allocation to the new one, we get

$$\mathbb{E}_{\Omega'_H} p - \mathbb{E}_{\Omega_H} p = \int_{\underline{p}'_1}^{\underline{p}_1} p f_H(p)dp - \int_{\underline{p}'_2}^{\underline{p}_2} p f_H(p)dp > 0$$

since by construction

$$\int_{\underline{p}'_1}^{\underline{p}_1} f_H(p)dp = \int_{\underline{p}'_2}^{\underline{p}_2} f_H(p)dp$$

and the interval $[\underline{p}'_2, \underline{p}'_1]$ is strictly to the left of $[\underline{p}_2, \underline{p}_1]$. From Lemma 8, $\mathbb{E}_{\Omega'_H} p > \mathbb{E}_{\Omega_H} p$ implies the planner prefers allocation Ω' over Ω.

[33]Things are slightly different if we have $p_0 \in (p'_2, p'_1)$. Then we have four new distribution coefficients but we also have two more equations: $\hat{f}_L(p_0-) = \hat{f}_L(p_0+)$ and $\Sigma_L(p_0)(\hat{f}'_L(p_0-) - \hat{f}'_L(p_0+)) = \delta$. We can use this system of four linear equations to pin down the four parameters.

4. Similarly, we can consider another transform which is to move the interval to the right: $(\underline{p}_3, \underline{p}_2) \to (\underline{p}'_3, \underline{p}'_2)$ where $(\underline{p}'_3, \underline{p}'_2) = (\underline{p}_3 + \epsilon_2, \underline{p}_2 + \epsilon_1)$. This can also lead to output increases. Keep on doing such transformations and eventually, we can have both the distance and the measure between $\underline{p}'_3$ and $\underline{p}'_1$ arbitrarily small while the new $(\underline{p}'_1, \underline{p}'_2, \underline{p}'_3)$ allocation strictly dominates the original $(\underline{p}_1, \underline{p}_2, \underline{p}_3)$ allocation.

2. For $N = 3$, no allocation dominates PAM

1. We now show by contradiction that no allocation dominates PAM for $N = 3$. Suppose on the contrary that there exists an allocation with cutoffs $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{p}_3$ which dominates the PAM allocation. Then by Lemma 8, we should have:

$$\int_{\tilde{p}_1}^{1} p f_H(p) dp + \int_{\tilde{p}_3}^{\tilde{p}_2} p f_H(p) dp > \int_{\underline{p}}^{1} p f_H^*(p) dp \tag{58}$$

and

$$\int_{\tilde{p}_2}^{\tilde{p}_1} p f_L(p) dp + \int_{0}^{\tilde{p}_3} p f_L(p) dp < \int_{0}^{\underline{p}} p f_L^*(p) dp. \tag{59}$$

From Step 1, we can first fix $\tilde{p}_3$ and make $\tilde{p}'_2$ move towards $\tilde{p}_3$, which is efficiency improving. $\tilde{p}_1$ could be extended to the left until it reaches $\hat{p}_1$: $\int_{\hat{p}_1}^{1} f_H(p) dp = \pi$. Since $\int_{\tilde{p}'_1}^{1} f_H(p) dp < \pi$, it must be the case that $\hat{p}_1 < \tilde{p}'_1$. If $\tilde{p}'_2$ is sufficiently close to $\tilde{p}_3$, we will have $\tilde{p}'_2 < \hat{p}_1$. By hypothesis:

$$\int_{\hat{p}_1}^{1} p f_H(p) dp > \int_{\tilde{p}'_1}^{1} p f_H(p) dp + \int_{\tilde{p}_3}^{\tilde{p}'_2} p f_H(p) dp > \int_{\underline{p}}^{1} p f_H^*(p) dp.$$

On the other hand, it is also efficiency improving by fixing $\tilde{p}_1$ and making $\tilde{p}'_2$ move towards $\tilde{p}_1$. Similarly define $\hat{p}_3$ as: $\int_{0}^{\hat{p}_3} f_L(p) dp = (1 - \pi)$ such that $\hat{p}_3 > \tilde{p}'_3$. By hypothesis,

$$\int_{0}^{\hat{p}_3} p f_L(p) dp < \int_{0}^{\underline{p}} p f_L^*(p) dp.$$

since we can make $\tilde{p}'_2$ sufficiently close to $\tilde{p}_1$.

2. The next step of the proof requires Lemma 10 below. The Lemma implies that we should have $\tilde{p}'_3 < \hat{p}_3 < \underline{p} < \hat{p}_1 < \tilde{p}'_1$ to guarantee that

$$\int_{\hat{p}_1}^{1} p f_H(p) dp > \int_{\underline{p}}^{1} p f_H^*(p) dp \quad \text{and} \quad \int_{0}^{\hat{p}_3} p f_L(p) dp < \int_{0}^{\underline{p}} p f_L^*(p) dp.$$

Therefore, inequalities (58) and (59) only hold when $\tilde{p}'_1 - \tilde{p}'_3 > \hat{p}_1 - \hat{p}_3 > 0$ which contradicts that fact that we can make the distance between $\tilde{p}'_1$ and $\tilde{p}'_3$ arbitrarily small while still keeping the inequalities (58) and (59). Hence, no allocation with $N = 3$ cutoffs could be better than the PAM allocation in terms of aggregate surplus.

3. For N cutoffs, the allocation is dominated by any allocation with $N - 2$ cutoffs.

Consider three adjacent cutoffs $\underline{p}_{n-1}, > \underline{p}_n > \underline{p}_{n+1}$ such that workers with $p \in (\underline{p}_{n-1}, \underline{p}_{n-2})$ and $p \in (\underline{p}_{n+1}, \underline{p}_n)$ are allocated to high type firms; workers with $p \in (\underline{p}_n, \underline{p}_{n-1})$ and $p \in (\underline{p}_{n+2}, \underline{p}_{n+1})$

are allocated to low type firms. Suppose the density functions are such that the market clears and the expectation of p's is p_0. Then we just need to choose κ such that

$$\int_{\underline{p}_{n-1}-\kappa}^{\underline{p}_{n-1}} f_H(p)dp = \int_{\underline{p}_{n+1}}^{\underline{p}_n} f_H(p)dp.$$

Now $\underline{p}_{n-1}$, $\underline{p}_n$ and $\underline{p}_{n+1}$ converge to $\underline{p}_{n-1}-\kappa$ but p_{n+2} is kept to be the same. The market clearing condition requires that

$$\int_{\underline{p}_{n+2}}^{\underline{p}_{n-1}-\kappa} \tilde{f}_L(p)dp = \int_{\underline{p}_n}^{\underline{p}_{n-1}} f_L(p)dp + \int_{\underline{p}_{n+2}}^{\underline{p}_{n+1}} f_L(p)dp.$$

Meanwhile, the martingale property condition requires that:

$$\int_{\underline{p}_1}^{1} pf_H(p)dp + \cdots + \int_{\underline{p}_{n-1}-\kappa}^{\underline{p}_{n-2}} pf_H(p)dp + \int_{\underline{p}_{n+2}}^{\underline{p}_{n-1}-\kappa} p\tilde{f}_L(p)dp + \cdots + \int_0^{\underline{p}_N} pf_L(p)dp = p_0.$$

Similar to Step 1, we have a system of two linear equations about two distributional coefficients and density $\tilde{f}_L$ could be solved. As before,

$$\mathbb{E}_{\Omega_H} p = \int_{\Omega_H} pf_H(p)dp$$

must become higher and this allocation with $N-2$ cutoffs will generate a higher aggregate payoff.

Finally, by the standard induction argument, we can conclude that the PAM allocation with one cutoff dominates any allocation with $N \geq 3$ cutoffs in aggregate surplus. ■

Lemma 10

Lemma 10 *Let $\hat{p}_1$ be such that $\int_{\hat{p}_1}^1 f_H(p)dp = \pi$, where $f_H(p)$ satisfies the Kolmogorov forward equation, then $\int_{\hat{p}_1}^1 pf_H(p)dp$ is increasing in $\hat{p}_1$. Let $\hat{p}_3$ be such that $\int_0^{\hat{p}_3} f_L(p)dp = (1-\pi)$, where $f_L(p)$ satisfies the Kolmogorov forward equation, then $\int_0^{\hat{p}_3} pf_L(p)dp$ is also increasing in $\hat{p}_3$.*

Proof. We just prove the case that $\hat{p}_1 > p_0$. The other cases are similar. Let $f_H(p) = C_H(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}$ where

$$\gamma_{H1} = -\frac{3}{2} + \eta_H \quad \text{and} \quad \gamma_{H2} = -\frac{3}{2} - \eta_H.$$

From Kolmogorov forward equation,

$$\int_{\hat{p}_1}^1 f_H(p)dp = \frac{1}{\delta}\int_{\hat{p}_1}^1 \frac{d^2}{dp^2}[\Sigma_H(p)f_H(p)] = \pi$$

or

$$\frac{\eta_H + \hat{p}_1 - \frac{1}{2}}{\hat{p}_1(1-\hat{p}_1)}\Sigma_H(\hat{p}_1)f_H(\hat{p}_1) = \delta\pi.$$

Notice that

$$\int_{\hat{p}_1}^1 pf_H(p)dp = \frac{1}{\delta}\int_{\hat{p}_1}^1 p\frac{d^2}{dp^2}[\Sigma_H(p)f_H(p)]dp$$

and could be simplified as:

$$\pi\hat{p}_1 + \frac{\pi\hat{p}_1(1-\hat{p}_1)}{\eta_H + \hat{p}_1 - \frac{1}{2}} = \frac{\pi\hat{p}_1(\eta_H + \frac{1}{2})}{\eta_H + \hat{p}_1 - \frac{1}{2}}$$

which is increasing in $\hat{p}_1$ since

$$\eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > \frac{1}{2}.$$

■

On the Job Human Capital Accumulation

Under the assumption of $\underline{p}^u = \underline{p}^e = \underline{p}$, the value functions could be written as:

$$\begin{aligned} W_y^u(p) &= \frac{\mu_y(p) - rV_y}{r+\delta+\lambda} + k_{y1}^u p^{1-\alpha_y^u}(1-p)^{\alpha_y^u} + k_{y2}^u p^{\alpha_y^u}(1-p)^{1-\alpha_y^u} \\ &- \frac{\lambda\frac{(s_y^u)^2}{(s_y^e)^2}}{(r+\delta+\lambda)[(\lambda+\delta+r) - \frac{(s_y^u)^2}{(s_y^e)^2}(r+\delta)]}[\mu_y(p) + \xi(p) - rV_y] \\ &+ \frac{\lambda}{(\lambda+\delta+r) - \frac{(s_y^u)^2}{(s_y^e)^2}(r+\delta)} W_y^e(p) \end{aligned}$$

and

$$W_y^e(p) = \frac{\mu_y(p) + \xi(p) - rV_y}{r+\delta} + k_{y1}^e p^{1-\alpha_y^e}(1-p)^{\alpha_y^e} + k_{y2}^e p^{\alpha_y^e}(1-p)^{1-\alpha_y^e}$$

where

$$\alpha_y^u = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta+\lambda)}{(s_y^u)^2}} \geq 1$$

$$\alpha_y^e = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{(s_y^e)^2}} \geq 1.$$

Boundary conditions

$$W_L^e(\underline{p}) = W_H^e(\underline{p}), \quad W_L^{e\prime}(\underline{p}) = W_H^{e\prime}(\underline{p}), \quad W_L^{e\prime\prime}(\underline{p}) = W_H^{e\prime\prime}(\underline{p})$$

would imply (by normalizing $V_L = 0$ as usual):

$$r\tilde{V}_H^e = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H^e(\alpha_L^e - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)}.$$

And from

$$W_L^u(\underline{p}) = W_H^u(\underline{p}), \quad W_L^{u\prime}(\underline{p}) = W_H^{u\prime}(\underline{p}), \quad W_L^{u\prime\prime}(\underline{p}) = W_H^{u\prime\prime}(\underline{p}),$$

another equilibrium payoff $\tilde{V}_H^u$ could be derived as:

$$\begin{aligned} r\tilde{V}_H^u &= (\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}) - \frac{B_H}{A_H}\frac{\lambda\xi_L}{r+\delta+\lambda}(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}) \\ &+ \frac{B_H}{A_H}\frac{\alpha_H^u(\alpha_L^u - 1)(D_H - D_L)\underline{p}}{\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)}, \end{aligned}$$

where

$$D_H = \frac{A_H}{B_H}\Delta_H - \frac{1-A_H}{B_H}\frac{\lambda\Delta_\xi}{r+\delta+\lambda}$$

$$D_L = \frac{A_L}{B_L}\Delta_L - \frac{1-A_L}{B_L}\frac{\lambda\Delta_\xi}{r+\delta+\lambda}$$

$$A_H = 1 - \frac{(s_H^u)^2}{(s_H^e)^2} \quad B_H = (\lambda+\delta+r) - \frac{(s_H^u)^2}{(s_H^e)^2}(r+\delta)$$

$$A_L = 1 - \frac{(s_L^u)^2}{(s_L^e)^2} \quad B_L = (\lambda+\delta+r) - \frac{(s_L^u)^2}{(s_L^e)^2}(r+\delta).$$

Proof of Proposition 3

Proof. Supermodularity is equivalent to $\Delta_H > \Delta_L$, and $\xi_H \simeq \xi_L$ is equivalent to $\Delta_\xi = \xi_H - \xi_L \to 0$. The proof can be divided into three parts. As a sufficient condition,

1. $$(\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}) - \frac{B_H}{A_H}\frac{\lambda\xi_L}{r+\delta+\lambda}(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}) < (\mu_{LH} - \mu_{LL})$$

2. $$\frac{B_H}{A_H}(D_H - D_L) < \Delta_H - \Delta_L$$

   and

3. $$\frac{\alpha_H^u(\alpha_L^u - 1)\underline{p}}{\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)} < \frac{\alpha_H^e(\alpha_L^e - 1)\underline{p}}{\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)}$$

should be satisfied simultaneously.

First of all, notice that $\frac{(s_H^u)^2}{(s_H^e)^2} > \frac{(s_L^u)^2}{(s_L^e)^2}$ since $\Delta_H > \Delta_L$. As a result, $\frac{A_H}{B_H} < \frac{A_L}{B_L}$ and $\frac{1-A_H}{B_H} > \frac{1-A_L}{B_L}$. The first inequality holds since $\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL} < \mu_{LH} - \mu_{LL}$ and $\frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}) - \frac{B_H}{A_H}\frac{\lambda\xi_L}{r+\delta+\lambda}(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}) > 0$. The second inequality could be proved similarly.

For the last inequality, we just need to compare:

$$\alpha_H^u(\alpha_L^u - 1)[\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)]$$

and

$$\alpha_H^e(\alpha_L^e - 1)[\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)].$$

To prove 3, it suffices to show

$$\alpha_H^u(\alpha_L^u - 1)(\alpha_L^e - \alpha_H^e) > alpha_H^e(\alpha_L^e - 1)(\alpha_L^u - \alpha_H^u).$$

The direct proof is not easy. But notice from the expressions of α's:

$$(\alpha_L^e - \alpha_H^e)(\alpha_L^e + \alpha_H^e - 1) = 2(r+\delta)[\frac{\sigma^2}{(\Delta_L + \Delta_\xi)^2} - \frac{\sigma^2}{(\Delta_H + \Delta_\xi)^2}]$$

and

$$(\alpha_L^u - \alpha_H^u)(\alpha_L^u + \alpha_H^u - 1) = 2(r+\delta+\lambda)[\frac{\sigma^2}{\Delta_L^2} - \frac{\sigma^2}{\Delta_H^2}].$$

Hence, when $\Delta_\xi = 0$,

$$\frac{\alpha_L^e - \alpha_H^e}{\alpha_L^u - \alpha_H^u} = \frac{r+\delta}{r+\delta\lambda}\frac{\alpha_L^u + \alpha_H^u - 1}{\alpha_L^e + \alpha_H^e - 1}.$$

The original inequality is transformed to compare:

$$(r+\delta)\alpha_H^u(\alpha_L^u - 1)(\alpha_L^u + \alpha_H^u - 1)$$

and

$$(r+\delta+\lambda)\alpha_H^e(\alpha_L^e - 1)(\alpha_L^e + \alpha_H^e - 1).$$

Meanwhile, we have:

$$(r+\delta)\alpha_H^u(\alpha_L^u - 1)\alpha_L^u = (r+\delta)\alpha_H^u\frac{2(r+\delta+\lambda)}{\Delta_L^2}$$
$$> (r+\delta+\lambda)\alpha_H^e(\alpha_L^e - 1)\alpha_L^e = (r+\delta+\lambda)\alpha_H^e\frac{2(r+\delta)}{\Delta_L^2}$$

and

$$(r+\delta)\alpha_H^u(\alpha_L^u - 1)(\alpha_H^u - 1) = (r+\delta)(\alpha_L^u - 1)\frac{2(r+\delta+\lambda)}{\Delta_H^2}$$
$$> (r+\delta+\lambda)\alpha_H^e(\alpha_L^e - 1)(\alpha_H^e - 1) = (r+\delta+\lambda)(\alpha_L^e - 1)\frac{2(r+\delta)}{\Delta_H^2}$$

since $\alpha_y^u > \alpha_y^e$. This implies:

$$\alpha_H^u(\alpha_L^u - 1)(\alpha_L^e - \alpha_H^e) > \alpha_H^e(\alpha_L^e - 1)(\alpha_L^u - \alpha_H^u)$$

and therefore,

$$\frac{\alpha_H^u(\alpha_L^u - 1)\underline{p}}{\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)} < \frac{\alpha_H^e(\alpha_L^e - 1)\underline{p}}{\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)}.$$

Notice from the above proof, 3 holds only when Δ_ξ is small and will not hold as Δ_ξ becomes sufficiently large.

Finally, we can conclude that $\tilde{V}_H^u < \tilde{V}_H^e$ when $\xi_H \simeq \xi_L$, and as a result $\underline{p}^e < \underline{p}^u$. ■

No-deviation condition for the non-Bayesian learning example

Under the non-Bayesian learning case, suppose it is optimal for a p worker to choose firm y, the value function for this worker should be such that (from Hamilton-Jacobi-Bellman equation):

$$(r+\delta)W_y(p) = w_y(p) + \lambda_y p W_y'(p).$$

Suppose there is a cutoff $\underline{p}$ such that workers with $p > \underline{p}$ are matched with H firms and vice versa.

Then the absence of deviation implies that a $p > \underline{p}$ worker has no incentive to deviate, rematch with a L firm and switch back after dt time:

$$W_H(p) > \tilde{W}_L(p) = \mathbb{E}\left\{\int_t^{t+dt} e^{-(r+\delta)(s-t)} w_L(p_s)ds + e^{-(r+\delta)dt} W(p_{t+dt})\right\}.$$

For dt sufficiently small, p_{t+dt} is still close to p such that it is optimal for a p_{t+dt} worker to choose firm H as well. It is immediate to see that:

$$\lim_{dt\to 0} \frac{W_H(p) - \tilde{W}_L(p)}{dt} = w_H(p) - w_L(p) + (\lambda_H - \lambda_L)pW_H'(p),$$

and hence no deviation implies that:

$$w_H(p) - w_L(p) + (\lambda_H - \lambda_L)pW_H'(p) > 0$$

for all $p > \underline{p}$. Let $p \to \underline{p}+$ and we have by applying the value matching condition:

$$w_H(\underline{p}+) - w_L(\underline{p}-) + (\lambda_H - \lambda_L)\underline{p}W_H'(\underline{p}+) = \lambda_L \underline{p}(W_L'(\underline{p}-) - W_H'(\underline{p}+)) \geq 0$$

or equivalently $W_L'(\underline{p}-) \geq W_H'(\underline{p}+)$. On the other hand, a $p < \underline{p}$ worker also has no incentive to deviate, rematch with a H firm and switch back after dt time. Similarly, no deviation implies that:

$$w_L(p) - w_H(p) + (\lambda_L - \lambda_H)pW_L'(p) > 0$$

for all $p < \underline{p}$. Let $p \to \underline{p}-$ and it could be shown:

$$w_L(\underline{p}-) - w_H(\underline{p}+) + (\lambda_L - \lambda_H)\underline{p}W_L'(\underline{p}-) = \lambda_H \underline{p}(W_H'(\underline{p}+) - W_L'(\underline{p}-)) \geq 0$$

or equivalently $W_H'(\underline{p}+) \geq W_L'(\underline{p}-)$. Therefore, at $\underline{p}$, it must be the case that $W_H'(\underline{p}) = W_L'(\underline{p})$ and no-deviation condition coincides with the smooth-pasting condition.

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