Competing Teams[*]

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Abstract

In many economic applications of matching, the teams that form compete later in market structures with strategic interactions or with knowledge spillovers. For instance, pharmaceutical companies assemble R&D teams to develop new drugs and compete for patents; similarly, oligopolistic firms hire their skilled workforce in a competitive labor market and then compete in product markets. Such post-match competition introduces externalities at the matching stage: a team's payoff depends not only on their members' attributes but also on those of other matched teams. This paper develops a large market model of matching with externalities, in which first teams form, and then they compete. We analyze the sorting patterns that ensue under competitive equilibrium as well as their efficiency properties. Our main results show that insights substantially differ from those of the standard model without externalities (Becker (1973)): there can be multiple competitive equilibria with different sorting patterns; both optimal and competitive equilibrium matching can involve randomization; and competitive equilibrium can be inefficient with a matching that can drastically deviate from the optimal one. We also shed light on the economic relevance of our matching model with externalities. We analyze two economic applications that illustrate how our model can rationalize the trend in within- and between-firm inequality, and also the evolution of markups of sectors where firms have market power.

Keywords. Matching with Externalities. Sorting. Strategic Interaction. Knowledge Spillovers. Wage Inequality. Market Power.

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1 Introduction

The success of a firm, a team, or a partnership often depends on how it fares relative to its competitors. For instance, in a patent race between pharmaceutical companies, where the winner takes all the benefit, a competing team affects the other teams' performance negatively. If the competitor discovers the blockbuster drug, the rest gets nothing. Competing teams may also affect the outcome positively whenever the performance of the competitor generates knowledge spillovers that boost the own firm's performance. The success in the discovery of the structure of DNA for which James Watson and Francis Crick at Cambridge gained credit would not have been possible without the spillovers from the team led by Rosalind Franklin and Maurice Wilkins at Kings College.[1]

It has long been recognized that externalities between firms have important implications for the provision of effort and therefore for the efficient allocation of resources.[2] In this paper, we build further on the insights from this literature, but focus instead on the effects of these externalities on *team composition*, an equally crucial determinant of performance. If a pharmaceutical firm gets the best scientists, they are more likely to make new discoveries and hence the firm is more likely to obtain a patent. The fact that firms spend so much time and resources carefully choosing their skilled workers – often poaching them away from competitors – is direct evidence that team composition is an important strategic tool in the competition with other firms. Consulting firms, banks and law firms try to hire the best young talent; university departments constantly attempt to attract the most productive academics; and research divisions in technology companies lure the best engineers.

This paper sheds light on robust qualitative features of the effects of externalities on matching. We develop a tractable model with a large number of heterogenous agents, just like the standard matching model (Becker (1973)), but now in the presence of externalities. In a competitive labor market, a continuum of firms hire heterogeneously skilled workers to form teams. Once the teams are formed, those firms then compete in an output market with externalities. Since the performance of a team or firm in this downstream competition depends on the composition of the other teams, this interdependence feeds back into the labor market matching problem. Indeed, the hiring decision now takes into account not only the complementarities between the workers hired, but also the feedback effects coming from the composition of other teams that form. We allow for different ways to structure competition among firms after hiring their teams. Indeed, we consider economy-wide externalities, where each firm exerts an external effect on all other firms – as in models with knowledge spillovers. And we also consider pairwise externalities where firms compete with one other firm only, and where the pairs of teams that compete against each other are set either at random – as in settings where the competitor is initially unknown – or deterministically ex ante – as in oligopolistic markets where each firm knows the identity of its competitors. With this variety of settings, we encompass many potential economic applications of our framework.

We show that externalities have profound implications and derive insights that have no counterpart in the standard model. First, we show that despite the presence of complementarities both the optimal and the equilibrium matching may be stochastic – for example, a fraction of the population matches in a positively assortative way (PAM) and the rest negatively assortative (NAM). Second, we show that there can be multiple equilibria with different sorting patterns. Third, we provide interpretable conditions under which the optimal matching

[1]Watson and Crick received the Nobel Prize for the celebrated model of DNA as a double helix, fruit of their brilliant intuition and the meeting of different but complementary minds. Yet, Watson was motivated by a talk by Wilkins on the molecular structure of DNA in 1951, and together with Crick, before coming up with their model, they had access to Franklin's X-ray photographs that documented the helical structure. For an account of this story see "The Discovery of the Molecular Structure of DNA – The Double Helix," Official Website of the Nobel Prizes.

[2]Most notably, the literature on tournaments, contests, and patent races has extensively focused on important aspects such as long-term, repeated interaction (Che and Yoo (2001)) and the optimal provision of effort (Che and Gale (2003)).

differs, sometimes in a drastic way, from the equilibrium one. Intuitively, complementarities are no longer the sole determinant of sorting, as they interact with the externalities to pin down who matches with whom.

Progress in characterizing the equilibrium allocation has been hindered by problems of existence of competitive equilibrium as well as by the combinatorial complexity of the matching problem that ensues.[3] In our tractable setup, we sidestep some of these problems and provide conditions under which we can construct a competitive equilibrium, and show that there could be multiple ones including cases entailing stochastic matching. All this in a relatively elementary fashion using standard tools.

These insights are robust – they obtain under binary or continuum of characteristics and under relatively weak assumptions – and cannot arise without externalities. More importantly, they have profound implications for the optimal labor force composition in firms. One striking feature is that when a competitive equilibrium is inefficient, the optimal matching and thus team composition can look drastically different from the equilibrium matching. For example, the outlook of the market is very different if the planner's matching is NAM (diversity within teams, homogeneity between teams) while the competitive equilibrium matching is PAM (homogeneity within teams, diversity between teams). These different outcomes can arise even for small changes in the technology, for example, when the differential externality under PAM and NAM becomes slightly stronger.[4]

To assess the economic relevance of our model, we tackle some economic applications that are of interest in Macro/Labor Economics and Industrial Organization. We claim that our insights help us understand important economic phenomena, and we derive a couple of findings that are in line with recent empirical evidence that has received attention in the literature. In our main application to the macroeconomics of knowledge spillover, we highlight a mechanism that can account for the evolution of wage inequality. Recent evidence establishes that nearly all of the increase in inequality is driven by between-firm inequality and not by within-firm inequality.[5] At the same time, the literature shows that knowledge spillovers are an important determinant of the firm size distribution.[6] We link the wage inequality literature with the firm inequality literature by adding a matching stage where workers sort into firms. Knowledge spillovers drive between-firm inequality and worker complementarities drive within-firm inequality, though both are obviously jointly determined in equilibrium.

We find that, in the unique competitive equilibrium with stochastic matching, increased complementarity between workers leads to more positive sorting within firms. This implies that the composition of workers within firms is more alike while the composition between firms looks more distinct. We can then provide conditions under which between-firm wage inequality can increase drastically while within-firm wage inequality barely changes. Moreover, due to the externalities, equilibrium is inefficient. The planner favors an allocation with less positive sorting and hence a more mixed composition of workers within firms, which results in lower wage inequality. Observe that this smooth evolution of wage inequality cannot arise in a standard matching model because it requires a mixture between PAM and NAM allocations. Indeed, this is determined by the presence of externalities.

We also investigate the relationship between market power and the composition of skills in the firm. Recent

[3]See the seminal paper by Koopmans and Beckmann (1957). Their "quadratic assignment problem," which need not have a competitive equilibrium, has generated a huge literature in Operations Research and Combinatorial Optimization. Despite its apparent simplicity, it is considered to be one of the most difficult NP-Hard problems (in the sense that unless one proves that P=NP, one cannot even obtain f-approximation algorithms for any constant f (see the recent survey Loiola, de Abreu, Bonaventura-Netto, Hahn, and Querido (2007)). This speaks to the overall difficulty of the topic of matching with externalities.

[4]The discontinuity of the equilibrium allocation in the properties of the technology is of course well-known from the assignment game without externalities (Becker (1973)): as the cross-partial derivative of the match surplus switches from positive to negative, the allocation discontinuously jumps from PAM to NAM. The novelty here is that it is driven by the presence of externalities.

[5]See Card, Heining, and Kline (2013) for Germany, Song, Price, Guvenen, Bloom, and von Wachter (2015) and Barth, Bryson, Davis, and Freeman (2014) for the US, Benguria (2015) for Brazil, and Vlachos, Lindqvist, and Hakanson (2015) for Sweden.

[6]See Lucas and Moll (2014), Perla and Tonetti (2014), König, Lorenz, and Zilibotti (2016) and Eeckhout and Jovanovic (2002), amongst others.

evidence shows a sharp rise in market power in US firms. We apply the model to an oligopolistic output market setting where competing firms hire their workforce on the economy wide labor market. We show that the degree of complementarity between workers affects the equilibrium distribution of markups in the output market. More complementarity leads to more dispersion of markups as well as higher markups. This establishes that changes in the technology of labor productivity affect the composition of skills across firms and, equally importantly, that labor market composition affects the extent of the inefficiency due to market power. We believe this is novel.

Finally, we provide a policy discussion on interventions that can alleviate the inefficiency due to externalities. To make it more concrete, we use a stylized version of our model as a metaphor for matching in professional sports teams, and we shed light on the impact of three distinct policy measures that are actually used: taxes and subsidies, salary caps and a 'rookie draft.' In most European professional sports leagues such as soccer or basketball, there is little intervention in the team composition. The result is that in most leagues, a few teams get all the top players and the lion share of audiovisual and commercial revenues. Many of these teams are at the top of the competitions year after year. In contrast, most sports leagues in the US take radical policy measures to ensure what is known as balanced competition: there is active intervention to ensure that the difference between teams is limited, and all teams are composed of both superstars as well as more modest players.[7]

As mentioned, this paper contributes to the analysis of matching with externalities, an important topic that has received scant attention in the matching literature, despite the pervasiveness of externalities in economic applications. Its importance was recognized in the seminal matching paper by Koopmans and Beckmann (1957), who analyze a variation of their matching problem between locations and plants in the presence of transportation costs between locations, which generate externalities in the optimal assignment. They show that in their model a competitive equilibrium does not exist, and left the problem open for future research. Sasaki and Toda (1996) provided a suitable concept of stability in matching with externalities, and analyzed its implications for the marriage model and assignment games. A recent paper by Pycia and Yenmez (2017) generalizes the analysis of stable matchings to many-to-many and many-to-one matching problems, and show several properties of core allocations, including some comparative statics. What distinguishes our paper from the rest of the literature is our focus on large markets and equilibrium and optimal sorting patterns. We study both the optimal matching problem from a planner's perspective and a decentralized version using a standard notion of market equilibrium with externalities. Our parsimonious model and the equilibrium notion affords a fairly complete solution to the problem in most instances and an explicit comparison between the equilibrium allocation and the planner's solution. Moreover, we shed light on the intuition underlying the inefficiencies that we derive, and flesh out in detail some economic applications of our framework.

The rest of the paper proceeds as follows. In the next section we start with a simple example. Section 3 describes the model. Section 4 contains the main results regarding sorting patterns and the inefficiency that the externalities can generate. In Section 5 we develop our main economic application, which is the role of sorting in the presence of economy-wide knowledge spillovers. We also analyze market power and the effects of alternative policies aimed at correcting the inefficiency. Section 6 concludes. The Appendix contains all the proofs as well as some additional extensions omitted from the text.

[7] While those policies do not completely eradicate the existence of leading teams, the record of the winning teams provides evidence that those policies are designed to correct a market outcome that tends towards PAM in the direction of NAM, which is essentially diametrically opposed. It is worth noting that policies agreed upon by team owners might also be aimed at extracting rents from players, but it is not clear why this would lead to a change in the sorting pattern.

2 An Example

To illustrate some of the main results of the paper, consider the following simple two-stage matching problem.[8] There is a unit measure of agents, half of them with a high productive attribute H and half with a low one L. In the first stage, agents match pairwise, and thus form teams. They have linear utility and are free to make transfers among each other. In the second stage, the formed pairs randomly match with each other and 'compete.' The payoff structure from this (reduced-form) competition is as follows: if two teams with the same composition compete, then each obtains a payoff of 1, while if they have different composition each obtains 0. Finally, if an agent is unmatched, then his payoff is normalized to zero. We will examine the competitive equilibria of the first stage and the sorting patterns that can emerge.

We first show that there is a PAM equilibrium (only HH and LL teams form) with supporting wages 1/4 for both H and L. To see this, note that if H conjectures PAM, then in the second stage his team will match with equal probability with a team HH and LL. So if they match with (hire) another H in the first stage they obtain $(1/2) \times 1 + (1/2) \times 0 - 1/4 = 1/4$, while if they match with (hire) an L, then they obtain $0 - 1/4 = -1/4$. Hence, they strictly prefer to form a team with another H. A similar analysis applies to any agent with characteristic L, who strictly prefers to match with another L. Thus, PAM along with wages equal to 1/4 is a competitive equilibrium. The aggregate output is $(1/2) \times (1/2)$ (one half measure of teams, each with expected output 1/2).

There is also a NAM equilibrium (only HL teams form) with supporting wages 1/2 for both H and L. To see this, note that both types of agents prefer to hire an agent of the opposite characteristic if they conjecture NAM. This is because in this case they match with probability one with a mixed team HL, so they obtain 1 if they also form a mixed team, and 0 otherwise. The aggregate output of this equilibrium is $(1/2) \times 1$.

Finally, there is an equilibrium where agents randomize in their choice of partners and wages are given by 1/3 for each type of agent. Suppose that agents conjecture that matching is PAM with probability $\alpha \in (0, 1)$ and NAM with $1 - \alpha$ (equivalently, a fraction α of each type matches in a PAM way and $1 - \alpha$ in a NAM way). Consider an agent with H: if he matches with another H, then the teams' expected output is $\alpha/2$ (with probability α matching is PAM and the formed HH team is matched with another HH team with probability 1/2, while in any other event the team's output is 0) and thus each team member obtains $\alpha/4$; similarly, if he matches with an L, the resulting team's expected output is $(1 - \alpha) \times 1$. A similar analysis holds for L. From the incentive constraints of H and L, a necessary condition for an equilibrium where matching is stochastic and given by α is that each type of agent is indifferent between hiring someone of the same characteristic or the opposite one. Formally, for each type the incentive constraint is $(\alpha/2) - (1/3) = (1 - \alpha) - (1/3)$. Adding the incentive constraints of H and L we obtain the condition $(\alpha/2) + (\alpha/2) = 2(1 - \alpha)$, which holds if and only if $\alpha = 2/3$, completing the construction of a competitive equilibrium in which matching is stochastic. Aggregate expected output is $(1/2) \times (1/3)$ (one half measure of teams, each with expected output 1/3).

The planner in this setting – assuming she can intervene in the first stage but not in the second – can choose any way to match the agents pairwise, which can be summarized as the choice of the fraction $\alpha^P \in [0, 1]$ of agents that she matches as PAM, and the rest as NAM. Thus, her problem is $\max_{\alpha^P \in [0,1]}(1/2) \times ((\alpha^P/2)^2 + (\alpha^P/2)^2 + (1 - \alpha^P)^2)$. This is simply the measure of teams 1/2 multiplied by the sum of expected payoffs of the teams (for example, there are $\alpha^P/2$ teams HH and each obtains, under α^P, expected output $\alpha^P/2$, and similarly with the other terms). It is easy to check that the maximum is achieved at $\alpha^P = 0$, so NAM is efficient.

If instead we assume that a team generates an expected output of 1 if matched with a team of a different

[8] We are grateful to a referee for suggesting the basic example with random matching of this section.

composition, and 0 otherwise, then following the same steps as above one can show that there is a unique equilibrium that is efficient and entails stochastic matching with $\alpha = 2/3$.

So far we have assumed that competing teams are randomly assigned pairwise. Another alternative would be to add an exogenous initial stage in which half of the agents of each type are assigned pairwise, with each pair being future competitors. Then each agent in the competing teams matches with (hires) a partner, thus completing the formation of the two teams that will face off downstream.

To verify that similar results as in the random assignment case obtain, assume that before the interaction starts, half of the population is assigned pairwise in a PAM way. That is, half of the H's are matched together and similarly for half of the L's. (Assuming NAM instead yields similar results and is thus omitted.) Each of these pairs are competitors. Consider now the matching stage, where each of these competitors hires a partner and then compete. If two competing teams have the same composition, then each obtains 1, and 0 otherwise.

Proceeding as before, we can show that there is a competitive equilibrium with PAM and also one with NAM, each supported by wages 1/2 for both H and L. To see this, consider an agent with H who conjectures PAM, and so he assumes that his team will compete against an HH team (recall that he was initially assigned to a competing H). By hiring another H he obtains $1 - (1/2) = 1/2$, while hiring an L yields $0 - (1/2) = -1/2$. The same applies to an agent with L. Hence, there is an equilibrium with PAM and aggregate output equal to 1/2, and a similar logic yields one with NAM and aggregate output equal to 1/2. Finally, there is a stochastic equilibrium matching with $\alpha = 1/2$ and wages given by 1/4, with aggregate output of 1/4. To see this, notice that an agent with H obtains $\alpha - (1/4)$ when hiring another H (since with probability α the initial member H of the competing teams hires another H), and $1 - \alpha - (1/4)$ when hiring an agent with L, and similarly for the incentive constraint an agent with attribute L. Hence, adding the constraints yields $\alpha = 1 - \alpha$, and thus $\alpha = 1/2$ is the equilibrium stochastic matching. Regarding the planner, she solves $\max_{\alpha^P}(1/2) \times ((\alpha^P)^2 + (1 - \alpha^P)^2)$. To see where the objective function comes from, notice that under matching α^P there are $\alpha^P/4$ teams HH, $\alpha^P/4$ teams LL, and $(1 - \alpha^P)/2$ teams HL. Now, a team HH, who already has an H in its competing team, competes with HH with probability α and with HL with probability $1 - \alpha$: thus, its expected payoff is α^P, and similarly for a team LL. Finally, a team HL or LH competes with an indentical team with probability $1 - \alpha^P$ and with a different one with α^P, so its expected payoff is $1 - \alpha^P$. Thus, the planner's objective function is $(\alpha^P/4) \times \alpha^P + (\alpha^P/4) \times \alpha^P + ((1 - \alpha^P)/4) \times (1 - \alpha^P) = (1/2) \times ((\alpha^P)^2 + (1 - \alpha^P)^2)$, and the optimal matchings are $\alpha^P = 1$ and $\alpha^P = 0$. Hence, the competitive equilibria with PAM and NAM are both efficient while the stochastic matching competitive equilibrium is inefficient.

These examples reveal that competition in the second stage turns the matching problem in the first stage into one with externalities: the composition of teams in the market affects the payoff of any given team, as each team competes against another one in the second stage. As a result, the first stage can have *multiple* equilibria with drastically different sorting patterns, including one in which matching is *stochastic*.[9] Moreover, equilibrium can be *inefficient*. None of these results hold in the standard case (as in Becker (1973)) without externalities.

[9] In this example, both PAM and NAM can emerge in equilibrium because the externality changes the team's payoff function from supermodular to submodular. This effect cannot arise in the standard model, where a pair's payoff is determined only by the members' characteristics and the production complementarities.

3 The Model

3.1 Overview

We consider an economy with a large number of heterogeneous agents who match pairwise. For instance, this could be a labor market where skilled workers form teams or partnerships. Equivalently, one could envision a large number of identical firms that hire pairs of heterogeneous workers and make zero profits. Absent externalities, if agents can perfectly transfer utility, then this would be a standard matching problem (e.g., as in Becker (1973)).

Implicit in our model, however, is another stage after matching in which the formed teams compete. Continuing with the labor market example, firms, after they hire their workers, compete in an output market.

Competition among teams can take a different form depending on the economic application under consideration. Indeed, each team could compete with exactly one other team, whose identity could be known before the second stage, or it could be drawn at random from the pool of teams. Alternatively, competition might take place among all of them. Our model will encompass all these alternatives in a reduced form by assuming that the payoff function of each team depends not only on its composition but also on the composition of other teams.

The crucial feature that our model captures is that competition in the second stage feeds back into the formation of teams in the first stage. This turns the team formation problem into a matching problem with externalities, which creates a wedge between equilibrium and optimal matchings.

3.2 The General Framework

There is a unit-measure continuum of agents. Each agent is indexed by a characteristic $x \in [0,1]$, whose distribution in the population is given by a cdf $F : \mathbb{R} \to [0,1]$. The cdf F has either a finite support (discrete case), in which case it is an increasing step function, or it has support $[0,1]$ (continuous case), in which case we assume that it is strictly increasing and continuous on $[0,1]$. Following the standard assumption in matching models that focus on sorting (e.g., see Chade, Eeckhout, and Smith (2017)), agents match pairwise and thus form teams of size two. A (deterministic) matching is thus a (measurable) one-to-one function $\mu : [0,1] \to [0,1]$ that is measure-preserving (matches measurable sets of $[0,1]$ of equal F-measure).[10] The most important instances for our purposes are the matching μ that is increasing (PAM), denoted by μ_+, and the decreasing one (NAM), denoted by μ_-. In the continuous case μ_+ and μ_- are given by $\mu_+(x) = x$ and $\mu_-(x) = F^{-1}(1 - F(x))$ for all x.

Let $\mathbb{M}$ be the set of matchings μ. Match payoff or output is given by a function $\mathcal{V} : [0,1]^2 \times \mathbb{M} \to \mathbb{R}_+$ such that, if an agent with characteristic x matches with one with characteristic x' and matching is given by μ, then the match payoff of team (x, x') is $\mathcal{V}(x, x'|\mu)$. The function $\mathcal{V}(\cdot, \cdot|\mu)$ is twice continuously differentiable for each $\mu \in \mathbb{M}$. Agents value match output and their preferences are quasilinear in money, so utility is perfectly transferable among agents.[11] For simplicity, we assume that unmatched agents produce zero and we normalize their payoff to zero as well. The dependence of $\mathcal{V}$ on μ captures the effects of a second stage where the formed teams compete. The precise functional form of $\mathcal{V}$ will vary across applications and will depend on the precise nature of competition in the second stage, which we will describe in more detail in the next subsection.

We focus on the competitive equilibria of this matching problem with externalities. Our definition of competitive equilibrium is a fairly standard "textbook" one (see for example Mas-Colell, Whinston, and Green (1995), or Chapter 6 in Arrow and Hahn (1971)). When choosing the composition of teams, agents take as given both

[10] We endow $[0,1]$ with its Borel σ-field and measurable in this paper should be understood as Borel measurable.

[11] We leave for future research the more complex analysis when utility is imperfectly transferable, which would call for an extension of the general model in Legros and Newman (2007) to the case with externalities.

market wages (i.e., they are price takers) as well as the matching (which is the source of the externalities). This implies that each firm behaves as if its own choice does not affect the candidate equilibrium allocation, a conjecture that is consistent with our large economy environment. More precisely, a competitive equilibrium of the matching problem consists of a wage function $w : [0,1] \to \mathbb{R}$ and a matching function $\mu : [0,1] \to [0,1]$ such that, for all x, $\mu(x) \in \text{argmax}_{x'} \mathcal{V}(x, x'|\mu) - w(x')$ (that is, each agent with characteristic x finds it optimal to match with partner $\mu(x)$), agents obtain nonnegative payoffs, and the market clears.

As mentioned, there are two interpretations of a competitive equilibrium in this model. One is that any given agent hires a partner of a certain characteristic at the market wage for that characteristic. Agents must be indifferent between hiring a partner and keeping the output net of the partner's wage, or being hired and receiving the wage corresponding to his characteristic. The alternative interpretation is that there is a large number of identical firms or entities, and each of them makes wage offers to two agents, who will then split the entire surplus due to a zero profit condition. In both cases, identical agents get paid identical wages (equal treatment).

The planner's objective is to find the matching that maximizes the sum (integral) of the match outputs across all teams, given that utility is transferable. Since the planner weakly prefers to match everyone, her problem is to find a $\mu \in \mathbb{M}$ that maximizes $\int_0^1 \mathcal{V}(x, \mu(x)|\mu) dF(x)$. We denote the maximizer μ^P. If $\mathcal{V}(x, x'|\mu) \equiv \hat{\mathcal{V}}(x, x')$ for all (x, x'), then this problem has a well-known solution in the following cases: if $\hat{\mathcal{V}}$ supermodular in (x, x'), then the optimal matching is PAM, and if it is submodular then the optimal matching is NAM.

If we allow matching to be stochastic, then a matching is now a measure π on (the Borel σ-field of) $[0,1]^2$ such that its marginals coincide with F, that is, $\pi(E \times [0,1]) = \int_E dF$ and $\pi([0,1] \times E) = \int_E dF$ for each Borel set $E \subset [0,1]$. Denote by $\mathcal{M}$ the set of such measures, and with some abuse of notation, let $\mathcal{V}(x, x'|\cdot) : \mathcal{M} \to \mathbb{R}_+$ for each $(x, x') \in [0,1]^2$. A competitive equilibrium consists of a wage function w and a matching $\pi \in \mathcal{M}$ such that each agent with characteristic x is indifferent among all the x' in the support of partners with whom x can match under π. That is $\mathcal{V}(x, x'|\pi) - w(x')$ is constant and nonnegative for all such x'. In turn, the planner's problem is to choose $\pi \in \mathcal{M}$ that maximizes $\int_{[0,1]^2} \mathcal{V}(x, x'|\pi) d\pi(x, x')$, and the maximizer is denoted by π^P.

3.3 Competing Teams' Assignment

We will focus on cases where competition in the second stage takes places in one of the following forms:[12]

1) Pairwise Competing Teams with Local Spillovers: The interaction in the second stage takes place between pairs of teams. We will explore the following instances of this case:

1.*i*) Ex-Post Random Assignment of Teams: After teams are formed, they are randomly matched pairwise. Since ex-ante all teams are potential competitors, the composition of all teams are payoff relevant, but ex post each team will compete with only one team. For any μ, let $G(\mu) = \{(s, s')|s' = \mu(s), s \in [0,1]\}$ be its graph, and let $V : [0,1]^2 \times G(\mu) \to \mathbb{R}_+$ be given by $V(x, x'|s, \mu(s))$, which is the output that a team with (x, x') obtains if it competes against a team with $(s, \mu(s))$. We will assume that the (measurable) function V is symmetric in its first and second argument, and also in its third and fourth argument.[13] Then for all $(x, x') \in [0,1]^2$

$$\mathcal{V}(x, x'|\mu) \equiv \int_0^1 V(x, x'|s, \mu(s)) dF(s). \tag{1}$$

[12] To avoid confusion, we use the term 'assignment' to denote how teams are matched in the second stage, and reserve the term 'matching' to denote how team members match in the first stage.

[13] Symmetry is standard in one-sided matching problems without externalities and rules out task-specific productivity, e.g., where the same worker is more productive if assigned to task 1 (say, manager) than to task 2 (say, mechanic). See Kremer and Maskin (1996) for the analysis of matching without the symmetry assumption.

The assumptions on V imply that $\mathcal{V}(\cdot,\cdot|\mu)$ is symmetric in (x,x'). An intuitive interpretation of the functional form of $\mathcal{V}$ given by (1) is that each team competes against teams of a given composition a fraction of time represented by their presence in the overall population. An example is sports competition, where each team plays every other team, and hence the sum of the outcomes of all competitors is equal to the expected value of competing with a random team multiplied by a constant (the measure of teams). For another example, consider firms that, after hiring their teams of skilled workers in first stage, in the second stage bid for contracts without knowing ex ante the identity of their competitor.

1.*ii*) Ex-Ante Deterministic Assignment of Teams: Teams compete pairwise in the second stage and each team knows in advance its opponent. So if (x,x') knows that, given a matching μ, it will compete against a team with composition $(s,\mu(s))$, then the only relevant part of $G(\mu)$ for each team is the point (pair) that represents the competitor's composition. The way we model this is by positing an exogenous initial assignment $\eta \in \mathbb{M}$ that takes place before the first stage, in which half of the population with composition given by F is assigned pairwise.[14] Each assigned pair consists of competitors that will interact in the second stage, and each member of this pair matches in the first stage with a partner from the one-half measure of agents remaining in the population. The resulting teams then compete in the second stage. In the example above, $s=\eta(x)$ and thus agents with x and s are competitors; under the matching μ, x conjectures it will compete with $(s,\mu(s))$ and chooses a partner x'. Then for all $(x,x') \in [0,1]^2$

$$\mathcal{V}(x,x'|\mu) \equiv V(x,x'|\eta(x),\mu(\eta(x))), \tag{2}$$

where with some abuse of notation we have omitted η from $\mathcal{V}$. Once again we will assume that the function V is symmetric in its first and second argument, and in its third and fourth argument. Note, however, that in this case we cannot assert that $\mathcal{V}(\cdot,\cdot|\mu)$ is symmetric in (x,x') since the right side is the composition of V and η, and hence $V(x,x'|\eta(x),\mu(\eta(x)))$ need not be equal to $V(x',x|\eta(x'),\mu(\eta(x')))$. That is, in this case an asymmetry ensues between those agents who are initially assigned to competing teams and those each of them hires.

A natural interpretation of this type of exogenous deterministic assignment is that there is a large number of local markets, each with two firms that hire workers and then compete downstream. This captures applications in which firms hire workers in competitive markets and then compete in oligopolistic product markets (for example, Coca Cola and Pepsi, or Visa and MasterCard), thus potentially subsuming a large number of economic applications in Industrial Organization. Whenever we deal with this case, we will assume that the assignment η is PAM, so $\eta(x)=x$, since the analysis for NAM is similar.

2) Competing Teams with Aggregate Spillovers: All the teams compete against each other in the second stage, and competition entails spillover effects that enter the payoff of each team as a common aggregate externality. Let $\xi : \mathbb{M} \to \mathbb{R}$ and let $S : [0,1]^2 \times \mathbb{R} \to \mathbb{R}_+$. Then for all $(x,x') \in [0,1]^2$

$$\mathcal{V}(x,x'|\mu) \equiv S(x,x',\xi(\mu)). \tag{3}$$

We will assume in this case that the function $S(\cdot,\cdot,\xi(\mu))$ is symmetric in (x,x'), and thus so is $\mathcal{V}(\cdot,\cdot|\mu)$. The functional form of the aggregate externality component ξ depends on the application at hand. An intuitive one is

[14] If F has a density f, then, for each x, $f(x)$ is divided by two, and all agents in one of the halves match pairwise. We could also model ex-ante random assignment of teams, in which ex ante all the agents are randomly matched with a competitor. The problem is that in this case two agents with the same characteristics can be matched with competitors with different ones, thus precluding the useful equal treatment property commonly exploited in matching models.

where the composition of the labor force of all firms determines the production of knowledge within firms, which generates spillover effects (positive externalities) on all the firms.[15]

In all these cases, we have described $\mathcal{V}$ under the assumption that matching is deterministic, but it is straightforward to modify it if instead is stochastic, and we will do so below on a needed basis. Also, notice that the assignment of competing teams is exogenously given and cannot be altered either by the firms or the planner. This restriction is what generates externalities that cannot be completely internalized, thereby leading to inefficiencies in the equilibrium composition of teams. As in *any* model with externalities, the problem trivializes if there are no 'missing markets.'[16] Intuitively, if teams could *choose* their competitor and set up the appropriate transfers among them, then the externality problem could be eliminated as the allocation of competing teams in the second stage would be efficient. It is hardly plausible that in a large market setup such as our model firms will internalize the externalities in this way, especially in the case of aggregate spillovers, where firms compete among all of them and not pairwise, and thus it is difficult to see how they could internalize the externalities via transfers. Moreover, there could also be a technological (such as differentiated products), legal (such as patent legislation) or geographical constraints that restrict firms to compete only in a specific sector or location, as it would be too costly for them to, for example, switch from consumer marketing and retail to cement production. For completeness, we discuss this issue in Appendix A.5, and illustrate how *endogenous assignment* of competing teams plus transfers between teams can restore equilibrium efficiency.

4 Main Results

4.1 Binary Characteristics

We start with a benchmark case in which agents' characteristics are binary: $x \in \{\underline{x}, \overline{x}\}$, where $0 \leq \underline{x} < \overline{x} \leq 1$ and where exactly half of the agents are of type $\overline{x}$ and half are of type $\underline{x}$.[17] This case is rich enough to present, in an elementary and intuitive fashion, the main insights that emerge in our matching setting with externalities.

There are three possible team configurations: a team can consist of two agents with type $\overline{x}$, or with two $\underline{x}$ members, or it can be a mixed team with one member of each characteristic. In cases 1.i) and 2) above, mixed teams $(\underline{x}, \overline{x})$ and $(\overline{x}, \underline{x})$ are treated symmetrically. In the case of deterministic assignment of competing teams (case 1.ii) above), however, we will need to distinguish between a team with $(\underline{x}, \overline{x})$ and one with $(\overline{x}, \underline{x})$. For example, when the η assignment is PAM, we have that in the case $(\underline{x}, \overline{x})$, before matching takes place to form teams, each agent with $\underline{x}$ is assigned to another $\underline{x}$ who will be part of their competing team. In turn, in the case $(\overline{x}, \underline{x})$ each agent with $\overline{x}$ is assigned to an agent with $\overline{x}$.

We will proceed in a general way in this section and allow for a stochastic matching π, which in this setting is characterized by a number $0 \leq \alpha \leq 1$. This number represents the fraction of the population that matches à la PAM, with $1-\alpha$ matching in a NAM way. Clearly, the corner $\alpha = 1$ represents μ_+, and $\alpha = 0$ represents μ_-. In this way, α spans all the possible matchings in this economy. Also for notational economy, in this section we will set $\mathcal{V}(x, x'|\alpha) \equiv \mathcal{V}(x, x'|\pi)$, $\mathcal{V}(x, x'|1) \equiv \mathcal{V}(x, x'|\mu_+)$, and $\mathcal{V}(x, x'|0) \equiv \mathcal{V}(x, x'|\mu_-)$.

[15] The function ξ could also vary depending on the composition of the team at hand. We will see an instance of this with stochastic matching in Section 5.1.

[16] Indeed, in line with standard Coasian arguments, if teams can set up contracts among them, then they will find a way to achieve efficiency. In all of our applications this does not seem likely to happen.

[17] The equal number of agents with high and low characteristics is made for convenience. Otherwise in the case of negative sorting one needs to keep track of the measure of agents with the characteristic present in more than half of the population who match among themselves once cross matches are exhausted. Since this extension does not lead to new insights, and since it will be relaxed in the case with a continuum of characteristics below, we focus here on the uniform case.

Competitive Equilibria. The binary case permits a complete description of the set of competitive equilibria and their sorting properties. To this end, we introduce the function $\Gamma : [0, 1] \to \mathbb{R}$ given by

$$\Gamma(\alpha) = \mathcal{V}(\overline{x}, \overline{x}|\alpha) + \mathcal{V}(\underline{x}, \underline{x}|\alpha) - \mathcal{V}(\underline{x}, \overline{x}|\alpha) - \mathcal{V}(\overline{x}, \underline{x}|\alpha), \tag{4}$$

which represents the gain/loss from rematching two teams as PAM instead of as NAM if the matching is α. Although $\mathcal{V}(x, x'|\cdot)$, and thus Γ, can be nonlinear functions of α, it turns out that it is linear in several cases of interest. Indeed, in the case of pairwise competing teams with random assignment of teams (case $1.i$) in the previous section), or with ex-ante deterministic assignment (case $1.ii$)), or when there are aggregate spillovers (case 2)) but the aggregate externality depends only on the characteristics of the pairs matched under μ and not on α (otherwise it can be nonlinear, as we illustrate below), this function can be written as follows:

$$\mathcal{V}(x, x'|\alpha) = \alpha\mathcal{V}(x, x'|1) + (1 - \alpha)\mathcal{V}(x, x'|0). \tag{5}$$

That is, $\mathcal{V}(x, x'|\alpha)$ is the expected match output for a team with composition X when it is assigned to a competing team in a PAM way with probability α (or aggregate spillovers are as in PAM, which occurs with probability α) and obtains $\mathcal{V}(x, x'|1)$, or in a NAM way with probability $1 - \alpha$ and obtains $\mathcal{V}(x, x'|0)$.[18]

Using (5) we can express equation (4) as

$$\Gamma(\alpha) = \alpha\Gamma(1) + (1 - \alpha)\Gamma(0). \tag{6}$$

In cases where Γ is not linear in α, we will assume that it is a continuous function.

A wage function w in this setup reduces to a pair of wages $\underline{w} \equiv w(\underline{x})$ and $\overline{w} \equiv w(\overline{x})$. A competitive equilibrium with PAM and wages $(\underline{w}, \overline{w})$ must satisfy the following incentive constraints:

$$\mathcal{V}(\overline{x}, \overline{x}|1) - \overline{w} \geq \mathcal{V}(\overline{x}, \underline{x}|1) - \underline{w} \tag{7}$$

$$\mathcal{V}(\underline{x}, \underline{x}|1) - \underline{w} \geq \mathcal{V}(\underline{x}, \overline{x}|1) - \overline{w}. \tag{8}$$

Adding both constraints reveals that a necessary condition for a PAM equilibrium is $\Gamma(1) \geq 0$, or, equivalently, that $\mathcal{V}(\cdot|1)$ is supermodular in (x, x').[19]

Similarly, a competitive equilibrium with NAM and wages $(\underline{w}, \overline{w})$ satisfies

$$\mathcal{V}(\overline{x}, \underline{x}|0) - \underline{w} \geq \mathcal{V}(\overline{x}, \overline{x}|0) - \overline{w} \tag{9}$$

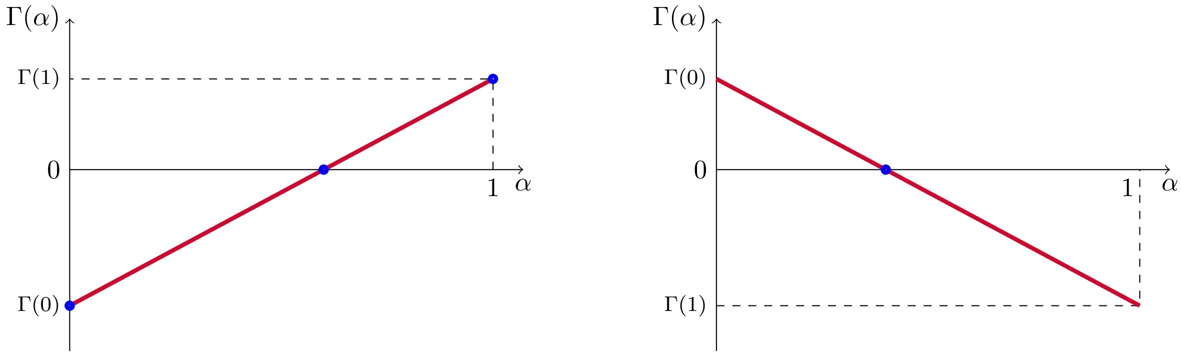
$$\mathcal{V}(\underline{x}, \overline{x}|0) - \overline{w} \geq \mathcal{V}(\underline{x}, \underline{x}|0) - \underline{w}, \tag{10}$$

and the corresponding necessary condition is $\Gamma(0) \leq 0$ or $\mathcal{V}(\cdot|0)$ submodular in (x, x').

Finally, in a competitive equilibrium with $\alpha \in (0, 1)$ and wages $(\underline{w}, \overline{w})$, agents must be indifferent between

[18] With ex-post random assignment of competing teams, under α the 1/2 measure of teams consists of $\alpha/4$ teams with two members with $\overline{x}$, $\alpha/4$ teams with two members with $\underline{x}$, and $(1-\alpha)/2$ mixed teams. Hence, the expected payoff of a team with composition (x, x') is, using equation (1),$\mathcal{V}(x, x'|\alpha) = (\alpha/2)V(x, x'|\underline{x}, \underline{x}) + (\alpha/2)V(x, x'|\overline{x}, \overline{x}) + (1-\alpha)V(x, x'|\underline{x}, \overline{x}) = \alpha((V(x, x'|\underline{x}, \underline{x}) + V(x, x'|\overline{x}, \overline{x}))/2) + (1-\alpha)V(x, x'|\underline{x}, \overline{x}) = \alpha\mathcal{V}(x, x'|1) + (1-\alpha)\mathcal{V}(x, x'|0)$. With ex-ante deterministic assignment η PAM it is immediate, since $\mathcal{V}(\overline{x}, \overline{x}|\alpha) = \alpha V(\overline{x}, \overline{x}|\overline{x}, \overline{x}) + (1-\alpha)V(\overline{x}, \overline{x}|\overline{x}, \underline{x}) = \alpha\mathcal{V}(\overline{x}, \overline{x}|1) + (1-\alpha)\mathcal{V}(\overline{x}, \overline{x}|0)$, and similarly for $(\underline{x}, \underline{x})$, $(\underline{x}, \overline{x})$, and $(\overline{x}, \underline{x})$. Finally, with aggregate spillovers that do not depend on α, we have $\mathcal{V}(x, x'|\alpha) = \alpha S(x, x', \xi(1)) + (1-\alpha)S(x, x', \xi(0)) = \alpha\mathcal{V}(x, x'|1) + (1-\alpha)\mathcal{V}(x, x'|0)$.

[19] Recall that a function f defined on a lattice of $\mathbb{R}^2$ is supermodular if given any two points (x, y) and (x', y'), $f(x \vee x', y \vee y') + f(x \wedge x', y \wedge y') \geq f(x, y) + f(x', y')$; it is submodular if the inequality is reversed; and it is modular if it holds with equality.



(a) Multiple Equilibria: $\alpha = 0$, interior, and 1. (b) Unique Interior Equilibrium.

Figure 1: Equilibrium Set. If Γ is all above zero then only PAM exists, and if below only NAM. There is an interior equilibrium if Γ changes sign.

hiring a low or a high type. That is, the following equations must hold:

$$\mathcal{V}\left(\overline{x},\overline{x}|\alpha\right) - \overline{w} = \mathcal{V}(\overline{x},\underline{x}|\alpha) - \underline{w} \tag{11}$$

$$\mathcal{V}\left(\underline{x},\underline{x}|\alpha\right) - \underline{w} = \mathcal{V}(\underline{x},\overline{x}|\alpha) - \overline{w}, \tag{12}$$

and the corresponding necessary condition is that $\Gamma(\alpha) = 0$, that is, $\mathcal{V}(\cdot,\cdot|\alpha)$ is modular in (x, x').

We now show that if Γ is continuous in α, then the necessary conditions derived for a competitive equilibrium with PAM, NAM, and stochastic matching are *sufficient* for equilibrium existence.

Proposition 1 *If Γ is continuous in α, then a competitive equilibrium exists. It exhibits PAM if $\Gamma(1) \geq 0$, NAM if $\Gamma(0) \leq 0$, and it is interior with $0 < \alpha < 1$ if $\Gamma(\alpha) = 0$.*

Figure 1 depicts the competitive equilibria for the case in which Γ is linear in α. Except for the nongeneric case in which $\Gamma(\alpha) = 0$ for all $\alpha \in [0, 1]$, there is either a unique competitive equilibrium or three of them. There are nonexistence examples in the matching literature in the presence of externalities (e.g., see the quadratic example in Koopmans and Beckmann (1957)). Interestingly, in our binary case a competitive equilibrium *always* exists, and there can be *multiple* ones including one with *stochastic matching* $\alpha \in (0, 1)$. Multiplicity (with different sorting patterns) and interiority cannot arise without externalities (as in Becker (1973)). In particular, multiplicity emerges when the externalities changes the match output from being a submodular to a supermodular function in team composition as α goes from 0 to 1. In turn, equilibrium is unique if the complementarity properties of the match output does not change with α, or if it is neither submodular at $\alpha = 0$ nor supermodular at $\alpha = 1$.

Example 1. As an illustration, let $\mathcal{V}(x, x'|\alpha) = \zeta + k(x, x')\ell(\alpha)$, where we assume that k positive and symmetric in (x, x'), the aggregate spillover effect is multiplicative, with ℓ continuous and strictly increasing in α, and ζ is positive and large enough to ensure that match payoff is nonnegative for all teams and for all values of α. Then $\Gamma(\alpha) = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}))\ell(\alpha)$. Assume that k is strictly supermodular in (x, x'), and thus $k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}) > 0$. Then if $\ell(1) > 0 > \ell(0)$, there are three equilibria: a PAM equilibrium $\alpha = 1$, a NAM equilibrium $\alpha = 0$, and an interior one. If $\ell(1) > 0$ and $\ell(0) > 0$, there is only a PAM equilibrium, and similarly for the other cases. And if ℓ is instead continuous and strictly decreasing in α with $\ell(0) > 0 > \ell(1)$, there there is a unique equilbrium that is interior. Note in passing that in this case if ℓ is a nonlinear function of α, then Γ is also a nonlinear function of α.

The Planner's Problem. The planner takes the structure of competition after matching as given, and her objective is to choose the matching $\alpha \in [0,1]$ that maximizes the total expected output of the economy. Formally

$$\max_{\alpha\in[0,1]} \frac{1}{2}\left(\frac{\alpha}{2}\mathcal{V}(\overline{x},\overline{x}|\alpha)+\frac{\alpha}{2}\mathcal{V}(\underline{x},\underline{x}|\alpha)+\frac{(1-\alpha)}{2}\mathcal{V}(\underline{x},\overline{x}|\alpha)+\frac{(1-\alpha)}{2}\mathcal{V}(\overline{x},\underline{x}|\alpha)\right).$$

To explain the objective function, note that there is a measure 1/2 of teams, of which a fracion $\alpha/2$ contains two agents with $\overline{x}$, and each of these teams obtains $\mathcal{V}(\overline{x},\overline{x}|\alpha)$; a fraction $\alpha/2$ are teams with two agents with $\underline{x}$ and each of these teams obtains $\mathcal{V}(\underline{x},\underline{x}|\alpha)$; a fraction $(1-\alpha)/2$ are of composition $(\underline{x},\overline{x})$ and each of these teams obtains $\mathcal{V}(\underline{x},\overline{x}|\alpha)$; and a fraction $(1-\alpha)/2$ are of composition $(\overline{x},\underline{x})$ and each of these teams obtains $\mathcal{V}(\overline{x},\underline{x}|\alpha)$.

We will focus on the case in which Γ is linear in α, as it contains most of the insights of the planner's problem, and then discuss the nonlinear case. Using (5) and the definition of Γ, the problem can be written as

$$\max_{\alpha\in[0,1]} \frac{1}{2}\left(\frac{\alpha^2}{2}A+\frac{\alpha}{2}B+C\right), \tag{13}$$

where $A \equiv \Gamma(1)-\Gamma(0)$, $B \equiv \Gamma(0)+(\mathcal{V}(\underline{x},\overline{x}|1)+\mathcal{V}(\overline{x},\underline{x}|1)-\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\overline{x},\underline{x}|0))$, and $C \equiv (\mathcal{V}(\underline{x},\overline{x}|0)+\mathcal{V}(\overline{x},\underline{x}|0))/2$. The following proposition characterizes the solution α^p to the planner's problem in terms of A, B, and C.

Proposition 2 *Assume that Γ is linear in α and either $A \neq 0$ or $B \neq 0$.*[20] *The optimal matching is as follows:*
(i) If $A \geq 0$, then the planner chooses $\alpha^p = 1$ if $A+B \geq 0$ and $\alpha^p = 0$ if $A+B < 0$;
(ii) If $A < 0$ and $B \leq 0$, then the planner chooses $\alpha^p = 0$;
(iii) If $A < 0$, $B > 0$, and $B+2A \geq 0$, then the planner chooses $\alpha^p = 1$;
(iv) If $A < 0$, $B > 0$, and $B+2A < 0$, then the planner chooses $\alpha^p = -B/2A \in (0,1)$.

The intuition is straightforward. The planner's objective function is a quadratic function of α, and thus it is either convex or concave: if convex, then the solution is at a corner (part (*i*)), while if concave, it is at a corner if the objective function is monotone (parts (*ii*) and (*iii*)), and it is interior otherwise.

When $\mathcal{V}$ is nonlinear in α, as it can be in the case of aggregate spillovers, there is no sweeping characterization of the planner's problem as in the linear case. But it is possible to provide simple sufficient conditions on $\mathcal{V}$ such that the optimal matching is interior, which is the surprising result in Proposition 2. To this end, rewrite the planner's objective as follows (recall that in this case $\mathcal{V}(\cdot,\cdot|\alpha)$ is symmetric in (x,x') and thus $\mathcal{V}(\underline{x},\overline{x}|\alpha) = \mathcal{V}(\overline{x},\underline{x}|\alpha)$):

$$\frac{1}{4}\left(\alpha\mathcal{V}(\overline{x},\overline{x}|\alpha)+\alpha\mathcal{V}(\underline{x},\underline{x}|\alpha)+2(1-\alpha)\mathcal{V}(\underline{x},\overline{x}|\alpha)\right)=\frac{1}{4}\left(\alpha\Gamma(\alpha)+2\mathcal{V}(\underline{x},\overline{x}|\alpha)\right). \tag{14}$$

Since $\mathcal{V}$ is nonnegative, so is the planner's objective function. As a result, the optimal matching is interior if the objective function is strictly increasing at $\alpha = 0$ and strictly decreasing at $\alpha = 1$. Formally, if suffices that

$$\Gamma(0)+2\mathcal{V}_\alpha(\underline{x},\overline{x}|0) > 0, \tag{15}$$

[20]This is just to avoid the nongeneric case in which all matchings are optimal.

which holds if each term is nonnegative and one of them positive; and

$$\Gamma(1) + (\mathcal{V}_\alpha(\overline{x},\overline{x}|1) + \mathcal{V}_\alpha(\underline{x},\underline{x}|1)) < 0, \tag{16}$$

which holds if each term is nonpositive and one of them negative.

Intuitively, the planner's solution trivializes when there are no externalities, for then $\mathcal{V}(x,x'|\cdot)$ is independent of α. As a result, the planner's objective function becomes linear, and the optimal matching is either PAM ($\alpha^P = 1$) or NAM ($\alpha^P = 0$), depending on whether $\mathcal{V}$ is supermodular or submodular in (x,x'). Unlike the standard case, when externalities are present an *interior* matching can be optimal. Moreover, in the standard case a marginal change in complementarities that changes $\mathcal{V}$ from being supermodular to submodular changes the matching from PAM to NAM, that is, from one corner to the other one. With externalities such a change does not have the same impact, for now the sorting pattern depends in a more complex way on the properties of $\mathcal{V}$, as the planner weighs not only the complementarities but also the externality effect.

Example 2. For a simple numerical illustration of an interior solution to the planner's problem, assume again that $\mathcal{V}(x,x'|\alpha) = \zeta + k(x,x')\ell(\alpha)$, and suppose that $k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}) < 0$. Consider first the linear case $\ell(\alpha) = a + b\alpha$, with $b > 0$, $a < 0$, and $a + b > 0$. Then $A = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}))b$, $B = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}))a + 2k(\underline{x},\overline{x})b$, and thus $B + 2A = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}))(a+b) + (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}))b$. Then it is easy to verify that $A < 0$, $B > 0$, and that $B + 2A < 0$ if and only if $2k(\underline{x},\overline{x}) > ((a+2b)/(a+b))(k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}))$, in which case the optimal matching is interior. Assume now that ℓ is nonlinear with $\ell' > 0$, $\ell(0) < 0$, and $\ell(1) > 0$. Then $\Gamma(0) = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}))\ell(0) > 0$ and $\mathcal{V}_\alpha(\underline{x},\overline{x}|0) = k(\underline{x},\overline{x})\ell'(0) > 0$, and thus (15) holds. Also, $\Gamma(1) = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}))\ell(1) < 0$ while $\mathcal{V}_\alpha(\overline{x},\overline{x}|1) + \mathcal{V}_\alpha(\underline{x},\underline{x}|1) = (k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}))\ell'(1) > 0$, and (16) holds if and only if $2k(\underline{x},\overline{x}) > (1 + (\ell'(1)/\ell(1)))(k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}))$, which generalizes the condition just derived for the linear case, and can be pinned down from primitives.

Comparing Equilibrium and Efficient Matching. To understand the efficiency properties of equilibria, it is instructive to focus on the planner's marginal incentives to increase α, that is, the first derivative of her objective function, which we will denote by Γ^p. For simplicity, we first analyze the case of Γ linear in α and then discuss the general case. In the linear case the derivative Γ^p is given by

$$\begin{aligned}\Gamma^p(\alpha) &= \frac{1}{2}\left(\alpha A + \frac{B}{2}\right)\\ &= \frac{1}{2}\left(\alpha\Gamma(1) + (1-\alpha)\Gamma(0) - \Gamma(0) + \frac{\mathcal{V}(\underline{x},\overline{x}|1) + \mathcal{V}(\overline{x},\underline{x}|1) - \mathcal{V}(\underline{x},\overline{x}|0) - \mathcal{V}(\overline{x},\underline{x}|0)}{2} + \frac{\Gamma(0)}{2}\right)\\ &= \frac{1}{2}\left(\Gamma(\alpha) - \frac{D}{2}\right),\end{aligned} \tag{17}$$

where the first equality in (17) follows from differentiation of the planner's objective with respect to α, the second from replacing A and B and adding and subtracting $\Gamma(0)$, and the third from replacing $\alpha\Gamma(1) + (1-\alpha)\Gamma(0)$ by $\Gamma(\alpha)$ and from defining D as

$$D \equiv \mathcal{V}(\overline{x},\overline{x}|0) + \mathcal{V}(\underline{x},\underline{x}|0) - \mathcal{V}(\underline{x},\overline{x}|1) - \mathcal{V}(\overline{x},\underline{x}|1). \tag{18}$$

At the corners, we have $\Gamma^p(1) = (\Gamma(1) - (D/2))/2$ and $\Gamma^p(0) = (\Gamma(0) - (D/2))/2$.[21]

The constant $D/2$ summarizes the difference between the *private* and *social* incentives to increase α, and contains useful information about the efficiency properties of equilibria. The constant D measures the difference between the value of matching two teams in a PAM way when the equilibrium is NAM minus the value of matching them in a NAM way when the equilibrium is PAM. Indeed, we immediately obtain that an *interior* equilibrium is *inefficient* except in the nongeneric case in which $D = 0$ (as in the second example in Section 2, where there was a unique equilibrium that was efficient). This is because for an interior equilibrium $\alpha \in (0,1)$ we must have $\Gamma(\alpha) = 0$, but then the planner's marginal incentive to increase α is given by $\Gamma^p(\alpha) = -D/2$, which is generically not equal to zero and thus she prefers either a bigger or smaller α^p depending on the sign of D. Similarly, assume that $\Gamma(0) \leq 0$, so there is an equilibrium with NAM. Then if D is negative and large the planner will choose $\alpha^p \neq 0$. Writing in full the planner's marginal incentives at $\alpha = 0$ we obtain, after simplification

$$\Gamma^p(0) = \frac{1}{4}\left(\Gamma(0) + [\mathcal{V}(\underline{x}, \overline{x}|1) - \mathcal{V}(\overline{x}, \underline{x}|1) - V(\underline{x}, \overline{x}|0) - \mathcal{V}(\overline{x}, \underline{x}|0)]\right),$$

where the first term $\Gamma(0)$ summarizes the complementarities in $\mathcal{V}$ under $\alpha = 0$, and the second term in square brackets reflects the externality effect from increasing α away from zero. So a NAM equilibrium is inefficient if the externality effect is strong enough, and it is efficient otherwise. A similar analysis holds PAM.

Beyond the linear Γ case, we can see the wedge between the private and social incentives by using the planner's objective function (14), which yields

$$\Gamma^p(\alpha) = \frac{1}{4}(\Gamma(\alpha) + \alpha\Gamma'(\alpha) + 2\mathcal{V}_\alpha(\underline{x}, \overline{x}|\alpha)).$$

In an interior equilibrium $\Gamma(\alpha) = 0$, and the planner's marginal incentives to increase α is given by $\Gamma^p(\alpha) = (1/4)(\alpha\Gamma'(\alpha) + 2\mathcal{V}_\alpha(\underline{x}, \overline{x}|\alpha))$, which is generically nonzero and can be positive or negative depending on the primitives. For example, in the multiplicative case above this wedge is $\alpha\Gamma'(\alpha) + 2\mathcal{V}_\alpha(\underline{x}, \overline{x}|\alpha) = (\alpha(k(\overline{x}, \overline{x}) + k(\underline{x}, \underline{x})) + (1-\alpha)2k(\underline{x}, \overline{x}))\ell'(\alpha)$, whose sign depends on the sign of ℓ': if positive (negative), the planner's optimal choice is a larger (smaller) α^P than the value of α in the interior competitive equilibrium.

Summarizing, unlike the standard case, equilibrium with binary characteristics can only be efficient if it is a corner one and the externality effect is either small or reinforces the sign of complementarities in match output. In all other cases, equilibrium is *inefficient*. We shall see that similar results hold beyond the binary case.

4.2 Continuum of Characteristics

The binary case affords a complete derivation of the set of competitive equilibria, optima, and the main properties that make matching under externalities different from its counterpart without externalities. These properties are multiplicity of competitive equilibria, stochastic matching – either in competitive equilibrium or as the planner's optimal matching – and inefficiency. The general finite case clearly adds combinatorial complexity to the model, but an educated guess is that the main insights derived in the binary case will still obtain. Indeed, in Appendix A.3 we analyze the case with three characteristics (low, medium, and high), derive the necessary and sufficient conditions for PAM, NAM, and stochastic matching, as well as the main properties of the planner's problem (which continues to be a quadratic form). Moreover, we fully solve an example that is robust to the number of characteristics and that clearly illustrates the main insights (multiplicity, stochastic matching, and inefficiency).

[21] We are grateful to an anonymous referee for suggesting the use of D to compare the solutions of the planner and the market.

Instead of continuing with the finite case, in this section we extend most of these insights to the case when the agents' characteristic x lie in an interval $[\underline{x}, \overline{x}]$, continuously distributed according to a cdf F. This is a case that is commonly used in economic applications of matching without externalities, and this section provides a tractable extension with externalities, which we will use in the economic applications in the next section.

Following the pattern of the binary case, we will provide sufficient conditions for competitive equilibria with PAM and NAM, as well as for multiplicity of equilibria. A nice feature of the continuum case is that wages have a closed form solution that is uniquely pinned down, and its properties (monotonicity, curvature, interpretation) can be easily described. Moreover, we will show that the planner's solution can be "interior" (either stochastic or deterministic but away from PAM and NAM), and that competitive equilibrium can be inefficient.

Competitive Equilibria with PAM and NAM. Consider first the cases of either ex-post random pairwise assignment of competing teams or when all teams compete under aggregate spillovers. In these cases the match output function $\mathcal{V}(\cdot,\cdot|\mu)$ is symmetric in (x, x') for any given μ, and we will also assume throughout the analysis that it is twice continuously differentiable in its first and second arguments.

Let us first construct a competitive equilibrium with PAM, that is, $\mu_+(x) = x$ for all x. The problem an agent with characteristic x faces when the market wage function is w is

$$\max_{x'} \mathcal{V}(x, x'|\mu_+) - w(x').$$

The first-order condition of this problem is simply $\mathcal{V}_2(x, x'|\mu_+) = w'(x')$, where (henceforth) the notation $\mathcal{V}_n$ denotes the derivative of the function with respect to its n-th argument, and similar for second derivatives. To transform this into an equilibrium condition we posit that it must hold along the assignment μ_+, that is, each agent matches with an agent with the same characteristic, and thus $\mathcal{V}_2(x, x|\mu_+) = w'(x)$. Integrating yields the wage function w given by $w(x) = w(\underline{x}) + \int_{\underline{x}}^{x} \mathcal{V}_2(s, s|\mu_+)ds$. Since $\mathcal{V}(\cdot,\cdot|\mu_+)$ is symmetric in (x, x'), it follows that under PAM partners divide match output equally, and so $w(x) = 0.5\mathcal{V}(x, x|\mu_+)$ for all x.[22]

We claim that (w, μ_+), that is, the derived wage function and PAM, constitute a competitive equilibrium if and only if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular in (x, x'). Necessity follows as in the binary case: simply take the incentive constraints for x and x' (of not mimicking each other) and add them up. Regarding sufficiency, it will follow if each agent with characteristic x finds it (globally) optimal to choose a partner with the same characteristic when he conjectures that the prevailing matching in the market is PAM and he faces wages given by w. Without loss of generality, consider two agents with characteristics x and x', with $x' < x$; then $\mathcal{V}(x, x|\mu_+) - w(x) \geq \mathcal{V}(x, x'|\mu_+) - w(x')$ if and only if

$$\mathcal{V}(x, x|\mu_+) - \mathcal{V}(x, x'|\mu_+) \geq \int_{x'}^{x} \mathcal{V}_2(s, s|\mu_+)ds,$$

where the inequality follows from the definition of the wage function. But the left side is equal to $\int_{x'}^{x} \mathcal{V}_2(x, s|\mu_+)ds$, and thus we need to show that

$$\int_{x'}^{x} (\mathcal{V}_2(x, s|\mu_+) - \mathcal{V}_2(s, s|\mu_+))\, ds \geq 0,$$

which follows from the supermodularity of $\mathcal{V}(\cdot,\cdot|\mu_+)$ in (x, x'). The argument for $x' \geq x$ is symmetric. Thus, each agent finds it optimal to choose a partner of the same characteristic. Hence, a competitive equilibrium

[22] To see that the two expressions for w coincide, it sufficies to point out that $w'(x) = 0.5(\mathcal{V}_1(x, x|\mu_+) + \mathcal{V}_2(x, x|\mu_+)) = 0.5\,(2\mathcal{V}_2(x, x|\mu_+)) = \mathcal{V}_2(x, x|\mu_+)$ using the symmetry of $\mathcal{V}(\cdot,\cdot|\mu_+)$ in (x, x').

(w, μ_+) with PAM exists when $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular in (x, x').[23] The equilibrium wage function is strictly increasing in x, and it is convex if $\mathcal{V}_{22}(x, x|\mu_+) \geq 0$ for all x.[24]

A similar argument shows that a NAM equilibrium exists if and only if $\mathcal{V}(\cdot,\cdot|\mu_-)$ is submodular in (x, x') when agents conjecture that there is NAM in the market. Necessity is as usual, and to prove sufficiency, let $\mu_-(x) = F^{-1}(1 - F(x))$ be the matching function and let the wage function w be $w(x) = w(\underline{x}) + \int_{\underline{x}}^{x} \mathcal{V}_2(\mu_-^{-1}(s), s|\mu_-)ds$. Let $\mu_-(x) = x'$ and consider $x'' < x'$; then choosing x' is a global optimum for x if and only if

$$\mathcal{V}(x, x'|\mu_-) - \mathcal{V}(x, x''|\mu_-) \geq \int_{x''}^{x'} \mathcal{V}_2(\mu_-^{-1}(s), s|\mu_-)ds.$$

But the left side is equal to $\int_{x''}^{x'} \mathcal{V}_2(x, s|\mu_-)ds$, and thus we must show that

$$\int_{x''}^{x'} \left(\mathcal{V}_2(x, s|\mu_-) - \mathcal{V}_2(\mu_-^{-1}(s), s|\mu_-)\right) ds \geq 0.$$

Since μ_- is decreasing, so is μ_-^{-1}. Then $x = \mu_-^{-1}(x')$ implies that for any $x'' \leq s \leq x'$ we have $\mu_-^{-1}(s) \geq \mu_-^{-1}(x') = x$, and thus the result follows from the submodularity of $\mathcal{V}$, since the integrand in the expression above is positive as $\mathcal{V}_2$ is decreasing in its first argument. The argument for $x'' \geq x'$ is analogous. Hence, choosing a partner in accordance with μ_- is optimal for each agent, and as a result (w, μ_-) constitutes a competitive equilibrium. As in the PAM case, one can verify that w is convex if $\mathcal{V}_{22}(\mu_-^{-1}(x), x|\mu_-) \geq 0$ for all x.[25]

As in the binary case, we can have multiple equilibria with *different* assignments. In particular, a PAM and a NAM equilibrium can coexist if $\mathcal{V}(\cdot,\cdot|\mu)$ switches from supermodular to submodular in (x, x') when μ changes from μ_+ to μ_-. The following example illustrates the presence of *multiple* competitive equilibria.

Example 3. Let $\mathcal{V}$ be given by $\mathcal{V}(x, x'|\mu) = \zeta + k(x, x')e(\mu)$ for all (x, x') and μ, so there is an aggregate spillover effect that is multiplicative, with $e(\mu_+) > 0 > e(\mu_-)$, and $\zeta > 0$ and large enough to ensure that match payoff is nonnegative for all teams and for $\mu \in \{\mu_+, \mu_-\}$. Also, assume that $k_{12} \geq 0$. Then there is a competitive equilibrium with PAM and with wages given by $w(x) = w(\underline{x}) + e(\mu_+) \int_{\underline{x}}^{x} k_2(s, s)ds$ for all x. Similarly, there is an equilibrium with NAM with $w(x) = w(\underline{x}) + e(\mu_-) \int_{\underline{x}}^{x} k_2(\mu_-^{-1}(s), s)ds$ for all x. So as in the binary case, we can have multiple equilibria. For a closed-form, let $k(x, x') = xx'$, F uniform on $[0, 1]$, $e(\mu_+) = 1$, and $e(\mu_-) = -1$. Then in a competitive equilibrium with PAM the wage function w is given by $w(x) = 0.5(\zeta + x^2)$, which is strictly increasing and strictly convex in x. Under NAM, $\mu_-(x) = 1 - x$ and thus $w(x) = w(0) - 0.5x(2 - x)$ with $w(0) = 0.5a + 0.25$ (since $w(x) + w(1 - x) = a - x(1 - x)$ for all x), which is strictly decreasing and strictly convex. Both types of equilibria coexist in this setting since the externality switches $\mathcal{V}$ from supermodular to submodular in (x, x') depending on the matching that agents conjecture that will prevail in the market.

Consider now the case of ex-ante deterministic assignment of teams. That is, before teams are formed, half of the agents with characteristic x are matched with a future competitor with characteristic $\eta(x) = x$.

Let us construct a competitive equilibrium with PAM, that is, with $\mu_+(x) = x$ for all x. The problem of an

[23] An alternative proof in the PAM case is as follows: since the wage equals half the match output, if x chooses a partner of the same characteristic then his payoff is $0.5\mathcal{V}(x, x|\mu_+)$. If he chooses $x' > x$ then his payoff is $\mathcal{V}(x, x'|\mu_+) - 0.5\mathcal{V}(x', x'|\mu_+)$, and this is less than $0.5\mathcal{V}(x, x|\mu_+)$ if and only if $\mathcal{V}(x', x'|\mu_+) + \mathcal{V}(x, x|\mu_+) \geq 2\mathcal{V}(x, x'|\mu_+)$, that is, if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular.

[24] This follows from differentiating twice $w(x) = 0.5\mathcal{V}(x, x|\mu_+)$ and using $\mathcal{V}_{12}(x, x|\mu_+) \geq 0$ for all x.

[25] Here $w''(x) = \mathcal{V}_{12}(\mu_-^{-1}(x), x|\mu_-)(\mu_-^{-1})'(x) + \mathcal{V}_{22}(\mu_-^{-1}(x), x|\mu_-)$, with $V_{12}(\mu_-^{-1}(x), x|\mu_-) \leq 0$ and $(\mu_-^{-1})'(x) < 0$ for all x.

agent with x facing a wage function w is

$$\max_{x'} V(x, x'|x, x) - w(x').$$

We will assume that the function V is twice continuously differentiable in its arguments, which is more than sufficient for our purposes. The interpretation of this setup is as stated in Section 3.3: an agent with characteristic x is assigned, before the team formation stage, to a competitor with the same characteristic, and he conjectures PAM in this market, that is, that his competitor will team up with another agent with characteristic x.

From the first-order condition evaluated at μ_+ we obtain $V_2(x, x|x, x) = w'(x)$, and thus $w(x) = w(\underline{x}) + \int_{\underline{x}}^{x} V_2(s, s|s, s)ds$. We now show that this wage function along with the PAM assignment μ_+ constitute a competitive equilibrium if V is supermodular in its first two arguments, that is $V_{12} \geq 0$, and if $V_{23} + V_{24} \geq 0$, for which it suffices that V is supermodular in its second and third arguments, and also in its second and fourth arguments.[26] To see this, consider $x' < x$; then $V(x, x|x, x) - w(x) \geq V(x, x'|x, x) - w(x')$ if and only if

$$V(x, x|x, x) - V(x, x'|x, x) \geq \int_{x'}^{x} V_2(s, s|s, s)ds.$$

Since the left side is equal to $\int_{x'}^{x} V_2(x, s|x, x)ds$, this inequality is equivalent to

$$\int_{x'}^{x} (V_2(x, s|x, x) - V_2(s, s|s, s))\, ds \geq 0.$$

But

$$\int_{x'}^{x} (V_2(x, s|x, x) - V_2(s, s|s, s))\, ds \geq \int_{x'}^{x} (V_2(x, s|x, x) - V_2(s, s|x, x))\, ds \geq 0,$$

where the first inequality follows from $V_{23} + V_{24} \geq 0$, and the second from $V_{12} \geq 0$. The argument for $x \geq x'$ is analogous. Hence, a competitive equilibrium with PAM exists under the stated assumptions on V.

A similar argument yields a competitive equilibrium with NAM μ_- given by $\mu_-(x) = F^{-1}(1 - F(x))$. To see this, note that the problem of an agent with characteristic x is

$$\max_{x'} V(x, x'|x, \mu_-(x)) - w(x').$$

Proceeding as before, the first-order condition is $V_2(x, x'|x, \mu_-(x)) = w'(x')$ and thus the equilibrium condition is $V_2(\mu_-^{-1}(x'), x'|\mu_-^{-1}(x'), x') = w'(x')$. Integrating yields w given by $w(x) = w(\underline{x}) + \int_{\underline{x}}^{x} V_2(\mu_-^{-1}(s), s|\mu_-^{-1}(s), s)ds$. We claim that (w, μ_-) constitute a competitive equilibrium if $V_{12}+V_{23} \leq 0$, and $V_{24} \geq 0$. To prove it, it suffices to show that under these conditions it is a global optimum for an agent with characteristic x to choose $x' = \mu_-(x)$. Consider $x'' < x'$ (the argument for $x'' \geq x'$ is analogous); then x' is an optimal choice for x if and only if

$$V(x, x'|x, \mu_-(x)) - V(x, x''|x, \mu_-(x)) \geq \int_{x''}^{x'} V_2(\mu_-^{-1}(s), s|\mu_-^{-1}(s), s)ds.$$

Since the left side is equal to $\int_{x''}^{x'} V_2(x, s|x, \mu_-(x))ds = \int_{x''}^{x'} V_2(x, s|\mu_-^{-1}(x'), x')ds$, this inequality is equivalent to

$$\int_{x''}^{x'} \left(V_2(x, s|\mu_-^{-1}(x'), x') - V_2(\mu_-^{-1}(s), s|\mu_-^{-1}(s), s)\right) ds \geq 0.$$

[26] This holds if, for example, V is multiplicatively separable of the form $k(x, x')z(y, y')$ with k and z increasing in each argument.

Since $x = \mu_-^{-1}(x')$, it suffices to show that under the stated assumptions about V we have that $V_2(\mu_-^{-1}(\cdot), s|\mu_-^{-1}(\cdot), \cdot)$ is an increasing function. Differentiating yields $(V_{21} + V_{23})(\mu_-^{-1})' + V_{24} \geq 0$, which holds if $V_{21} + V_{23} \leq 0$ and $V_{24} \geq 0$. Thus, a competitive equilibrium with NAM exists under the stated assumptions.

We summarize all these results in the following proposition:

Proposition 3 *(i) If competing teams are either ex-post randomly assigned or if they all compete under aggregate spillovers, then there is a competitive equilibrium (w, μ_+) that exhibits PAM (a competitive equilibrium (w, μ_-) that exhibits NAM) if and only if $\mathcal{V}(\cdot, \cdot|\mu_+)$ is supermodular (if $\mathcal{V}(\cdot, \cdot|\mu_+)$ is submodular).*

(ii) If competing teams are ex-ante deterministically assigned by a PAM η, then there is a competitive equilibrium (w, μ_+) that exhibits PAM (a competitive equilibrium (w, μ_-) that exhibits NAM) if $V_{12} \geq 0$ and $V_{23}+V_{24} \geq 0$ (if $V_{12} + V_{23} \leq 0$, and $V_{24} \geq 0$).

In short, we have shown that the insights regarding existence of competitive equilibria with PAM and NAM, and that they may coexist, extend to the case with a continuum of types. In addition, with a continuum of characteristics we can derive properties of the wage function that supports PAM or NAM under each of the competing team assignment environments. A useful feature of our analysis is that it permits the derivation of sorting and wage properties under externalities in a tractable way that resembles the analysis without externalities. This should prove helpful in economic applications of our model, as we will illustrate in the next section.

We close with a brief discussion of stochastic matching with a continuum of types. A full analysis of this case is much more challenging than in the binary case and well beyond the scope of this paper. We can, however, say a few things about it. The most important one is that, as in the binary case, the necessary and sufficient condition for a measure π to be part of a competitive equilibrium with stochastic matching is that, conditional on π, the function $\mathcal{V}(\cdot, \cdot|\pi)$ is modular in (x, x'). In differential terms, this is equivalent to finding a measure π such that $\mathcal{V}_{12}(x, x'|\pi) = 0$ for all (x, x'). To prove necessity, consider any two agents with characteristics x and x' facing a wage function w. Among all the indifference conditions for these types, the following must hold:

$$\mathcal{V}(x, x|\pi) - w(x) = \mathcal{V}(x, x'|\pi) - w(x')$$
$$\mathcal{V}(x', x'|\pi) - w(x') = \mathcal{V}(x', x|\pi) - w(x).$$

Adding them up we obtain that $\mathcal{V}(\cdot, \cdot|\pi)$ must be modular in (x, x'). To prove sufficiency, assume that $\mathcal{V}_{12}(x, x'|\pi) = 0$ for all (x, x'). Let $w(x) = w(\underline{x}) + \int_{\underline{x}}^{x} \mathcal{V}_2(x, s|\pi)ds$. Since $\mathcal{V}$ is modular in its first two arguments, the integrand is independent of x. Then an agent with characteristic x solves $\max_{x'} \mathcal{V}(x, x') - w(x')$ and the first-order condition is $\mathcal{V}_2(x, x'|\pi) = \mathcal{V}_2(x'x'|\pi)$, which holds for all x' by the modularity premise. Hence, each agent is indifferent about whom to hire, and is therefore willing to randomize according to π. This proves sufficiency.

As mentioned, characterizing the set of competitive equilibrium is beyond our scope, but we can construct an admittedly simple example in the spirit of the binary case where a fraction α of the population matches in a PAM way and $1 - \alpha$ in a NAM way. Assume that F is uniform on $[0, 1]$, and that $\mathcal{V}(x, x'|\pi) = \zeta + k(x, x')\ell(\alpha)$, with $k_{12} > 0$, $\ell(0) < 0$, $\ell(1) > 0$, and $\ell' > 0$, and ζ large enough to make payoffs nonnegative. The interpretation is that the aggregate spillover depends only on the measure of teams matched à la PAM. Then there exists a value of α such that $\mathcal{V}_{12}(x, x'|\pi) = k_{12}(x, x')\ell(\alpha) = 0$. Although not conclusive, the example suggests that competitive equilibria with stochastic matching can also exist with a continuum of characteristics.

The Planner's Problem. A full characterization of the planner's problem like the one given for the binary case is not available for the continuum case, as it requires a nontrivial extension of optimal transport theory –

hitherto unknown – to handle problems where the measure being chosen also appears in the integrand, e.g., this is the case in $\max_{\pi \in \mathcal{M}} \int_{[0,1]^2} \mathcal{V}(x, x'|\pi) d\pi(x, x')$. It is possible, however, to provide interesting insights about the efficient matching without solving the full-blown problem, but instead building on the analysis of the binary case.

Consider the following restricted planner's problem, where the feasible set is any combination $\alpha \in [0, 1]$ of PAM and NAM. More precisely, let f be the density associated with F and suppose that, for each x, $\alpha f(x)$ is matched in a PAM way and $(1-\alpha)f(x)$ in a NAM way. This restricted problem is

$$\max_{\alpha \in [0,1]} \frac{1}{2} \left(\alpha \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, x|\alpha) dF(x) + (1-\alpha) \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\alpha) dF(x) \right),$$

where to avoid tedious details and repetition we focus on the case in which $\mathcal{V}$ is linear in α, that is, $\mathcal{V}(x, x'|\alpha) = \alpha \mathcal{V}(x, x'|\mu_+) + (1-\alpha)\mathcal{V}(x, x'|\mu_-)$, and $\mathcal{V}(\cdot, \cdot|\alpha)$ is symmetric in (x, x'). (A similar analysis can be performed in the case in which $\mathcal{V}$ is nonlinear in α and $\mathcal{V}(\cdot, \cdot|\alpha)$ is not symmetric in (x, x'), following the same arguments given in the binary case.) Simple algebra allows us to rewrite it as follows:

$$\max_{\alpha \in [0,1]} \frac{1}{2} \left(\alpha^2 A' + \alpha B' + C' \right),$$

where

$$A' = \left(\int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, x|\mu_+) dF(x) - \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_+) dF(x) \right) - \left(\int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, x|\mu_-) dF(x) - \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_-) dF(x) \right)$$

$$B' = \left(\int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, x|\mu_-) dF(x) - \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_-) dF(x) \right) + \left(\int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_+) dF(x) - \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_-) dF(x) \right)$$

$$C' = \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_-) dF(x).$$

Notice that these expressions are straightforward analogues of those in the planner's problem in the binary case. It is clear that if the planner were restricted to choose between combinations of PAM and NAM, then an adaptation of Proposition 2 would give a sharp characterization of the optimal matching in this restricted problem. Without this restriction, we can still use the restricted problem to show the following result:

Proposition 4 *If $A' < 0$, $B' > 0$, and $B' + 2A' < 0$, then the efficient matching is neither PAM nor NAM.*

This proposition reveals that the planner may prefer either another deterministic matching that is neither PAM nor NAM, or a stochastic matching. In particular, it shows that a stochastic matching can dominate both PAM and NAM. As we will see shortly, this can happen even if $\mathcal{V}(\cdot, \cdot|\mu_+)$ is supermodular in (x, x') or if $\mathcal{V}(\cdot, \cdot|\mu_-)$ is submodular in (x, x'), something that cannot arise without externalities.

INNEFICIENCY OF COMPETITIVE EQUILIBRIUM. We can now use Proposition 4 to show that PAM and NAM equilibria can be inefficient. To see this, assume that the conditions in Proposition 3 (i) or (ii) are satisfied and there is a competitive equilibrium with PAM. Consider now the planner's choice between just PAM or NAM: under PAM welfare is $0.5 \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, x|\mu_+) dF(x)$ while under NAM is $0.5 \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_-) dF(x)$. To show that the PAM equilibrium is inefficient it suffices to show that welfare is strictly higher under NAM (obviously, this does not imply that the planner will choose NAM). By adding and subtracting $0.5 \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x, \mu_-(x)|\mu_+) dF(x)$ to the welfare comparison between PAM and NAM, we obtain that NAM dominates PAM if and only if

$$\int_{\underline{x}}^{\overline{x}} \mathcal{V}(x,x|\mu_+)dF(x) - \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x,\mu_-(x)|\mu_+)dF(x) < \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x,\mu_-(x)|\mu_-)dF(x) - \int_{\underline{x}}^{\overline{x}} \mathcal{V}(x,\mu_-(x)|\mu_+)dF(x). \quad (19)$$

The left side can be interpreted as the efficiency gains from PAM instead of from NAM when everybody conjectures that the prevailing matching is PAM. The right side represents the effect of the externality in the welfare under NAM versus PAM. If the externality effects is strong enough, then PAM is dominated by NAM and any competitive equilibrium exhibiting PAM is inefficient.

Slightly more general, consider the restricted problem where the planner chooses among α combinations of PAM and NAM. Then following the same steps as in the binary case, one can easily provide conditions under which the optimal matching in this restricted problem is stochastic (since this is mainly a change in notation, we omit the details). Obviously, this is not a proof that the optimal matching in the full-blown problem can be stochastic, a result whose derivation must await the above mentioned extension in optimal transport theory.

5 Economic Relevance

This section sheds light on the economic relevance of the theory. We analyze economic applications that show that our framework can accommodate several applied settings and can also be used to study questions that have received ample attention recently in the Macro/Labor and Industrial Organization literatures. The objective of this section is twofold: first, to show the economic relevance of the model and how it contributes in a novel way to answer open economic questions in the literature; and second, to illustrate the workings of the model under our alternative specifications. In Section 5.1 we develop our main application of a model of competing teams in the presence of economy-wide externalities or *aggregate spillovers*. The model generates results that are consistent with the empirical evidence on the evolution of inequality within and between firms. In Section 5.2, we analyze team formation in a competitive input market when firms compete downstream in an oligopolistic output market, where competing firms are *deterministically* assigned. Finally, in Section 5.3, we apply the setting with *random assignment* to the design of competitions of sports teams and we analyze the impact of policy interventions.

5.1 Knowledge Spillovers

This application on knowledge spillovers contributes to the Macro/Labor literature on firm heterogeneity by showing how technological change affects the equilibrium allocation of skills *within* and *between* firms. Spillovers in the downstream market affect how firms hire and compose the skills of their teams. The team composition in turns affect the firm's investment in capital (knowledge). The equilibrium interaction between the input and the (non-competitive) output market can provide crucial insights into an important current question on the evolution of between- and within-firm inequality in the last decades. In particular, evidence for different countries shows that the increase in wage inequality in recent decades can nearly exclusively be explained by the increase in between-firm inequality while there is hardly any change in within-firm inequality.[27] We show that our model is capable of providing a rationale for these facts.

Romer (1986) and Lucas (1988) endogenous growth models introduced the idea that Total Factor Productivity (TFP) is not exogenous, but depends on the decisions of other agents in the economy. For example, any given

[27] See Card, Heining, and Kline (2013) for Germany, Song, Price, Guvenen, Bloom, and von Wachter (2015) and Barth, Bryson, Davis, and Freeman (2014) for the US, Benguria (2015) for Brazil, and Vlachos, Lindqvist, and Hakanson (2015) for Sweden.

worker's productivity is higher if the economy's work force is more productive. This can be modeled by letting TFP be a function of the aggregate investment, whether it be in education or in technology. This has been taken further by Jovanovic and Rob (1989), Eeckhout and Jovanovic (2002), Lucas and Moll (2014), Perla and Tonetti (2014), Benhabib, Perla, and Tonetti (2017), and König, Lorenz, and Zilibotti (2016), who observe that those spillovers may affect agents differentially, which naturally leads to inequality and to a distribution of firms.

We build on this literature, by adding a matching stage in which teams form, the theme of this paper. This allows us to capture within-firm heterogeneity in addition to the existing between-firm heterogeneity. We then ask how technological change – which takes the form of an increase in production complementarities – affects the skill distribution. In the first stage, teams of two workers form in a competitive labor market. In the second stage, teams make investment decisions where the return on investment is a function of the distribution of investment in the entire economy. The externality in the second stage is general, with non-internalized effects across all agents in the economy. We show how technological change affects the composition of skills, both in the equilibrium allocation as well as in the planner's solution. An increase in complementarities leads to *more* positive sorting in equilibrium and thus to *less* variance of skills within firms and *more* variance between firms. Similarly, the evidence on wages shows that the variance of wages within firms has remained relatively constant while the variance of wages between firms has increased, a stylized fact our model can rationalize as well.[28] These insights cannot be reconciled with the standard matching models without externalities (Becker (1973)). Indeed, without externalities sorting is not affected *at all* by a marginal increase in complementarities, except in the case in which technology switches from supermodular to submodular, in which case the allocation switches between PAM and NAM, a knife-edge case. Our model with externalities rationalizes allocations with mixed sorting, where technological change has a smooth impact on the allocation towards *more* positive or negative sorting.

We consider this setup under two distinct configurations. We first analyze spillovers with binary types in which firms copy technology from more productive firms. The advantage of the binary setting is that we can explicitly solve for an interior equilibrium. This version generates the empirically observed predictions on skill composition and wage inequality. Then we study a variation of the model with a continuum of types and illustrate the subtle mechanics of constructing a competitive equilibrium in this more complex setup.

Spillovers from Copying and the Evolution of Inequality. For tractability, we first focus on the binary set up with aggregate spillovers. In the matching stage, firms hire two workers of type $x \in \{\underline{x}, \overline{x}\}$ in a competitive labor market. And in the second stage, firms make an investment decision k, the payoff of which depends on their team composition, which is equal to the sum of the characteristics of its members, $x_1 + x_2$. For notational economy, we will denote this sum by $\overline{X} \equiv 2\overline{x}$, $\hat{X} \equiv \overline{x} + \underline{x}$, and $\underline{X} \equiv 2\underline{x}$.

Firms receive spillovers from other firms and these spillovers vary with the rank the firm has in the capital distribution and the composition X. We model this rank dependence by assuming that the spillover S results from copying the technology of higher ranked firms. The higher is the own capital of a firm, the fewer higher ranked firms there are and the less there is to copy. The cost of investment is quadratic and is inversely proportional to X^γ, where higher γ yields stronger complementarity between worker characteristics. The composition of workers affects the optimal investment: a firm with composition X that invests k obtains a direct return $A\lambda k$ as well as a spillover $AS(k, X)k$, where with a slight abuse of notation we denote by A the economy-wide (positive) TFP and λ is a positive constant (this is to ensure that there is a direct positive gross benefit from investment even if

[28] These results are not as straightforward to show as they seem since the distribution of team composition also changes when there is an complementarities increase, complicating the comparative statics analysis. We provide conditions under which they ensue.

the firm does not enjoy any spillover effect from other firms). Using the notation from the binary case in Section 4.1, consider an arbitrary matching $\alpha \in [0,1]$, where as usual α is the fraction of teams matched according to PAM, and denote by $\mathcal{V}(X|\alpha)$ the payoff of a team with composition X given a prevailing matching α.

We assume that $\mathcal{V}(X|\alpha)$ is given by

$$\mathcal{V}(X|\alpha) = \max_{k\geq 0} \left(A(\lambda + S(k,X))k - \frac{k^2}{2X^\gamma} \right), \tag{20}$$

where $\lambda > 0$. For values $\underline{\kappa} < \hat{\kappa} < \overline{\kappa}$, the spillover function S is defined as follows:

$$S(k,\overline{X}) = 0 \quad \forall k, \tag{21}$$

$$S(k,\hat{X}) = \begin{cases} 1 - \frac{\alpha}{2} - (1-\alpha) & \text{if } k \in [0,\overline{\kappa}) \\ 0 & \text{if } k \geq \overline{\kappa}, \end{cases} \tag{22}$$

$$S(k,\underline{X}) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } k \in [0,\hat{\kappa}) \\ 1 - \frac{\alpha}{2} - (1-\alpha) & \text{if } k \in [\hat{\kappa},\overline{\kappa}) \\ 0 & \text{if } k \geq \overline{\kappa}. \end{cases} \tag{23}$$

The intuition behind the function S is that a firm with a given composition can, by choosing k, learn some of the production technology of firms with better composition. Firms with high k have less to copy than firms with low k. Notice that the magnitude of the positive spillover a firm enjoys depends on the measure of teams with higher k. The higher the k chosen the smaller that measure.

We will look for a second-stage equilibrium where each firm with composition $\underline{X}$ chooses $\underline{\kappa}$, $\hat{X}$ chooses $\hat{\kappa}$, and $\overline{X}$ chooses $\overline{\kappa}$, where $\underline{\kappa} = A\underline{X}^\gamma(\lambda + 1 - (\alpha/2))$, $\hat{\kappa} = A\hat{X}^\gamma(\lambda + (\alpha/2))$, and $\overline{\kappa} = A\overline{X}^\gamma\lambda$ (these expressions come from the maximization problem (20), ignoring the kinks in S). As a result, the candidate $\mathcal{V}(\cdot|\alpha)$ is given by $\mathcal{V}(\underline{X}|\alpha) = A^2\underline{X}^\gamma(\lambda + 1 - \alpha/2)^2/2$, $\mathcal{V}(\hat{X}|\alpha) = A^2\hat{X}^\gamma(\lambda + \alpha/2)^2/2$, and $\mathcal{V}(\overline{X}|\alpha) = A^2\overline{X}^\gamma\lambda^2/2$. We now provide sufficient conditions for these choices to be a second-stage equilibrium of the market.

Lemma 1 *Let* $\alpha \in [0,1]$, $\underline{\kappa} = A\underline{X}^\gamma(\lambda + 1 - (\alpha/2))$, $\hat{\kappa} = A\hat{X}^\gamma(\lambda + (\alpha/2))$, *and* $\overline{\kappa} = A\overline{X}^\gamma\lambda$. *If* $\lambda \geq 1$, $\gamma \geq 1$, *and* $\underline{x}/\overline{x} < 1/3$, *then* $0 < \underline{\kappa} < \hat{\kappa} < \overline{\kappa}$ *and* $(\underline{\kappa},\hat{\kappa},\overline{\kappa})$ *is a second-stage equilibrium.*

We can now analyze the competitive equilibria in the labor market in the first stage. In this case, the function $\Gamma(\alpha) = \mathcal{V}(\overline{X}|\alpha) + \mathcal{V}(\underline{X}|\alpha) - 2\mathcal{V}(\hat{X}|\alpha)$ that is instrumental in computing equilibria is given by

$$\begin{aligned} \Gamma(\alpha) &= \frac{A^2\overline{X}^\gamma\lambda^2}{2} + \frac{A^2\underline{X}^\gamma\left(\lambda + 1 - \frac{\alpha}{2}\right)^2}{2} - 2\frac{A^2\hat{X}^\gamma\left(\lambda + \frac{\alpha}{2}\right)^2}{2} \\ &= \frac{A^2\overline{X}^\gamma\lambda^2}{2}\left(1 + \left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1 + \frac{1-\frac{\alpha}{2}}{\lambda}\right)^2 - 2^{1-\gamma}\left(1 + \frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1 + \frac{\alpha}{2\lambda}\right)^2\right), \end{aligned} \tag{24}$$

where the third equality follows by multiplying and dividing the second and third term by the first one and then simplifying. It is clear from (24) that Γ strictly decreases in α (the second term in the expression in parenthesis strictly decreases in α while the third strictly increases but has a minus sign in front of it). Hence, it follows from the analysis in Section 4.1 that, given the second-stage equilibrium in Lemma 1, there exists a *unique* equilibrium at the matching stage, which is either PAM, or NAM, or interior. We now provide some simple parametric conditions under which the unique equilibrium is interior and the proportion of teams matched in a PAM way is increasing in the complementarity parameter γ.

Proposition 5 *Assume the second-stage equilibrium described in Lemma 1. If $\lambda \geq 1$, $1 \leq \gamma < 1 + 2(\log(1 + (1/2\lambda))/\log 2)$, and $\underline{x}/\overline{x}$ is sufficiently small, then there is a unique competitive equilibrium in the first stage, which is interior (i.e., $\alpha \in (0,1)$). Moreover, the equilibrium α is strictly increasing in γ.*

The proposition highlights three features that cannot arise without externalities. First, there is a competitive equilibrium with mixing. Second, positive sorting as measured by the fraction of teams matched according to PAM is a strictly increasing function of complementarities, represented by the technology parameter γ. This comparative statics result also reveals that the distribution of skills across teams becomes more spread out with technological change, that is, with an increase in γ. This accords well with the evidence that most of the increase in skill inequality can be explained by the increase in between-firm inequality while within-firm inequality has remained constant.[29] Finally, as remarked in Section 4.1, the competitive equilibrium is inefficient.[30]

As an illustration we consider a closed form example. Assume $\underline{x} = 0$ and $\lambda = 1$. Then $\Gamma(\alpha) = 0$ if and only if $1 - 2^{1-\gamma}(1 + (\alpha/2))^2 = 0$, which yields

$$\alpha = 2\left(2^{\frac{1}{2}(\gamma-1)} - 1\right), \tag{25}$$

and this is positive if $\gamma > 1$, and less than one if $\gamma < 1 + 2(\log(3/2)/(\log 2)) \cong 2.17$. Differentiating with respect to γ reveals immediately that α is *strictly increasing* in γ.

We now compute the between- and within-firm variance of wages in our model and show when the between-firm variance strictly increases in γ while the within-firm variance can remain relatively constant, consistent with the stylized fact that has been documented recently (see the references in footnote 27). For clarity, denote by $\alpha^\star$ the equilibrium value of α, and assume that parameters are such that $\alpha^\star$ is strictly increasing in γ. We know from the analysis in Section 4.1 that wages are given by $\underline{w} = 0.5\mathcal{V}(\underline{X}|\alpha^\star) = A^2\underline{X}^\gamma(\lambda + 1 - (\alpha^\star/2))^2/4$ and $\overline{w} = 0.5\mathcal{V}(\overline{X}|\alpha^\star) = A^2\overline{X}^\gamma\lambda^2/4$. Then $\Delta w \equiv \overline{w} - \underline{w} = (A^2/4)(\lambda^2\overline{X}^\gamma - \underline{X}^\gamma(\lambda + 1 - (\alpha^\star/2))^2)$, where we have already included the equilibrium value $\alpha^\star$, which is a function of γ. Since $\alpha^\star$ is strictly increasing in γ, it is clear that $\partial\Delta w/\partial\gamma > 0$ under weak conditions, such as $\overline{X} \geq 1$ (to see this, simply take $\overline{X}^\gamma$ out of the parenthesis and differentiate; both terms are then strictly increasing in γ).

We show in Appendix A.8 that the within-firm variance is given by the following expression:

$$\text{Var}[w|\alpha^\star] = \frac{A^4\lambda^4}{128}(1 - \alpha^\star)\left(\overline{X}^\gamma - \underline{X}^\gamma\left(1 + \frac{1 - \frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2. \tag{26}$$

Note that the within-firm variance can increase or decrease in γ, since although Δw increases, the change is tempered by the decrease in $1 - \alpha^\star$ when γ increases. In particular, if $\underline{x} = 0$, $\overline{x} = 1$, and $\lambda = 1$, the within-firm variance of wages is equal to $(1 - \alpha^\star)2^{2\gamma} = (3 - 2^{\frac{1}{2}(\gamma+1)})2^{2\gamma}$ times $A^4/128$, where we have used the expression $\alpha = 2(2^{\frac{1}{2}(\gamma-1)} - 1)$ from the example above. Then the variance is strictly concave and nonmonotone in γ, first increasing for values of γ near 1, and then decreasing after reaching a peak at $\gamma \cong 1.52$. Thus, for values of γ near the peak, the within-firm variance barely changes. The more general insight is that in this set up, *the within-firm variance need not change much with an increase in γ.*

Let us turn now to the between-firms variance. To compute it, we need to take into account that the fraction

[29] Vlachos, Lindqvist, and Hakanson (2015) has detailed information on aptitude tests for Sweden and show that skill inequality has increased between firms but not within.

[30] See Appendix A.9 for the analysis of the planner's problem in this application, which also features a unique interior solution α that is strictly increasing in γ under similar parametric conditions.

of PAM/NAM teams changes with γ. We show in Appendix A.8 that the variance between firms is:

$$\begin{aligned}\text{Var}[w_i + w_j|\alpha^\star] =& \frac{A^4\lambda^4}{128}\left(\frac{\alpha^\star}{2}\left(3\overline{X}^\gamma - \underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2 + \frac{\alpha^\star}{2}\left(\overline{X}^\gamma - 3\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right.\\ &\left.+(1-\alpha^\star)\left(\overline{X}^\gamma + \underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right). \qquad (27)\end{aligned}$$

To see that it is easy to generate cases where this variance is strictly increasing, assume $\underline{x} = 0$, so that $\underline{X} = 0$, and $\overline{x} > 1/2$, so that $\overline{X} > 1$. Then simple algebra yields

$$\text{Var}[w_i + w_j|\alpha^\star] = \frac{A^4\lambda^4}{256}\overline{X}^{2\gamma}(8\alpha^\star + 2),$$

which is clearly *strictly increasing* and strictly convex in γ. By continuity, the same holds for $\underline{x}$ small.[31] The more general insight is that in this set up, *the between-firm variance changes significantly with an increase in* γ.

The planner's solution, which is in Appendix A.9, also features an interior solution for α^p. Moreover, the optimal α^p is increasing in the complementarity γ. Technological change therefore induces the planner to choose *more* positive sorting. This leads to more between-firm inequality and hence total inequality. We also find that the equilibrium is inefficient. The optimal α^p is smaller than the equilibrium $\alpha^\star$. Hence the planner chooses *less* assortative matching than the market, and therefore there is too much between-firm inequality.

In short, our model is capable of generating properties of the distribution of skills and wages within and between firms that are consistent with documented stylized facts. In particular, in equilibrium, technological change in the form of larger complementarities can have a negligible effect on within-firm inequality in wages while between-firm inequality significantly increases, as the empirical evidence suggests. This also occurs for the planner's problem, though the planner chooses less positive sorting and hence less inequality than the market.

Spillovers from a Patent Race. To illustrate the mechanics of our model with a continuum of types and aggregate spillovers, we now analyze a more complex setting with spillovers with a continuum of characteristics and construct a PAM equilibrium. We assume that x is distributed uniformly on $[0, 1]$, and that team composition enters match output as the sum of the characteristics (x, x') of its members, denoted by $X = x + x'$. Since we do not focus on the comparative statics of complementarities in this case, we set $\gamma = 1$ for simplicity. The uniform distribution of x induces a distribution of team composition X, which we denote by G. Under PAM, the measure one half of pairwise teams is distributed uniformly on $[0, 2]$, since $X = 2x$, $x \in [0, 1]$, and thus $G(X) = X/2$.[32]

In stage two, each team makes an investment decision, a choice of k. Output is a function of the distribution of k in the economy. What is different here is that we assume that the spillover effect is increasing in the rank that the investment of a team has in the distribution of k. This could be due, for example, to a patent race or a

[31] Replacing $\alpha^\star$ by its closed form expression and calculating the solution numerically reveals that the increase in between-firm variance when γ increases is a general property.

[32] Similarly, under NAM all teams formed consist of a pair $(x, 1-x)$ and hence all teams have $X = x + 1 - x = 1$ and are identical, that is, $G(X)$ is degenerate at $X = 1$.

first-mover advantage.[33] The problem of a team with X when matching is μ is

$$\mathcal{V}(X|\mu) = \max_{k\geq 0}\left(AH(k|\mu)k - \frac{k^2}{2X}\right), \tag{28}$$

where $H(\cdot|\mu)$ is the cdf of k in the economy when the matching is μ, and represents the spillover effects in this economy.[34] Notice that the objective function is strictly supermodular in (k, X), and hence the optimal solution, denoted by $\kappa^*(X)$ for each X and where we have omitted μ as an argument to simplify the notation, is increasing in X, strictly so when it is interior. Using the first-order condition of the optimization problem of a team with composition X, it must be the case that, for each X,

$$H(\kappa^*(X)|\mu) + \kappa^*(X)H'(\kappa^*(X)|\mu) = \frac{\kappa^*(X)}{AX}. \tag{29}$$

Using that κ^* is monotone, equilibrium in the second stage demands that the H be consistent with the distribution of teams formed in the first stage, given by G. Formally, for all $k \geq 0$ we must have

$$H(k|\mu) = \mathbb{P}[\kappa^*(X) \leq k] = \mathbb{P}[X \leq \kappa^{*-1}(k)] = G(\kappa^{*-1}(k)),$$

plus the boundary conditions $H(0|\mu) = 0$ and $H(\sup \kappa^*(X)|\mu) = 1$.

Assume that in the first stage we have PAM, and thus $G(X) = X/2$. Then $H(k) = \kappa^{*-1}(k)/2$ for all investment levels k in the support of κ^*. Setting $k = \kappa^*(X)$ in (29), and thus $X = \kappa^{*-1}(k) = 2H(k)$, and using the equilibrium condition $H(k) = \kappa^{*-1}(k)/2$, we obtain the following ordinary differential equation under PAM:

$$H(k|\mu_+) + kH'(k|\mu_+) = \frac{k}{A2H(k|\mu_+)} \quad \Longleftrightarrow \quad H^2(k|\mu_+) + kH(k|\mu_+)H'(k|\mu_+) = \frac{k}{2A}. \tag{30}$$

Solving this equation and using the boundary condition $H(0|\mu_+) = 0$, we obtain the equilibrium distribution H of investments in the second stage of the problem, given by

$$H(k|\mu_+) = \left(\frac{k}{3A}\right)^{\frac{1}{2}}, \tag{31}$$

which is equal to 0 at $k = 0$ and equals 1 at $k = 3A$. Inserting (31) into the first-order condition (29) we obtain the equilibrium investment function κ^* in the second stage

$$\kappa^*(X) = \frac{3}{4}AX^2, \tag{32}$$

which is strictly increasing in X and equals 0 at $X = 0$ and $3A$ at $X = 2$, as consistency requires. Finally, we can insert (31)–(32) into the objective function of problem (28) to obtain the following match output function:

$$\mathcal{V}(X|\mu_+) = \frac{3}{32}A^2X^3. \tag{33}$$

Since this is strictly supermodular in (x, x') (it depends on its sum $X = x + x'$ and $\mathcal{V}(\cdot|\mu_+)$ is strictly convex

[33] We use the simplest possible formulation because it permits a closed-form solution. For a more general formulation of spillover effects of this sort in a different context, see Eeckhout and Jovanovic (1998).

[34] The spillover function does not have to be a cdf and could be a more general function that depends on the distribution of investments and teams formed. Making it a cdf simplifies the equilibrium analysis.

in X), we can now rationalize the conjecture of each team in the second stage that the prevailing matching is PAM. Under PAM output is divided equally among team members given the symmetry of the problem, and thus μ_+ along with $w(x) = (3/64)A^2(2x)^3$ constitute a competitive equilibrium in the first stage. This completes the construction of a competitive equilibrium with PAM in this economy, where the distribution of investments in the second stage crucially depends on the formation of teams in the first stage, the general theme of this paper.[35]

5.2 Market Power

We now illustrate our model with an application to oligopolistic markets. Recent evidence establishes that market power, as measured by the markup of price over marginal cost, has risen steeply in the last few decades. Most of the rise in average markups is driven by an increase in the upper percentiles, i.e. due to higher dispersion and more skewness in the markup distribution (see De Loecker and Eeckhout (2017)).

It is well known that market power in the output market affects the market for inputs, such as labor. With more market power, output produced decreases and as a result, so does the demand for labor. This lowers both wages and labor force participation. In this application, we show how the characteristics of the labor market can also affect the extent of market power and markups. In particular, we establish how a change in the degree of complementarity between workers affects the markup distribution, leading to higher and more spread out markups. We believe this novel both in the Industrial Organization and Macroeconomic literatures.

In this application we consider a continuum of agents' characteristics. Since firms in most oligopolistic markets know who their competitors are, we assume the deterministic assignment of competing teams in stage two described in Section 3.3. This setup dictates that spillovers and market power tend to arise in narrowly defined sectors, yet firms hire on the economy-wide labor market. Indeed, this is precisely the case in markets where firms have market power and compete in a specific product market, say Coca Cola and Pepsi in soft drinks, and Visa and MasterCard in credit cards. Yet all these firms compete jointly in the upstream labor market when hiring marketing and sales professionals.

We analyze a market structure with there is a large number of sectors each with two firms that compete à la Cournot in a product market in stage two; in stage one, they hire skilled labor (which determines their marginal cost) in a competitive labor market with heterogeneous workers with characteristic $x \in [\underline{x}, \overline{x}]$ distributed with cdf F. We append to the canonical Cournot duopoly model with linear demand and constant marginal cost a matching stage where workers match with the firms. More precisely, we assume (as in the ex-ante deterministic assignment in Section 3.3) that half of the population of agents are initially assigned according to PAM with a future competitor. That is, there will be a continuum of 'locations' containing pairs of 'firms' (each with one agent) that will compete downstream à la Cournot. After this initial stage, each firm hires a 'partner' in a competitive market. At the end of this stage, in each location there will be a pair of firms, each with two agents.

In each location, the demand for the product is linear, given by $p = a - b(q_i + q_j)$, with $a > 0$ and $b > 0$, where q_i and q_j are the outputs of the two firms.[36] The cost of production for firm $k = i, j$ when the output level q_k and the firm composition is (x_k, x'_k) is given by $C(x_k, x'_k, q_k) = c(x_k, x'_k)q_k$, where (x_k, x'_k) is the work force composition of firm $k = i, j$. As a result, each firm $k = i, j$ maximizes $pq_k - c(x_k, x'_k)q_k$ with respect to q_k. The cost-per-unit function c is given by $c(x_k, x'_k) = \nu - \beta x_k x'_k$, with $\nu > \beta\overline{x}^2$, $\beta > 0$. That is, firms with better team

[35] If instead agents conjecture NAM, then $H(\cdot|\mu_-)$ is degenerate at some value of k since all teams have composition $X = 1$. One can show that all teams choosing $\kappa^* = A$ is a second-stage equilibrium with $\mathcal{V}(1|\mu_-) = A^2/2$. It then follows that welfare under PAM, given by $(1/2)\int_0^2 \mathcal{V}(X|\mu_+)(1/2)dX = (3/32)A^2$, is lower than under NAM, given by $(1/2)(A^2/2)$. Since PAM is dominated by at least another matching, it follows that the equilibrium constructed above is inefficient.

[36] We assume that a and b is the same accross sectors but we could easily relax it and allow for sector-specific demand parameters.

composition (higher x_k and x'_k) have lower marginal cost. Moreover, c is strictly submodular, that is, $c_{12} = -\beta$, with "degree" of submodularity indexed by β. To ensure interior solutions we will assume that $a > 2c(\underline{x}, \underline{x})$.

As is well-known, we obtain that in any given sector the unique Nash equilibrium quantities are $q_i = (a - 2c(x_i, x'_i) + c(x_j, x'_j))/(3b)$ and $q_j = (a - 2c(x_j, x'_j) + c(x_i, x'_i))/(3b)$, with equilibrium price $p = (a + c(x_i, x'_i) + c(x_j, x'_j))/3$. The profit function for each firm is

$$V(x_i, x'_i|x_j, x'_j) = \frac{(a - 2c(x_i, x'_i) + c(x_j, x'_j))^2}{9b} = \frac{(a - 2(\nu - \beta x_i x'_i) + \nu - \beta x_j x'_j)^2}{9b} \tag{34}$$

$$V(x_j, x'_j|x_i, x'_i) = \frac{(a - 2c(x_j, x'_j) + c(x_i, x'_i))^2}{9b} = \frac{(a - 2(\nu - \beta x_j x'_j) + \nu - \beta x_i x'_i)^2}{9b}. \tag{35}$$

At the matching stage, in a competitive equilibrium with PAM we have that $\mu_+(x) = x$, that is, each team (x, x) is paired with an identical team (since η is PAM too), and wages are given by $w(x) = w(\underline{x}) + \int_{\underline{x}}^{x} V_2(s, s|s, s)ds$. It follows from the profit functions above that $V_2(x, x|x, x) = -4c_2(x, x)(a - c(x, x))/(9b) > 0$. Hence, equilibrium wages are equal to

$$\begin{aligned} w(x) &= w(\underline{x}) - \frac{4}{9b} \int_{\underline{x}}^{x} c_2(s, s)(a - c(s, s))ds \\ &= w(\underline{x}) + \frac{4\beta}{9b} \int_{\underline{x}}^{x} s(a - \nu + \beta s^2)ds \\ &= w(\underline{x}) + \frac{4\beta}{9b} \left((a - \nu)\frac{x^2 - \underline{x}^2}{2} + \beta \frac{x^4 - \underline{x}^4}{4} \right), \end{aligned}$$

where the second equality uses the functional form of c, and the third follows by integration.

The following result shows that a PAM equilibrium exists and describes some equilibrium properties.

Proposition 6 *(i) If a is large enough, then there exists a competitive equilibrium with PAM. Wages increase in a and decrease in b, and firms with better composition of their labor force set higher markups.*

(ii) If a large enough the planner prefers PAM to NAM or to any convex combination of them.

In each part, the bound on a is derived in Appendix A.10, and it ensures that the downstream market demand is large enough to encourage bidding for the best workers in the first stage, and to induce the planner's preferences described in the second part of the proposition.

Regarding the properties of the competitive equilibrium in the matching stage, it is immediate that the wage function is strictly increasing and stricty convex in x, strictly increasing in a – a larger downstream market increases the incentives to hire the right workers and this drives wages up – and strictly decreasing in b – loosely, a less sensitive demand reduces the profitability of each sector, and thus of each firm in each sector, and this lowers wages as the marginal revenue of hiring someone of a better type decreases.

In the downstream interaction it is straightforward to show that, under PAM, sectors where firms have better labor force composition charge higher markups. To see this, let ϵ be the price-elasticity of demand in a given location in the Nash equilibrium of each sector. It is easy to show that at the equilibrium price and total quantity produced it is given by $\epsilon(x, x) = -0.5(a + 2c(x, x))/(a - c(x, x))$. The Lerner index ϱ of each firm is then

$$\varrho(x, x) \equiv \frac{p - c(x, x)}{p} = -\frac{1}{2\epsilon(x, x)} = \frac{a - \nu + \beta x^2}{a + 2\nu - 2\beta x^2}.$$

Intuitively, it follows from this formula that firms in sectors with better workforce composition – which are the ones with lower marginal cost of production – set higher markups. In particular, the firms with the best composition, namely, the ones with $(\overline{x}, \overline{x})$ set the highest markup.

More interestingly, note that $\varrho(x,x) \in [(a-\nu)/(a+2\nu), (a-\nu+\beta)/(a+2\nu-2\beta)]$ for all x. This interval expands when β increases, due to its impact on the marginal cost of production of firms. As a result, the spread of markups increases. As worker skills become more complementary in production (higher β), the model predicts a *higher dispersion* in markups, which also leads to higher average markups. These findings are consistent with the evidence on the evolution of markups: average markups have risen as well as the dispersion of markups.

The model provides a new insight in the literature on market power. The standard inefficiency of oligopoly due to market power is exacerbated by the complementarities in the input markets when firms choose the composition of their labor force, that is, when matching is *endogenous* and the equilibrium exhibits PAM (which is important for the properties of the distribution of markups). This insight contributes to the policy debate about the role of worker complementarities, and the labor market in general, in regulating firms with market power.

Part (*ii*) partially describes the planner's preferences under the stated conditions on the primitives: he prefers to match agents in a PAM way rather than in any convex combination of PAM and NAM. The bound on a that makes the result true depends on the primitives c and F (see Appendix A.10). It shows that within a restricted set of matching choices for the planner (PAM, NAM, or any convex combination thereof), PAM is efficient.

5.3 Policy Implications

We close this section with a stylized analysis of policy interventions. We do so in the context of sports competitions because some of the policies we analyze have been implemented. Sports competitions are inherently zero-sum contests where the competing teams exert a negative externality on each other. Because of the externality in the second stage, the team formation in the first stage will in general not be efficient.

Since teams tend to play all other teams – often multiple times – the setting with random assignment of teams is formally identical to the setting where we uniformly sum the outcomes of all realized matches.

For simplicity, we conduct our analysis using a simple instance of the binary case $x \in \{\underline{x}, \overline{x}\}$, enriched by the presence of uncertainty in the realized match output (besides the random assignment of competing teams), which is a natural feature of sports competitions. To do so, denote a team's realized performance by v, which is a positive random variable drawn from a distribution that depends on the team composition (x, x'). We will denote the mean of v conditional on (x, x') by $m(x, x')$. Consider a very simple setup where the team's (stochastic) outcome in a game against another team whose stochastic performance is v' (independent from v) is given by $z = av + bv'$ with $a \geq 0$. To capture the negative spillovers in the presence of uncertainty, it is sufficient to focus on the case where $b < 0$: better performance of the opponent generates negative spillovers. We can write the value of a team (x, x') conditional on matching with team $(\hat{x}, \hat{x}')$ as $V(x, x'|\hat{x}, \hat{x}') = (a+2b)m(x, x') + bm(\hat{x}, \hat{x}')$.[37] Since the externality is additive, one can show that there is a unique equilibrium and that the planner can focus on choosing just between PAM or NAM (these insights are independent of the sign of b). With random assignment, the expected match output for a team with composition (x, x') is, under PAM and NAM, given by

$$\begin{aligned}
\mathcal{V}(x, x'|\mu_+) &= (a+2b)m(x, x') + b\,\frac{m(\overline{x}, \overline{x}) + m(\underline{x}, \underline{x})}{2} \\
\mathcal{V}(x, x'|\mu_-) &= (a+2b)m(x, x') + b\,m(\underline{x}, \overline{x})
\end{aligned}$$

[37] Notice that this expected value need not be positive. An easy fix is to add a constant term $v_0 > 0$ large enough. Since this is irrelevant for the results we simply omit it.

The condition for the existence of a competitive equilibrium with PAM is $(a+2b)\left(m(\overline{x},\overline{x})+m(\underline{x},\underline{x})-2m(\underline{x},\overline{x})\right)\geq 0$, while for the planner to choose NAM is $(a+3b)\left(m(\overline{x},\overline{x})+m(\underline{x},\underline{x})-2m(\underline{x},\overline{x})\right)\leq 0$, which are satisfied provided $b\in\left[-\frac{a}{2},-\frac{a}{3}\right]$ and m is supermodular in X. Now we can establish the following result:

Proposition 7 *Let $z=av_i+bv_j$, with $a\geq 0$.*

1. *If $b\notin\left(-\frac{a}{2},-\frac{a}{3}\right)$, the competitive equilibrium is efficient;*
2. *If $b\in\left(-\frac{a}{2},-\frac{a}{3}\right)$, the competitive equilibrium is inefficient: if m is supermodular (submodular), the equilibrium exhibits PAM (NAM), while the planner's solution exhibits NAM (PAM).*

The intuition of this result is as follows. When $b>0$, the spillover effect is positive and both complementarities and externality go in the same direction and reinforce each other and equilibrium is efficient. If $b<0$ is negative enough, then both terms in V are negative and once again both effects reinforce each other. In turn, when $b<0$ but 'close' to zero, there is conflict between the two but complementarities dominate and thus efficiency ensues. A real conflict emerges when $b<0$ but not sufficiently so; in this case the externality effect is strong enough to offset the benefits from complementarities, and the competitive equilibrium (either PAM or NAM) is inefficient. So despite the presence of spillover effects, efficiency is only compromised on an interval of values of b.

Given the inefficiency stated in the proposition, it is a truism to wonder about policy remedies. To make the discussion more vivid, we discuss policies inspired by actual sports competitions, namely, taxes, a salary cap and a rookie draft.[38] To streamline the presentation, we have placed in Appendix A.12 the formal analysis of the assertions we make below, and we limit ourselves to discussing the findings.[39]

Taxes. If the planner wants to implement the NAM allocation, she can do so by taxing teams with composition $\overline{x}\,\overline{x}$ and $\underline{x}\,\underline{x}$ and by subsidizing teams with mixed composition. Let $\bar{t}$ be the tax on a team with one high characteristic member that hires an identical agent, and let $\overline{s}$ the subsidy if it hires a low characteristic agent. Likewise, denote by $\underline{t}$ the tax on the firm with an agent with low characteristic that hires an identical agent, and $\underline{s}$ the subsidy if it hires a high characteristic one. We consider budget balancing policies.

In Appendix A.12 we show that with sufficiently high taxes, the planner can implement the NAM allocation. What the taxes achieve is to drive a wedge between the private value of PAM and that of NAM. Not surprisingly, we find that the tax is proportional to the private surplus of a match under PAM relative to NAM.

Salary Cap. Suppose the planner imposes cap on the highest wage paid that teams cannot exceed. Denote the cap on the highest wage by C. To be interesting, we require $C\in[\underline{w},\overline{w})$. Consider a team that has to decide which two agents to hire. In a PAM equilibrium, a salary cap will not dissuade a team with two agents with $\overline{x}$ from hiring them. If anything, it can get the high characteristic agent at a lower wage. The most the cap can do is to reduce the incentives of a team with agents with $\underline{x}$ to bid for a high characteristic agent. In Appendix A.12 we show that the salary cap as described here will not change the inefficient equilibrium allocation.[40]

[38] See Palomino and Sákovics (2004) for the explicit analysis of team composition in sports competitions with competitive balance.

[39] As in the analysis above, we maintain the assumption that the planner's objective is to maximize the profits of the teams in the league. In reality, the planner may have an ulterior motive. For example, cities may subsidize teams by building stadiums (maybe because winning teams help politicians gain popularity and win elections), or teams are forced (possibly through legislation) to maximize consumer welfare in addition to profits.

[40] It should be noted though that what is known as a salary cap in the US sports competitions (NBA for example), is effectively a tax and subsidy scheme. Teams can if they want spend above a threshold total amount on salaries for the players, but for each dollar spent above the wage cap, an additional dollar must be paid to NBA who distributes it amongst all the teams. Effectively that means we have a policy in place with taxes (see above), with the caveat that in case where a tax is effectively paid, it is then distributed amongst the other teams.

Rookie Draft. Assume that the weakest teams can first pick the best players at a given, exogenously set wage rate. This set wage should be below the equilibrium wage to have any effect. Also, there must be a seniority difference amongst players. To address this issue in our model, suppose that there are two subgroups of players, seniors and rookies. For each subgroup, assume an equal number of agents with high and low characteristics.

We can interpret the problem as each senior hiring one rookie. Since teams in our model are very stylized, we assume that low characteristic seniors have the first pick in the rookie draft, given some exogenously set wages. Each of them chooses between $\underline{x}$ and $\overline{x}$, taking as given wages $\underline{w}_d$ and $\overline{w}_d$ (where d stands for draft).

In Appendix A.12, we derive an equilibrium whose outcome is that low characteristic seniors choose a high characteristic rookie in the draft. Thus, high characteristic seniors have no choice but to pick a rookie with a low characteristic. Moreover, we show that wages $\underline{w}_d$ and $\overline{w}_d$ can be chosen in a way that leaves both types of seniors better off under NAM than under the original equilibrium with PAM.[41] In other words, the rookie draft can implement the efficient NAM allocation even if the equilibrium outcome (without a rookie draft) is PAM.

6 Conclusion

In many market settings, the presumption that firms and teams operate in isolated output markets is tenuous. Often, there is strategic interaction between competing teams, for example due to knowledge spillovers, market power or patents. This generates externalities and has implications for the labor market. While it is well known that the inefficiencies in the output market affect the optimal provision of effort, in this paper, we argue that they also affect the composition of skilled workers in teams.

Our analysis reveals that the competitive equilibrium features of matching with externalities differ in several dimensions from the standard matching model that is a workhorse in economic applications. In particular, we show that there can be multiple equilibria with varying sorting patterns; both optimal and equilibrium matching can involve randomization; equilibrium can be inefficient with a matching that can drastically deviate from the optimal one; and match complementarities interact with externalities to determine sorting.

We derive these results under different assumptions regarding heterogeneity of agents (finite number of characteristics or a continuum), and also under different forms in which teams can compete downstream (aggregate spillovers, random asignment, and deterministic assignment of teams). Hence, our model encompasses a large number of economic applications of matching.

In addition to these insights, we argue that our framework is economically relevant. We show that a version of the model with general knowledge spillovers can account for the recent observed empirical fact that the rise in wage inequality is mainly driven by between-firm inequality rather than within-firm inequality. Our model combines the effect of general knowledge spillovers on the firm size distribution with within-firm complementarities, which generates increased between-firm sorting, and hence the predicted effect on between- and within-firm wage inequality. We believe this economic insight is novel and important. We also show how sorting can affect markups in an oligopolistic output market, thus providing a rationale for the empirically observed evolution of the distribution of markups. And we illustrate some policy interventions in the labor market that can correct the externalities from the output market, using a stylized sports competition version of the model.

Although there are many open questions for future research, we only mention three that seem important. Obviously, it would be interesting to have a full characterization of the planner's problem and competitive equi-

[41] In real world applications such as the NBA draft, rookie wages are set by the NBA and not by the market. So by checking that both types of seniors prefer the draft to the original PAM allocation, we are ensuring that no senior will vote against the change.

librium with stochastic matching when there is a continuum of characteristics. Also, analyzing the model under imperfectly transferable utility would enlarge the number of economic applications (e.g., contracting problems in market settings) that can be analyzed with externalities. Finally, we have abstracted from search frictions that are important in some labor markets, and that would be interesting to incorporate in the formation of teams.

We can also see at least two directions for empirical work. First, with sufficiently detailed data on team composition (research teams, sports teams, class rooms, etc.) and individual performance, one could identify the nature of the externalities in conjunction with the nature of the complementarities. Clearly, if a market setting is estimated with a model without externalities, the obtained estimates for complementarities will be biased. Second, the model can be estimated using wage data. Wages reflect the allocation and if the allocation is inefficient, this will be evident in the wage distribution. Even if the allocation is efficient (say PAM in equilibrium as well as PAM by the planner), wages nonetheless will incorporate the inefficiency and will not be set at private marginal product. Data on markups in output markets for example will therefore give an indication of the extent of the externality, and as a consequence of the extent to which wages are set inefficiently.

Appendix A

A.1 Proof of Proposition 1

We will first prove the result for the case of ex-post random assignment of competing teams and aggregate spillovers (cases 1.i) and 2) in Section 3.3), since in these cases $\mathcal{V}(\cdot,\cdot|\alpha)$ is symmetric in (x,x').

Assume $\Gamma(1)\geq 0$. We will construct an equilibrium with PAM. Inequalities (7)–(8) reveal that we need to find $\overline{w}$ and $\underline{w}$ that satisfy them. Set $\overline{w}=0.5\mathcal{V}\left(\overline{x},\overline{x}|1\right)$ and $\underline{w}=0.5\mathcal{V}\left(\underline{x},\underline{x}|1\right)$. These wages satisfy (7)–(8) and yield a nonnegative payoff to both $\underline{x}$ and $\overline{x}$. Hence, we have constructed a competitive equilibrium with PAM.

If $\Gamma(0)\leq 0$, then to construct an equilibrium with NAM we will set wages that satisfy inequalities (9)–(10), yield nonnegative payoffs to both $\underline{x}$ and $\overline{x}$, and are such that $\overline{w}+\underline{w}=\mathcal{V}(\underline{x},\overline{x}|0)$. If $\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\underline{x},\underline{x}|0)\geq 0$, then it is easy to verify $\underline{w}=0.5\mathcal{V}\left(\underline{x},\underline{x}|0\right)$ and $\overline{w}=\mathcal{V}(\underline{x},\overline{x}|0)-\underline{w}$ satisfy (9)–(10) and provide a nonnegative payoff to both $\underline{x}$ and $\overline{x}$. If $\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\underline{x},\underline{x}|0)<0$, then $\overline{w}=0.5\mathcal{V}\left(\overline{x},\overline{x}|0\right)$ and $\underline{w}=\mathcal{V}(\underline{x},\overline{x}|0)-\overline{w}$ do the job. Thus, in each case these wages along with $\alpha=0$ constitute a competitive equilibrium with NAM.

Assume that $\Gamma(\alpha)=0$ for some $0<\alpha<1$. Then (11)–(12) imply $\overline{w}-\underline{w}=\mathcal{V}(\overline{x},\overline{x}|\alpha)-\mathcal{V}(\overline{x},\underline{x}|\alpha)=\mathcal{V}(\underline{x},\overline{x}|\alpha)-\mathcal{V}(\underline{x},\underline{x}|\alpha)$. If $\overline{w}=0.5\mathcal{V}(\overline{x},\overline{x}|\alpha)$ and $\underline{w}=0.5\mathcal{V}(\underline{x},\underline{x}|\alpha)$, then the incentive constraints are satisfied with equality (using that under random assignment or aggregate spillovers $\mathcal{V}(\overline{x},\underline{x}|\alpha)=\mathcal{V}(\underline{x},\overline{x}|\alpha)$) and an agent with $\underline{x}$ receive the same nonnegative payoff if the agent hires another $\underline{x}$ or an $\overline{x}$, and similarly for an agent with $\overline{x}$. Hence, these wages along with the interior matching α constitute a competitive equilibrium.

Consider now ex-ante deterministic assignment of teams (case 1.ii) in Section 3.3) where η is PAM.

First we examine the case where workers are matched according to PAM. The incentive constraints are

$$V\left(\overline{x},\overline{x}|\overline{x},\overline{x}\right)-\overline{w}\geq V(\overline{x},\underline{x}|\overline{x},\overline{x})-\underline{w} \tag{36}$$

$$V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)-\underline{w}\geq V(\underline{x},\overline{x}|\underline{x},\underline{x})-\overline{w}, \tag{37}$$

and the necessary condition $\Gamma(1)\geq 0$ is now $V\left(\overline{x},\overline{x}|\overline{x},\overline{x}\right)+V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)-V(\overline{x},\underline{x}|\overline{x},\overline{x})-V(\underline{x},\overline{x}|\underline{x},\underline{x})\geq 0$. If $V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)\geq 0$, then it is easy to verify that $\overline{w}=\underline{w}+V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)$ and any $\underline{w}\in[0,\min\{V(\underline{x},\underline{x}|\underline{x},\underline{x}),V(\overline{x},\underline{x}|\overline{x},\overline{x})\}]$ satisfy (36)–(37) and yields nonnegative payoffs to both $\underline{x}$ and $\overline{x}$. If $V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)<0$ and $V\left(\overline{x},\overline{x}|\overline{x},\overline{x}\right)-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)\geq 0$, then $\overline{w}=V\left(\overline{x},\overline{x}|\overline{x},\overline{x}\right)-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)$ and $\underline{w}=0$ satisfy all the constraints. And if $V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)<0$ and $V\left(\overline{x},\overline{x}|\overline{x},\overline{x}\right)-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)<0$, then $\underline{w}=\overline{w}+V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)$ and any $\overline{w}\in[0,\min\{V(\overline{x},\overline{x}|\overline{x},\overline{x}),V(\underline{x},\overline{x}|\underline{x},\underline{x})\}]$ satisfy all the constraints. Hence, a competitive equilibrium with PAM exists when $\Gamma(1)\geq 0$.

Consider now a first stage matching according to NAM. Then the incentive constraints are

$$V(\overline{x},\underline{x}|\overline{x},\underline{x})-\underline{w}\geq V\left(\overline{x},\overline{x}|\overline{x},\underline{x}\right)-\overline{w}$$

$$V(\underline{x},\overline{x}|\underline{x},\overline{x})-\overline{w}\geq V\left(\underline{x},\underline{x}|\underline{x},\overline{x}\right)-\underline{w},$$

and the necessary condition $\Gamma(0)\leq 0$ becomes $V\left(\overline{x},\overline{x}|\overline{x},\underline{x}\right)+V\left(\underline{x},\underline{x}|\underline{x},\overline{x}\right)-V\left(\overline{x},\overline{x}|\overline{x},\underline{x}\right)-V(\underline{x},\overline{x}|\underline{x},\overline{x})\leq 0$. Since every team competes with a mixed team in this case, and the function V is symmetric in its first and second argument as well as in its third and fourth, it follows that the analysis of the NAM case is analogous to the one above for random assignment and aggregate spillovers: simply replace $\mathcal{V}(x,x'|0)$ by $V(x,x'|\underline{x},\overline{x})$ in the construction of the wages. Hence, a competitive equilibrium with NAM exists in this case if $\Gamma(0)\leq 0$.

Suppose now that $\Gamma(\alpha)=0$ for some $0<\alpha<1$, where $\Gamma(\alpha)=\mathcal{V}(\overline{x},\overline{x}|\alpha)+\mathcal{V}(\underline{x},\overline{x}|\alpha)-\mathcal{V}(\overline{x},\underline{x}|\alpha)-\mathcal{V}(\underline{x},\overline{x}|\alpha)$, and $\mathcal{V}(\overline{x},\overline{x}|\alpha) = \alpha V(\overline{x},\overline{x}|\overline{x},\overline{x}) + (1-\alpha)V(\overline{x},\overline{x}|\overline{x},\underline{x})$, $\mathcal{V}(\underline{x},\underline{x}|\alpha) = \alpha V(\underline{x},\underline{x}|\underline{x},\underline{x}) + (1-\alpha)V(\underline{x},\underline{x}|\underline{x},\overline{x})$, $\mathcal{V}(\overline{x},\underline{x}|\alpha) = \alpha V(\overline{x},\underline{x}|\overline{x},\overline{x}) + (1-\alpha)V(\overline{x},\underline{x}|\overline{x},\underline{x})$, and $\mathcal{V}(\underline{x},\overline{x}|\alpha) = \alpha V(\underline{x},\overline{x}|\underline{x},\underline{x}) + (1-\alpha)V(\underline{x},\overline{x}|\underline{x},\overline{x})$. From the incentive constraints of agents of each characteristic, which must hold with equality for them to be willing to randomize, it follows that $\overline{w} - \underline{w} = \mathcal{V}(\overline{x},\overline{x}|\alpha) - \mathcal{V}(\overline{x},\underline{x}|\alpha) = \mathcal{V}(\underline{x},\overline{x}|\alpha) - \mathcal{V}(\underline{x},\underline{x}|\alpha)$. If these differences are nonnegative, then $\overline{w} = \underline{w} + \mathcal{V}(\overline{x},\overline{x}|\alpha) - V(\overline{x},\underline{x}|\alpha) = \underline{w} + \mathcal{V}(\underline{x},\overline{x}|\alpha) - \mathcal{V}(\underline{x},\underline{x}|\alpha)$ and $\underline{w} \in [0, \min\{\mathcal{V}(\underline{x},\underline{x}|\alpha), \mathcal{V}(\overline{x},\underline{x}|\alpha)\}]$ satisfy all the constraints and yield the same nonegative payoff to an agent with $\underline{x}$ no matter who the agent ends up hiring, and similarly for an agent with $\overline{x}$. If instead these differences are negative, then set $\underline{w} = \overline{w} + V(\overline{x},\underline{x}|\alpha) - \mathcal{V}(\overline{x},\overline{x}|\alpha) = \overline{w} + \mathcal{V}(\underline{x},\underline{x}|\alpha) - \mathcal{V}(\underline{x},\overline{x}|\alpha)$ and $\overline{w} \in [0, \min\{\mathcal{V}(\overline{x},\overline{x}|\alpha), \mathcal{V}(\underline{x},\overline{x}|\alpha)\}]$. In each case, these wages along with matching $\alpha \in (0,1)$ constitute a competitive equilibrium with stochastic matching.

We have thus proven that if either $\Gamma(1) \geq 0$ or $\Gamma(0) \leq 0$, then a competitive equilibrium exists. The only case remaining is $\Gamma(1) < 0$ and $\Gamma(0) > 0$. Since Γ is continuous in α, it follows from the Intermediate Value Theorem there is an $\alpha \in (0,1)$ such that $\Gamma(\alpha) = 0$, so a competitive equilibrium exists by the construction above. □

A.2 Proof of Proposition 2

(*i*) The objective function is convex if $A \geq 0$. If it is strictly convex (or linear with $B \neq 0$), then the optimal solution is at a corner, so $\alpha^P \in \{0,1\}$, and which corner depends on whether $0.5(A+B)+C = 0.5(\mathcal{V}(\overline{x},\overline{x}|1) + \mathcal{V}(\underline{x},\underline{x}|0))$ is bigger or smaller than $C = 0.5(\mathcal{V}(\underline{x},\overline{x}|0) + \mathcal{V}(\overline{x},\underline{x}|0))$, that is, the comparison of the value of the aggregate expected output under PAM and NAM. This reduces to $A + B$ bigger than or less than zero.

(*ii*)–(*iii*) A necessary condition for an interior solution is that $A < 0$, so the planner's objective is strictly concave. But this is not sufficient since the solution can still be at a corner. If $B \leq 0$, then the planner's objective peaks at $\alpha^P = 0$ and NAM is optimal, while if $B + 2A \geq 0$ then it peaks at $\alpha^P = 1$ and PAM is optimal.

(*iv*) If $A < 0$, $B > 0$, and $B + 2A < 0$, then the planner's objective function is strictly concave and peaks at the interior value $\alpha^P = -B/2A$. Hence, an interior matching is optimal. □

A.3 The Ternary Case

In this section we will describe the model with three characteristics, derive the incentive constraints that define competitive equilibrium and set up the planner's problem. The analysis suggest that the main insights obtained in the binary case carry over to this case, especially regarding the possibility of a competitive equilibrium with stochastic matching, as well as multiplicity and inefficiency of competitive equilibrium. We will illustrate the results using a couple of examples similar to the ones presented in Section 2.

Assume that $x \in \{\underline{x}, \hat{x}, \overline{x}\}$, with $\underline{x} < \hat{x} < \overline{x}$, uniformly distributed (that is, a measure 1/3 of the agents has characteristic $\underline{x}$, $\hat{x}$, $\overline{x}$, respectively). In this setting there are four possible (deterministic) matchings to form a measure 1/2 of teams: μ_1, where a measure 1/6 of teams have composition $\underline{x}\,\underline{x}$, 1/6 have $\hat{x}\hat{x}$, and 1/6 have $\overline{x}\,\overline{x}$; μ_2, where a measure 2/6 of teams have composition $\underline{x}\overline{x}$ and 1/6 have $\hat{x}\hat{x}$; μ_3, where a measure 2/6 of teams have composition $\underline{x}\hat{x}$ and 1/6 have $\overline{x}\,\overline{x}$; and μ_4, where a measure 1/6 of teams have composition $\underline{x}\,\underline{x}$ and 2/6 have $\hat{x}\overline{x}$.

A stochastic matching is a vector $\pi = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_i \in [0,1]$, $i = 1,2,3,4$, $\sum_i \alpha_i = 1$, where α_i is the fraction of the population that matches according to μ_i, $i = 1,2,3,4$. Under π, there is a fraction $(\alpha_1 + \alpha_4)/6$ of teams $\underline{x}\,\underline{x}$, $(\alpha_1 + \alpha_2)/6$ with $\hat{x}\hat{x}$, $(\alpha_1 + \alpha_3)/6$ with $\overline{x}\,\overline{x}$, $\alpha_2/6$ with $\underline{x}\,\overline{x}$, $\alpha_3/6$ with $\underline{x}\,\hat{x}$, and $\alpha_4/6$ with $\hat{x}\,\overline{x}$.

Let $\mathcal{V}(x, x'|\pi)$ be the expected match output of a team with composition xx' when the matching is π, and similary, $\mathcal{V}(x, x'|\mu_i)$ is the corresponding expected output when the matching is μ_i, $i = 1,2,3,4$. For example, in the case of random assignment of competing teams, we have,

$$\mathcal{V}(x,x'|\pi) = \frac{\alpha_1+\alpha_4}{3}\mathcal{V}(x,x'|\underline{x},\underline{x}) + \frac{\alpha_1+\alpha_2}{3}\mathcal{V}(x,x'|\hat{x},\hat{x}) + \frac{\alpha_1+\alpha_3}{3}\mathcal{V}(x,x'|\overline{x},\overline{x}) + \frac{2\alpha_2}{3}\mathcal{V}(x,x'|\underline{x},\overline{x}) + \frac{2\alpha_3}{3}\mathcal{V}(x,x'|\underline{x},\hat{x}) + \frac{2\alpha_4}{3}\mathcal{V}(x,x'|\hat{x},\overline{x}) \tag{38}$$

while with aggregate spillovers and multiplicatively separable match output, we have $\mathcal{V}(x,x'|\pi) = \ell(\pi)k(x,x')$, where we could allow ℓ to vary with team composition, as in Section 5.1.

To derive a competitive equilibrium, we need to specify a matching and wages $\underline{w}$, $\hat{w}$, and $\overline{w}$, corresponding with hiring a partner with $\underline{x}$, $\hat{x}$, and $\overline{x}$. A competitive equilibrium with deterministic matching μ_i must satisfy the appropriate incentive constraints. For instance, if the matching is μ_1, the incentive constraints are

$$\mathcal{V}(\underline{x},\underline{x}|\mu_1) - \underline{w} \geq \mathcal{V}(\underline{x},\hat{x}|\mu_1) - \hat{w} \tag{39}$$

$$\mathcal{V}(\underline{x},\underline{x}|\mu_1) - \underline{w} \geq \mathcal{V}(\underline{x},\overline{x}|\mu_1) - \overline{w} \tag{40}$$

$$\mathcal{V}(\hat{x},\hat{x}|\mu_1) - \hat{w} \geq \mathcal{V}(\hat{x},\underline{x}|\mu_1) - \underline{w} \tag{41}$$

$$\mathcal{V}(\hat{x},\hat{x}|\mu_1) - \hat{w} \geq \mathcal{V}(\hat{x},\overline{x}|\mu_1) - \overline{w} \tag{42}$$

$$\mathcal{V}(\overline{x},\overline{x}|\mu_1) - \overline{w} \geq \mathcal{V}(\overline{x},\underline{x}|\mu_1) - \underline{w} \tag{43}$$

$$\mathcal{V}(\overline{x},\overline{x}|\mu_1) - \overline{w} \geq \mathcal{V}(\overline{x},\hat{x}|\mu_1) - \hat{w}. \tag{44}$$

and similarly for matchings μ_2, μ_3, and μ_4. As in the binary case under PAM, one can show that with μ_1, $\mathcal{V}(\cdot,\cdot|\mu_1)$ supermodular in (x,x') is necessary and sufficient for the existence of a competitive equilibrium. Similarly, with μ_2 the condition is $\mathcal{V}(\cdot,\cdot|\mu_2)$ submodular in (x,x'). As in the multiplicative example in the binary case, it is easy to construct a similar example with a PAM or NAM equilibrium and also illustrate that both can coexist if the aggregate externality switches the complementarity property of $\mathcal{V}$. We omit the details to avoid repetition.

Regarding a stochastic matching π, agents must be given incentives to randomize and thus the relevant incentive constraints must hold with equality. In particular, a competitive equilibrium with $\pi \gg 0$ requires that all the inequalities above hold as equalities, and this obtains if and only if $\mathcal{V}(\cdot,\cdot|\pi)$ is modular in (x,x'). Indeed, any interior solution $\pi \gg 0$ that solves the system of four equations in four unknowns (α_i, $i = 1,2,3,4$)

$$\mathcal{V}(\underline{x},\underline{x}|\pi) + \mathcal{V}(\hat{x},\hat{x}|\pi) = 2\mathcal{V}(\underline{x},\hat{x}|\pi)$$

$$\mathcal{V}(\underline{x},\underline{x}|\pi) + \mathcal{V}(\overline{x},\overline{x}|\pi) = 2\mathcal{V}(\underline{x},\overline{x}|\pi)$$

$$\mathcal{V}(\hat{x},\hat{x}|\pi) + \mathcal{V}(\overline{x},\overline{x}|\pi) = 2\mathcal{V}(\hat{x},\overline{x}|\pi)$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1,$$

is a competitive equilibrium with appropriately chosen wages.

For a simple example, assume random assignment of competing teams and also that a team obtains a payoff 1 if it is assigned to an identical team and 0 otherwise. We first claim that $\alpha_1 = 1$, and thus matching μ_1, is a competitive equilibrium with wages $(\underline{w},\hat{w},\overline{w}) = (\frac{1}{6},\frac{1}{6},\frac{1}{6})$. To see this, notice that $\mathcal{V}(\cdot,\cdot|\mu_1)$ is strictly supermodular. Even easier, notice that all the inequalities (39)–(44) are slack with the wages assumed (the left side of each is $1/3 - 1/6 = 1/6$ and the right side is $-1/6$). Thus, a competitive equilibrium with PAM exists. We next claim that $\pi = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ along with wages $(\underline{w},\hat{w},\overline{w}) = (\frac{1}{12},\frac{1}{12},\frac{1}{12})$ is a competitive equilibrium with stochastic matching. To see this, notice that under π the probability of being assigned to a team of the same composition is 1/6 for any team. Hence, π solves the system of equations above, and the wages (half of the output for each member) ensures that each agent is indifferent regarding whom to hire and thus willing to randomize. As a result, a competitive equilibrium with *stochastic matching* ensues. Since this equilibrium coexists with the PAM one, it follows that there are *multiple* competitive equilibria.

The planner's problem is as follows:

$$\max_{\pi} \frac{1}{2}\left(\frac{\alpha_1+\alpha_4}{3}\mathcal{V}(\underline{x},\underline{x}|\pi)+\frac{\alpha_1+\alpha_2}{3}\mathcal{V}(\hat{x},\hat{x}|\pi)+\frac{\alpha_1+\alpha_3}{3}\mathcal{V}(\overline{x},\overline{x}|\pi)+\frac{2\alpha_2}{3}\mathcal{V}(\underline{x},\overline{x}|\pi)+\frac{2\alpha_3}{3}\mathcal{V}(\underline{x},\hat{x}|\pi)+\frac{2\alpha_4}{3}\mathcal{V}(\hat{x},\overline{x}|\pi)\right),$$

subject to $\alpha_i \in [0,1]$, $i = 1,2,3,4$, and $\sum_i \alpha_i = 1$. If the objective function is strictly concave or strictly quasiconcave and there is an interior π that solve the first-order conditions with respect to α_i, $i = 1,2,3,4$, then this is the optimal matching. In some cases it is relatively easy to pin down the curvature of the planner's objective function. This is the case when competing teams are assigned randomly and $\mathcal{V}$ is given by (1). This is because the objective function becomes a quadratic function in π, and hence it is easy to check if it is concave or convex by evaluating the Hessian. The function is strictly concave if the Hessian is negative definite, and thus if the first-order conditions have an interior solution, then it is the efficient matching. Similarly for corner solutions and the strictly convex case. Another tractable case (both for the analysis of competitive equilibria and for the planner's problem), which leads to a quadratic objective function for the planner, is when there are aggregate spillovers and they are linear and multiplicative, and different for different team composition. For instance, let match output be a product of a function of (x, x') and a linear term that depends on the mass of teams whose composition is not (x, x') (similar to the case in Section 5.1), such as $\mathcal{V}(\underline{x},\underline{x}|\pi) = (1-((\alpha_1+\alpha_4)/3))k(\underline{x},\underline{x})$, etc., or the mass that is (x, x') such as $\mathcal{V}(\underline{x},\underline{x}|\pi) = (\alpha_1+\alpha_4)/3)k(\underline{x},\underline{x})$, etc.

For an illustration of the planner's problem, consider the example above with random matching in which a team obtains a payoff 1 if it is assigned to an identical team and 0 otherwise. The planner's objective function is

$$\frac{1}{2}\left(\left(\frac{\alpha_1+\alpha_4}{3}\right)^2+\left(\frac{\alpha_1+\alpha_2}{3}\right)^2+\left(\frac{\alpha_1+\alpha_3}{3}\right)^2+\frac{4\alpha_2^2}{9}+\frac{4\alpha_3^2}{9}+\frac{4\alpha_4^2}{9}\right),$$

which is clearly strictly convex and thus the optimal matching is at a corner. If $\alpha_1 = 1$, then the planner obtains 3/18, while if either α_2, α_3, or α_4 is equal to 1, then the planner obtains 5/18. Hence, a corner solution with either $\alpha_2 = 1$, $\alpha_3 = 1$, or $\alpha_4 = 1$ is optimal.[42] Since in this case there are competitive equilibria with stochastic matching and with PAM , it follows that both are *inefficient*.

In short, this example shows that with three types, multiple competitive equilibria can exist, can be inefficient, and can also entail stochastic matching. Note that the example is generalizable beyond the ternary case.

Besides suggesting that all the insights derived in the binary case extend to the ternary case, the main takeaway from this section is that moving from two to a larger but finite number of characteristics presents mainly combinatorial difficulties without providing new insights.[43] For this reason, we consider in the main body of the paper the binary case and the one with a continuum of characteristics, which affords a neat calculus-based approach to analyze the competitive equilibria of the model.

[42] For a strictly concave example, suppose instead that a team obtains a payoff 1 if assigned to a competing team of different composition. Then it is easy to show that the unique efficient matching is interior with $\pi = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$.

[43] To see some of the combinatorial hurdles, notice that one needs to calculate all the possible partitions of the population subject to the constraint that all agents with the same characteristic are matched to the same type of partner, so as to preserve equal treatment, and then calculate the distribution of different team compositions. With random matching, however, it is still true that the planner's problem will be a nicely behaved quadratic optimization program, for which efficient algorithms exist.

A.4 Proof of Proposition 4

Under these conditions, the planner's objective in the restricted problem is strictly concave in α and it is strictly increasing at $\alpha = 0$ and strictly decreasing at $\alpha = 1$. Thus, in the restricted problem the planner does not choose PAM or NAM. All the more when we allow the planner to choose among all posible matchings. $\square$

A.5 Endogenous Assignment of Competing Teams

Assume that, after the first stage where teams form, in the second stage teams *choose* the other team to compete with. One way to think about it is as a competitive equilibrium in the second stage where teams take as given the 'price of acquiring a competitor' of a given composition. Intuition suggests that this is now a standard Becker-like matching-among-teams problem, and thus the second-stage competitive equilibrium will be efficient (there was a 'missing market' that now has been allowed to open). But then in the first stage agents anticipate the equilibrium sorting pattern in the second stage, and hence they know that whoever they choose as a partner affects the incentives to choose a competing team in the second stage. In this way they can internalize the matching externalities, and this can lead to efficiency in the first stage as well.

Proving this assertion in general is beyond the scope of this paper, since it would require to have available a full solution for the planner's problem with a continuum of types. But we can use our binary setup to prove one case in detail and then describe how to prove the other cases (which proceed along the same lines).

As in Section 4.1, there is a measure one of agents, half of them with characteristic $\underline{x}$ and half with $\overline{x}$. Assume that matching in the first stage is stochastic and given by $\alpha \in (0, 1)$. That is, at the beginning of the second stage there is a measure of 1/2 of teams, $\alpha/2$ of them has composition $\overline{x}\,\overline{x}$, $\alpha/2$ has composition $\underline{x}\,\underline{x}$, and $1 - \alpha$ has composition $\underline{x}\,\overline{x}$. Assume that the 'price' of each of these teams is $\overline{t}$, $\underline{t}$, and $\hat{t}$, respectively, and that each team wants to 'acquire' a competitor taking these prices as given. That is, there is now a market in which the teams endogenously choose whom to compete with.

For definiteness, we consider a PAM assignment in the second stage. A team with composition $\overline{x}\,\overline{x}$ is willing to match with another team of the same composition if and only if

$$2V(\overline{x}, \overline{x}|\overline{x}, \overline{x}) - \overline{t} \geq V(\overline{x}, \overline{x}|\underline{x}, \overline{x}) + V(\underline{x}, \overline{x}|\overline{x}, \overline{x}) - \hat{t} \tag{45}$$

$$2V(\overline{x}, \overline{x}|\overline{x}, \overline{x}) - \overline{t} \geq V(\overline{x}, \overline{x}|\underline{x}, \underline{x}) + V(\underline{x}, \underline{x}|\overline{x}, \overline{x}) - \underline{t}. \tag{46}$$

The explanation of (45) is as follows: the total output that the competing teams generate is the sum of the outputs of each of them. So if both teams have composition $\overline{x}\,\overline{x}$, then total output is $2V(\overline{x}, \overline{x}|\overline{x}, \overline{x})$. If instead a team with $\overline{x}\,\overline{x}$ matches with one with $\underline{x}\,\overline{x}$, total output is $V(\overline{x}, \overline{x}|\underline{x}, \overline{x}) + V(\underline{x}, \overline{x}|\overline{x}, \overline{x})$. A similar explanation applies to the (46), which compares the gain of matching with another $\overline{x}\,\overline{x}$ with that of matching with $\underline{x}\,\underline{x}$.

For teams with $\underline{x}\,\overline{x}$ and $\underline{x}\,\underline{x}$ the incentive constraints are

$$2V(\underline{x}, \overline{x}|\underline{x}, \overline{x}) - \hat{t} \geq V(\underline{x}, \overline{x}|\overline{x}, \overline{x}) + V(\overline{x}, \overline{x}|\underline{x}, \overline{x}) - \overline{t} \tag{47}$$

$$2V(\underline{x}, \overline{x}|\underline{x}, \overline{x}) - \hat{t} \geq V(\underline{x}, \overline{x}|\underline{x}, \underline{x}) + V(\underline{x}, \underline{x}|\underline{x}, \overline{x}) - \underline{t} \tag{48}$$

$$2V(\underline{x}, \underline{x}|\underline{x}, \underline{x}) - \underline{t} \geq V(\underline{x}, \underline{x}|\overline{x}, \overline{x}) + V(\overline{x}, \overline{x}|\underline{x}, \underline{x}) - \overline{t} \tag{49}$$

$$2V(\underline{x}, \underline{x}|\underline{x}, \underline{x}) - \underline{t} \geq V(\underline{x}, \underline{x}|\underline{x}, \overline{x}) + V(\underline{x}, \overline{x}|\underline{x}, \underline{x}) - \hat{t} \tag{50}$$

Proceeding in an analogous way as in Section 4.1 (e.g., adding incentive constraints (45) and (47), etc.), one can show that PAM ensues if and only if the following conditions hold:

$$V(\overline{x},\overline{x}|\overline{x},\overline{x}) + V(\underline{x},\overline{x}|\underline{x},\overline{x}) \geq V(\overline{x},\overline{x}|\underline{x},\overline{x}) + V(\underline{x},\overline{x}|\overline{x},\overline{x})$$
$$V(\underline{x},\overline{x}|\underline{x},\overline{x}) + V(\underline{x},\underline{x}|\underline{x},\underline{x}) \geq V(\overline{x},\overline{x}|\underline{x},\underline{x}) + V(\underline{x},\underline{x}|\overline{x},\overline{x})$$
$$V(\underline{x},\overline{x}|\underline{x},\overline{x}) + V(\underline{x},\underline{x}|\overline{x},\overline{x}) \geq V(\underline{x},\overline{x}|\underline{x},\underline{x}) + V(\underline{x},\underline{x}|\underline{x},\overline{x}),$$

which is a supermodularity condition on $V(\cdot|\cdot)$ on the vectors $(\underline{x},\underline{x}) < (\underline{x},\overline{x}) < (\overline{x},\overline{x})$.

Assuming the PAM assignment in the second stage, the condition for a stochastic matching α in the first stage to be part of a competitive equilibrium is that each agent is indifferent between matching with an agent of the same or a different characteristic at wages $\underline{w}$ and $\overline{w}$, *taking into account that the assignment is PAM in the second stage.* Formally,

$$V(\overline{x},\overline{x}|\overline{x},\overline{x}) - \overline{w} = V(\underline{x},\overline{x}|\underline{x},\overline{x}) - \underline{w}$$
$$V(\underline{x},\underline{x}|\underline{x},\underline{x}) - \underline{w} = V(\underline{x},\overline{x}|\underline{x},\overline{x}) - \overline{w},$$

from which it follows that a necessary and sufficient condition for a competitive equilibrium with stochastic matching α is that V satisfies

$$V(\overline{x},\overline{x}|\overline{x},\overline{x}) + V(\underline{x},\underline{x}|\underline{x},\underline{x}) = 2V(\underline{x},\overline{x}|\underline{x},\overline{x}).$$

But this is *precisely* the condition for the planner – who can only alter matching in the first stage of team formation and takes the second stage assignment as given – to be indifferent between PAM and NAM and any mixture of the two. Thus, the competitive equilibrium with stochastic matching α is efficient in this case. For an illustration, it is immediate to check that all these conditions are satisfied if, as in the motivating example in Section 2, teams get a payoff 1 if they match with an identical team and zero otherwise.

A similar analysis can be done for the other combinations of matchings in the first stage and assignments in the second stage, such as PAM in the first stage when teams are formed and PAM in the second stage when competitors endogenously are assigned, or NAM in the first and in the second stage, etc.[44]

A.6 Proof of Lemma 1

We first show that $0 < \underline{\kappa} < \hat{\kappa} < \overline{\kappa}$. Note that $\overline{\kappa} > \hat{\kappa}$ if and only if $\overline{X}^{\gamma}\lambda > \hat{X}^{\gamma}(\lambda + (\alpha/2))$, which rearranges to $\overline{X} > (1 + (\alpha/(2\lambda)))^{\frac{1}{\gamma}}\hat{X}$. Since $\overline{X} = 2\overline{x}$ and $\hat{X} = \underline{x} + \overline{x}$, $\overline{k} > \hat{k}$ if and only if

$$\frac{\underline{x}}{\overline{x}} < \frac{2 - (1 + \frac{\alpha}{2\lambda})^{\frac{1}{\gamma}}}{(1 + \frac{\alpha}{2\lambda})^{\frac{1}{\gamma}}}. \tag{51}$$

If $\lambda \geq 1$ and $\gamma \geq 1$, the right side of the inequality is smallest at $\lambda = \gamma = \alpha = 1$, and it equals $1/3$. Similarly, $\hat{\kappa} > \underline{\kappa}$ if and only if $\hat{X}^{\gamma}(\lambda + (\alpha/2)) > \underline{X}^{\gamma}(\lambda + 1 - (\alpha/2))$, which rearranges to

$$\frac{\underline{x}}{\overline{x}} < \frac{1}{2\left(\frac{\lambda+1-\frac{\alpha}{2}}{\lambda+\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}} - 1}. \tag{52}$$

[44] The only subtlety in these 'corners' is that agents need to conjecture that there will be a positive mass of every team composition in the second stage to make sense of the 'right-hand side' of the incentive constraints in the first stage such as, for example, $V(\overline{x},\overline{x}|\overline{x},\overline{x}) - \overline{w} \geq V(\underline{x},\overline{x}|\underline{x},\overline{x}) - \underline{w}$ (since if PAM ensues in the first stage there will be no teams of composition $\underline{x}\,\overline{x}$ in the second). This can be formally justified via perturbations or trembles.

If $\lambda \geq 1$ and $\gamma \geq 1$, then the right side is smallest when $\lambda = \gamma = 1$ and $\alpha = 0$, in which case it becomes 1/3. Hence, $\underline{\kappa} < \hat{\kappa} < \overline{\kappa}$, and it is obvious that $\underline{\kappa} > 0$.

To show that these are equilibrium choices in the second stage, consider the choice of $\underline{\kappa}$ for a team with composition $\underline{X}$. Note that $\underline{\kappa} = \text{argmax}_k(\lambda + 1 - (\alpha/2)) - (k^2/2\underline{X}^\gamma)$; since the objective is strictly concave in k and $\underline{\kappa} < \hat{\kappa}$, it follows that $\underline{\kappa}$ yields a higher payoff to $\underline{X}$ than $\hat{\kappa}$ in the case where the spillover is $1 - (\alpha/2)$. But then all the more $\underline{\kappa}$ dominates $\hat{\kappa}$ when the spillover under $\hat{\kappa}$ is $1 - (\alpha/2) - (1 - \alpha) < 1 - (\alpha/2)$. Similarly, $\underline{\kappa}$ is a better choice for $\underline{X}$ than $\overline{\kappa}$ when the spillover is $1 - (\alpha/2)$, and thus it continues to dominate it if the spillover under $\overline{\kappa}$ is zero. We have thus shown that $\underline{\kappa}$ is an optimal choice for any team with composition $\underline{X}$.

A similar argument proves that $\hat{\kappa}$ is the optimal choice for any team with composition $\hat{X}$. This is because since $\hat{\kappa} = \text{argmax}_k(\lambda + 1 - (\alpha/2) - (1 - \alpha)) - (k^2/2\hat{X}^\gamma)$, the objective is strictly concave in k, and $\hat{\kappa} < \overline{\kappa}$, it follows that $\hat{\kappa}$ yields a higher payoff to $\hat{X}$ than $\overline{\kappa}$ in the case where the spillover is $1 - (\alpha/2) - (1 - \alpha)$. But then it also dominates it when the spillover under $\overline{\kappa}$ is zero.

Finally, it is straightforward that the optimal choice for any team with $\overline{X}$ is $\overline{\kappa}$, as the spillover is zero no matter what k the team chooses, and $\overline{\kappa}$ is the unconstrained maximum in this case. □

A.7 Proof of Proposition 5

The analysis in the text reveals that equilibrium is unique. It is interior if and only if $\Gamma(0) > 0$ and $\Gamma(1) < 0$. The expressions for $\Gamma(0)$ and $\Gamma(1)$ are given by

$$\begin{aligned}
\Gamma(0) &= \frac{A^2\overline{X}^\gamma\lambda^2}{2}\left(1 + \left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1 + \frac{1}{\lambda}\right)^2 - 2^{1-\gamma}\left(1 + \frac{\underline{x}}{\overline{x}}\right)^\gamma\right) \\
\Gamma(1) &= \frac{A^2\overline{X}^\gamma\lambda^2}{2}\left(1 + \left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1 + \frac{1}{2\lambda}\right)^2 - 2^{1-\gamma}\left(1 + \frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1 + \frac{1}{2\lambda}\right)^2\right).
\end{aligned}$$

Note that $\Gamma(0) > 0$ for all $\gamma \geq 1$ and $\underline{x}/\overline{x} < 1$. To see, this it suffices to write the last term of $\Gamma(0)$ as $(1/2)((1 + (\underline{x}/\overline{x}))/2)^\gamma < 1$. Regarding $\Gamma(1)$, it is necessary that the negative term offsets the two positive terms. This holds if $1 \leq \gamma < 1 + 2(\log(1 + (1/2\lambda))/\log 2)$ and $\underline{x}/\overline{x}$ sufficiently small. Note that when $\underline{x}/\overline{x} = 0$, $\Gamma(1) < 0$ under the stated condition on γ. By continuity it holds for $\underline{x}/\overline{x}$ sufficiently small, proving the assertion. Because $\Gamma(\alpha)$ is strictly decreasing in α, there exists a unique $\alpha \in (0, 1)$ such that $\Gamma(\alpha) = 0$. This allocation (the matching α) plus wages given by $\overline{w} = 0.5\mathcal{V}(\overline{X}|\alpha)$ and $\underline{w} = 0.5\mathcal{V}(\underline{X}|\alpha)$ constitute the unique competitive equilibrium.

To show that the equilibrium value α is strictly increasing in γ, let us write the equilibrium condition including γ as an argument, that is, $\Gamma(\alpha^*(\gamma), \gamma) = 0$. It follows from (24) that this is continuously differentiable in each argument; moreover, $\Gamma_\alpha(\alpha^*(\gamma), \gamma) < 0$. Therefore, $\alpha^*_\gamma(\gamma) = -\Gamma_\gamma(\alpha^*(\gamma), \gamma)/\Gamma_\alpha(\alpha^*(\gamma), \gamma)$, and this is positive if and only if $\Gamma_\gamma(\alpha^*(\gamma), \gamma) > 0$. Differentiating (24) with respect to γ and evaluating it at α^* yields

$$\Gamma_\gamma(\alpha^*(\gamma), \gamma) = \left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(\log\frac{\underline{x}}{\overline{x}}\right)\left(1 + \frac{1 - \frac{\alpha^*(\gamma)}{2}}{\lambda}\right)^2 + 2^{1-\gamma}\left(1 + \frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1 + \frac{\alpha^*(\gamma)}{2\lambda}\right)^2\left(\log 2 - \log\left(1 + \frac{\underline{x}}{\overline{x}}\right)\right),$$

where the first term is negative while the second is positive. Consider the limit of this expression as $\underline{x}/\overline{x} \to 0$. Since α^* converges to a number strictly between 0 and 1 as $\underline{x}/\overline{x}$ goes to zero, the second term converges to

$$2^{1-\gamma}\left(1 + \frac{\alpha^*(\gamma)}{2\lambda}\right)^2 \log 2 > 0.$$

Similarly, since α^* converges to a number strictly between 0 and 1 as $\underline{x}/\overline{x}$ goes to zero, it follows that the convergence of the first term depends on the limit of

$$\lim_{\frac{\underline{x}}{\overline{x}}\to 0} \left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma} \log \frac{\underline{x}}{\overline{x}},$$

which is of the $0 \cdot \infty$ type. Passing $(\underline{x}/\overline{x})^{\gamma}$ to the denominator we can transform it into a $0/0$ expression. Using L'Hôpital's rule we obtain the following limit

$$\lim_{\frac{\underline{x}}{\overline{x}}\to 0} \left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma} \log \frac{\underline{x}}{\overline{x}} = \lim_{\frac{\underline{x}}{\overline{x}}\to 0} \frac{\frac{1}{\frac{\underline{x}}{\overline{x}}}}{-\gamma\left(\frac{\underline{x}}{\overline{x}}\right)^{-(\gamma+1)}} = \lim_{\frac{\underline{x}}{\overline{x}}\to 0} \left(-\frac{1}{\gamma}\right)\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma} = 0.$$

Hence, the first term in Γ_γ converges to zero as $\underline{x}/\overline{x}$ goes to zero. As a result, for $\underline{x}/\overline{x}$ small, the equilibrium α is strictly increasing in γ, completing the proof of the proposition. □

A.8 Derivation of Within- and Between-Firm Variance

The within-firm variance is an average of the variances within each firm, so

$$\begin{aligned}
\mathrm{Var}[w|\alpha^\star] &= \frac{1}{2}\left(\frac{\alpha^\star}{2}\mathrm{Var}[w|\overline{X}] + \frac{\alpha^\star}{2}\mathrm{Var}[w|\underline{X}] + (1-\alpha^\star)\mathrm{Var}[w|\hat{X}]\right) \\
&= \frac{1}{2}(1-\alpha^\star)\mathrm{Var}[w|\hat{X}] \\
&= \frac{1}{2}(1-\alpha^\star)\left(\frac{1}{2}\left(\underline{w} - \frac{\underline{w}+\overline{w}}{2}\right)^2 + \frac{1}{2}\left(\overline{w} - \frac{\underline{w}+\overline{w}}{2}\right)^2\right) \\
&= \frac{1}{2}(1-\alpha^\star)\left(\frac{\Delta w}{2}\right)^2 \\
&= \frac{A^4\lambda^4}{128}(1-\alpha^\star)\left(\overline{X}^\gamma - \underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2,
\end{aligned}$$

where the second equality uses $\mathrm{Var}[w|\overline{X}] = \mathrm{Var}[w|\underline{X}] = 0$, as these teams consists of homogeneous agents and thus members get paid the same wage, and the rest follows from simple algebra.

Since the mean wage is simply $(\underline{w}+\overline{w})/2$, the variance between firms is:

$$\begin{aligned}
\mathrm{Var}[w_i + w_j|\alpha^\star] &= \frac{1}{2}\left(\frac{\alpha^\star}{2}\left(2\overline{w} - \frac{\underline{w}+\overline{w}}{2}\right)^2 + \frac{\alpha^\star}{2}\left(2\underline{w} - \frac{\underline{w}+\overline{w}}{2}\right)^2 + (1-\alpha^\star)\left(\underline{w}+\overline{w} - \frac{\underline{w}+\overline{w}}{2}\right)^2\right) \\
&= \frac{1}{2}\left(\frac{\alpha^\star}{2}\left(\frac{3\overline{w}-\underline{w}}{2}\right)^2 + \frac{\alpha^\star}{2}\left(\frac{3\underline{w}-\overline{w}}{2}\right)^2 + (1-\alpha^\star)\left(\frac{\underline{w}+\overline{w}}{2}\right)^2\right) \\
&= \frac{A^4\lambda^4}{128}\left(\frac{\alpha^\star}{2}\left(3\overline{X}^\gamma - \underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2 + \frac{\alpha^\star}{2}\left(\overline{X}^\gamma - 3\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right. \\
&\quad \left. +(1-\alpha^\star)\left(\overline{X}^\gamma + \underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right),
\end{aligned}$$

where the first equality is the definition of the variance of the sum of wages, and the rest follows by replacing $\overline{w}$ and $\underline{w}$ and algebraic manipulation.

A.9 Knowledge Spillovers: The Planner's Problem

Assume there is an equilibrium in the second stage as described in Lemma 1. Then the planner solves

$$\max_{\alpha} \frac{A^2}{4}\left(\frac{\alpha}{2}\overline{X}^{\gamma}\lambda^2 + \frac{\alpha}{2}\underline{X}^{\gamma}\left(\lambda+1-\frac{\alpha}{2}\right)^2 + (1-\alpha)\hat{X}^{\gamma}\left(\lambda+\frac{\alpha}{2}\right)^2\right).$$

The objective function is strictly concave in α if $\lambda > 0.25$. To show this, note that its derivative is

$$\lambda^2\overline{X}^{\gamma} + \underline{X}^{\gamma}\left(\lambda+1-\frac{\alpha}{2}\right)\left(\lambda+1-\frac{3}{2}\alpha\right) - 2\hat{X}^{\gamma}\left(\lambda+\frac{\alpha}{2}\right)\left(\lambda-1+\frac{3}{2}\alpha\right),$$

and thus the second derivative is, after some algebra,

$$\underline{X}^{\gamma}\left(-2(\lambda+1)+\frac{3}{2}\alpha\right) - \hat{X}^{\gamma}(4\lambda-1+3\alpha).$$

The first term is negative, and so is the second if $\lambda > 0.25$, making the objective is strictly concave in α.

Rewrite the derivative of the objective function as follows:

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{\lambda}-\frac{\alpha}{2\lambda}\right)\left(1+\frac{1}{\lambda}-\frac{3}{2\lambda}\alpha\right) - 2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{\alpha}{2\lambda}\right)\left(1-\frac{1}{\lambda}+\frac{3}{2\lambda}\alpha\right).$$

The efficient matching is interior if and only if this derivative is positive at $\alpha = 0$ and negative at $\alpha = 1$. At $\alpha = 0$ it is given by $\lambda^2\overline{X}^{\gamma} + \underline{X}^{\gamma}(\lambda+1)^2 - 2\hat{X}^{\gamma}\lambda(\lambda-1)$, which is positive if

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{\lambda}\right)^2 - 2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1-\frac{1}{\lambda}\right) > 0. \tag{53}$$

This is positive for any $\lambda \leq 1$, and thus also for λ in a neighborhood of 1. Alternatively, it holds for $\underline{x}/\overline{x}$ small, and either $\gamma > 1$ or λ close to one. In turn, at $\alpha = 1$, the derivative of the objective function is given by $\lambda^2\overline{X}^{\gamma} + \underline{X}^{\gamma}(\lambda^2 - 0.25) - 2\hat{X}^{\gamma}(\lambda+0.5)^2$, which is negative if

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1-\frac{1}{4\lambda^2}\right) - 2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{2\lambda}\right)^2 < 0, \tag{54}$$

and this holds if $\underline{x}/\overline{x}$ is sufficiently small and $\gamma < 1 + 2(\log(1+(1/2\lambda))/\log 2)$.

Hence, the planner's problem solution α^p is interior under mild conditions on primitives that satisfy (53)–(54). It is a valid solution if in addition it satisfies (51)–(52).

We claim that (51)–(54) hold if λ and γ are close to one (that is, in a neighbordhood of one) and $\underline{x}/\overline{x} < 1/3$. To see this, assume that $\gamma = \lambda = 1$. Then conditions (51) and (52) reduce to $\underline{x}/\overline{x} < 1/3$. Condition (53) is satisfied since it holds for any $\lambda \leq 1$. Finally, condition (54) reduces to $-5 - 6(\underline{x}/\overline{x}) < 0$. This shows that the claim is true at $\gamma = \lambda = 1$, and since the inequalities are strict, it follows that the claim holds in a neighborhood of one for each parameter, thereby completing the proof. *In fact, to satisfy all the conditions so far it suffices that $\lambda \geq 1/(2^{\gamma-1}-1) > 1$, $\gamma < 2$, and $\underline{x}/\overline{x}$ sufficiently small.*

In short, there exists a second stage equilibrium as described above, such that the planner's optimal matching in the first stage, given by α^p, is interior. Moreover, since the planner's objective is strictly concave under the parametric assumptions made, it follows that the interior α^p is unique.

It remains to show that the planner's optimal α^p increases in γ. The optimal α^p is the relevant root of the derivative of the objective function equal to zero (it is a quadratic). That is, it solves[45]

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{\lambda}-\frac{\alpha^p}{2\lambda}\right)\left(1+\frac{1}{\lambda}-\frac{3}{2\lambda}\alpha^p\right)-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{\alpha^p}{2\lambda}\right)\left(1-\frac{1}{\lambda}+\frac{3}{2\lambda}\alpha^p\right)=0. \tag{55}$$

We claim that the efficient α^p is strictly increasing in γ for all $\lambda \geq 1$ and $\gamma \geq 1$ so long as $\underline{x}/\overline{x}$ is sufficiently small. To prove it, note that the second term of (55) is zero at $\underline{x}/\overline{x} = 0$ and the third term strictly decreases in γ. Hence, the efficient α^p strictly increases in γ.

To see one case in closed form, assume that $\underline{x} = 0$ and $\lambda = 1$. Then the quadratic becomes:

$$1-2^{1-\gamma}\left(1+\frac{\alpha^p}{2}\right)\frac{3}{2}\alpha^p=0,$$

which rearranges to

$$(\alpha^p)^2+2\alpha^p-\frac{2}{3}2^{\gamma}=0.$$

The relevant root is

$$\alpha^p=-1+\sqrt{1+\frac{2}{3}2^{\gamma}},$$

which satisfies $\alpha^p > 0$, $\alpha^p < 1$ so long as $\gamma < \log 4.5/\log 2 \cong 2.17$, and it is increasing in γ. Since the result holds for $\underline{x} = 0$ in strict form, it also holds for $\underline{x}$ sufficiently small.

Intuitively, the efficient α^p for this example is different from the equilibrium $\alpha^\star$ derived in the text, so competitive equilibrium is inefficient. Because the efficient α^p is smaller than that in equilibrium, the *extent* of positive sorting in the competitive equilibrium is too high. The planner chooses fewer firms with PAM, and as a result, the equilibrium between-firm inequality is too high.

A.10 Proof of Proposition 6

(i) The only part that is not proven in the text is that a competitive equilibrium with PAM exists. The sufficient conditions on V for PAM stated in Section 4.2 do not hold, but we can show directly from the maximization problem of each agent that it is globally optimal to hire a partner of the same characteristic when he conjectures that the equilibrium in the market exhibits PAM. An agent with characteristic x facing a wage function $w(x) = w(\underline{x}) + ((4\beta)/(9b))\left((a-\nu)((x^2-\underline{x}^2)/2)+\beta((x^4-\underline{x}^4)/4)\right)$ solves

$$\max_{x'}\left(\frac{(a-\nu+2\beta xx'-\beta x^2)^2}{9b}-\frac{4\beta}{9b}\left((a-\nu)\frac{x'^2-\underline{x}^2}{2}+\beta\frac{x'^4-\underline{x}^4}{4}\right)-w(\underline{x})\right).$$

The first-order condition for an interior maximum is

$$\frac{4\beta}{9b}x(a-\nu+2\beta xx'-\beta x^2)-\frac{4\beta}{9b}\left((a-\nu)x'+\beta x'^3\right)=0,$$

[45]One can solve it explicitly in Mathematica since it is a quadratic formula in α, and confirm the monotonicity property.

which clearly holds at $x' = x$, so this is a critical point. Taking the second derivative of the objective function we obtain $(4/9)(\beta/b)(2\beta x^2 - (a-\nu) - 3\beta x'^2)$, which is negative if $a > \nu + 2\beta\overline{x}^2 - 3\beta\underline{x}^2$. Thus, under this parametric condition the objective function is strictly concave, and as a result $x' = x$ is a global maximum.[46]

(ii) We will prove a slightly more general result that holds for any symmetric and submodular function c; in particular, it holds for $c(x, x') = \nu - \beta x x'$. We will consider the restricted planner's problem where he chooses any convex combination between PAM and NAM. Using the profit function of each firm, we first show that the expression for A' in Proposition 4 is $A' = -(4/9b)\int_{\underline{x}}^{\overline{x}}(c(x,x) - c(x,\mu_-(x)))^2 dF(x) < 0$. To see this, notice that

$$\begin{aligned}
\mathcal{V}(x, x|\mu_+) &= \frac{(a - c(x,x))^2}{9b} \\
\mathcal{V}(x, \mu_-(x)|\mu_+) &= \frac{(a - 2c(x,\mu_-(x)) + c(x,x))^2}{9b} \\
\mathcal{V}(x, \mu_-(x)|\mu_-) &= \frac{(a - c(x,\mu_-(x)))^2}{9b} \\
\mathcal{V}(x, x|\mu_-) &= \frac{(a - 2c(x,x) + c(x,\mu_-(x)))^2}{9b}.
\end{aligned}$$

Inserting these equations into A' and simplifying them yields the expression above. As a result, the planner's objective function in the restricted problem is strictly concave in α and thus the solution is either a corner or an interior α. To find out where the solution lies, we compute B' from Proposition 4, which is equal to

$$B' = \frac{1}{9b}\left(2a\left(\int_{\underline{x}}^{\overline{x}} c(x,\mu_-(x))dF(x) - \int_{\underline{x}}^{\overline{x}} c(x,x)dF(x)\right) + \int_{\underline{x}}^{\overline{x}}\left(5c^2(x,x) + 3c^2(x,\mu_-(x)) - 8c(x,x)c(x,\mu_-(x))\right)dF(x)\right).$$

Since c is strictly submodular, the first term is positive, and thus $B' > 0$ for a large enough. As a result, it follows from the quadratic form of the planner's objective function that the solution to the restricted problem is either interior or PAM. As in Proposition 2, which case ensues depends on the sign of $B' + 2A'$, whose expression is

$$\begin{aligned}
B' + 2A' = \frac{1}{9b}\Bigg(&2a\left(\int_{\underline{x}}^{\overline{x}} c(x,\mu_-(x))dF(x) - \int_{\underline{x}}^{\overline{x}} c(x,x)dF(x)\right) \\
&+ \int_{\underline{x}}^{\overline{x}}\left(5c^2(x,x) + 3c^2(x,\mu_-(x)) - 8c(x,x)c(x,\mu_-(x))\right)dF(x) \\
&-8\int_{\underline{x}}^{\overline{x}}(c(x,x) - c(x,\mu_-(x)))^2 dF(x)\Bigg).
\end{aligned}$$

It is clear that if a is large enough then $B' + 2A' > 0$ and thus the optimal solution for the planner in the restricted problem is PAM. Although we have taken into account only firm profits, the same result holds if we add to it consumer surplus in each sector, which is also given by $(2/(9b))\int_{\underline{x}}^{\overline{x}}(a - c(x,x))^2 dF(x)$, which is the same as the sum of the profits of the firms in all the sectors (recall that in each sector there are two firms). $\square$

[46] If one instead assumes a general symmetric, twice continuously differentiable, strictly decreasing, convex, and strictly supermodular marginal cost function c, then one can show the existence of a PAM equilibrium for large enough a and negative enough c_{12}. Since this does not add much to the analysis, the details are omitted.

A.11 Proof of Proposition 7

The match outputs of a team with composition (x, x') under PAM and NAM are, respectively,

$$\begin{aligned}\mathcal{V}(x, x'|\mu_+) &= (a+2b)m(x,x') + b\left(\frac{m(\overline{x},\overline{x}) + m(\underline{x},\underline{x})}{2}\right) \\ &= (a+2b)m(x,x') + bm(\mu_+) \\ \mathcal{V}(x, x'|\mu_-) &= (a+2b)m(x,x') + bm(\underline{x},\overline{x}) \\ &= (a+2b)m(x,x') + bm(\mu_-),\end{aligned}$$

where we have set $m(\mu_+) \equiv (m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}))/2$ and $m(\mu_)) \equiv m(\underline{x},\overline{x})$. Since the complementarities in (x, x') are independent of the matching, the equilibrium sorting pattern depends on the sign of $(a+2b)(m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}) - 2m(\underline{x},\overline{x}))$, while the planner's optimal choice depends on the sign of $(a+2b)(m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}) - 2m(\underline{x},\overline{x})) - 2b(m(\mu_-) - m(\mu_+))$, which, using the expressions above, can be rewritten as $(a+3b)(m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}) - 2m(\underline{x},\overline{x}))$. It is now clear that equilibrium is inefficient if and only if the signs of $(a+2b)$ and $(a+3b)$ differ, and this can only occur if $b \in (-a/2, -a/3)$. If m supermodular (submodular) and b is in this range, then the equilibrium exhibits PAM but the planner chooses NAM (PAM). □

A.12 Taxes, Salary Cap, and Rookie Draft

First we derive the equilibrium wages, which under PAM are equal to half the match output and hence

$$\begin{aligned}\overline{w} &= \frac{1}{2}\left((a+2b)m(\overline{x},\overline{x}) + bm(\mu_+)\right) \\ \underline{w} &= \frac{1}{2}\left((a+2b)m(\underline{x},\underline{x}) + bm(\mu_+)\right).\end{aligned}$$

Similarly, the following are wages that support a NAM equilibrium:

$$\begin{aligned}\overline{w} &= \frac{1}{2}\left(\mathcal{V}(\underline{x},\overline{x}|\mu_-) + \frac{1}{2}\left(\mathcal{V}(\overline{x},\overline{x}|\mu_-) - \mathcal{V}(\underline{x},\underline{x}|\mu_-)\right)\right) = \frac{1}{2}\left((a+3b)m(\underline{x},\overline{x}) + \frac{1}{2}(a+2b)(m(\overline{x},\overline{x}) - m(\underline{x},\underline{x}))\right) \\ \underline{w} &= \frac{1}{2}\left(\mathcal{V}(\underline{x},\overline{x}|\mu_-) - \frac{1}{2}\left(\mathcal{V}(\overline{x},\overline{x}|\mu_-) - \mathcal{V}(\underline{x},\underline{x}|\mu_-)\right)\right) = \frac{1}{2}\left((a+3b)m(\underline{x},\overline{x}) - \frac{1}{2}(a+2b)(m(\overline{x},\overline{x}) - m(\underline{x},\underline{x}))\right).\end{aligned}$$

TAXES. Suppose the planner's allocation exhibits NAM. We claim that a pair of taxes $\overline{t}, \underline{t}$ on firms with composition $(\overline{x},\overline{x})$ and $(\underline{x},\underline{x})$, along with $\overline{s} = \underline{s} = 0$, implement the optimal allocation provided that

$$\overline{t} > \frac{1}{2}(a+2b)\left(m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}) - 2m(\underline{x},\overline{x})\right) \quad \text{and} \quad \underline{t} > \frac{1}{2}(a+2b)\left(m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}) - 2m(\underline{x},\overline{x})\right). \tag{56}$$

To see this, consider first an equilibrium allocation that is PAM. Given taxes and subsidies, both types of agents will prefer to hire a different type provided that

$$\begin{aligned}(a+2b)m(\overline{x},\overline{x})) + bm(\mu_+) - \overline{w} - \overline{t} &< (a+2b)m(\underline{x},\overline{x}) + bm(\mu_+) - \underline{w} + \overline{s} \\ (a+2b)m(\underline{x},\underline{x}) + bm(\mu_+) - \underline{w} - \underline{t} &< (a+2b)m(\underline{x},\overline{x}) + bm(\mu_+) - \overline{w} + \underline{s}.\end{aligned}$$

Substituting for the equilibrium wages and setting $\overline{s} = \underline{s} = 0$ to ensure budget balance we obtain (56). When

(56) holds, there will be no taxes actually paid, since the threat of the tax will deter the PAM allocation.

We also need to verify that, under a NAM equilibrium, there is no incentive to deviate to PAM. In that case, the taxes (again with $\overline{s} = \underline{s} = 0$) must satisfy:

$$(a+2b)m(\overline{x},\overline{x}) + bm(\underline{x},\overline{x}) - \overline{w} - \overline{t} < (a+2b)m(\underline{x},\overline{x}) + bm(\underline{x},\overline{x}) - \underline{w}$$
$$(a+2b)m(\underline{x},\underline{x}) + bm(\underline{x},\overline{x}) - \underline{w} - \underline{t} < (a+2b)m(\underline{x},\overline{x}) + bm(\underline{x},\overline{x}) - \overline{w}$$

which after substituting for the NAM equilibrium wages we obtain the bounds on taxes given in (56).

Salary Cap. Under PAM without salary cap we have

$$(a+2b)m(\overline{x},\overline{x}) + bm(\mu_+) + 2\overline{w} > (a+2b)m(\underline{x},\overline{x}) + bm(\mu_+) - \overline{w} - \underline{w}$$
$$(a+2b)m(\underline{x},\underline{x}) + bm(\mu_+) + 2\underline{w} > (a+2b)m(\underline{x},\overline{x}) + bm(\mu_+) - \overline{w} - \underline{w}.$$

If the salary cap $C < \overline{w}$ binds, these conditions become

$$(a+2b)m(\overline{x},\overline{x}) - 2C > (a+2b)m(\underline{x},\overline{x}) - C - \underline{w}$$
$$(a+2b)m(\underline{x},\underline{x}) - 2\underline{w} > (a+2b)m(\underline{x},\overline{x}) - C - \underline{w}.$$

It is clear that the cap will not have any effect on the team consisting of two agents with high characteristics, for now they have to pay them a lower salary and thus profits are higher. At best, the salary cap can affect the incentives of the team with two agents with low characteristics to hire a better one instead and have a more balanced composition. In short, the salary cap will not change the equilibrium from PAM to NAM.

Rookie Draft. Consider a low characteristic senior who conjectures that all the low characteristic seniors will choose a high characteristic rookie. He will do so as well so long as

$$(a+2b)m(\underline{x},\overline{x}) + bm(\underline{x},\overline{x}) - \overline{w}_d \geq (a+2b)m(\underline{x},\underline{x}) + bm(\underline{x},\overline{x}) - \underline{w}_d,$$

which reduces to

$$(a+2b)\left(m(\underline{x},\overline{x}) - m(\underline{x},\underline{x})\right) \geq \overline{w}_d - \underline{w}_d. \tag{57}$$

Notice that, since the original PAM equilibrium condition was

$$\overline{w} - \underline{w} \geq (a+2b)\left(m(\underline{x},\overline{x}) - m(\underline{x},\underline{x})\right),$$

it follows that under the draft $\overline{w}_d - \underline{w}_d \leq \overline{w} - \underline{w}$, so salaries are more compressed or less spread out.

A senior with $\underline{x}$ prefers the NAM that ensues under (57) to the PAM allocation in the original equilibrium if

$$(a+2b)m(\underline{x},\overline{x}) + bm(\underline{x},\overline{x}) - \overline{w}_d \geq (a+2b)m(\underline{x},\underline{x}) + bm(\mu_+) - \underline{w},$$

which reduces to

$$\overline{w}_d \leq \underline{w} + (a+2b)\left(m(\underline{x},\overline{x}) - m(\underline{x},\underline{x})\right) - \frac{b}{2}\left(m(\overline{x},\overline{x}) + m(\underline{x},\underline{x}) - 2m(\underline{x},\overline{x})\right). \tag{58}$$

Similarly, a senior with $\overline{x}$ prefers the new NAM allocation to the original PAM equilibrium allocation if

$$(a+2b)m(\underline{x},\overline{x})+bm(\underline{x},\overline{x})-\underline{w}_d \geq (a+2b)m(\overline{x},\overline{x})+bm(\mu_+)-\overline{w},$$

or, equivalently,

$$\overline{w}_d \leq \overline{w}-(a+2b)\left(m(\overline{x},\overline{x})-m(\underline{x},\overline{x})\right)-\frac{b}{2}\left(m(\overline{x},\overline{x})+m(\underline{x},\underline{x})-2m(\underline{x},\overline{x})\right). \tag{59}$$

It is easy to find nonnegative wages $\overline{w}_d$ and $\underline{w}_d$ such that (57), (58), and (59) are satisfied. For example, if one sets (57) with equality, then inserting it in (58) and using (59) we obtain

$$\overline{w}_d \leq \min\left\{\underline{w},\overline{w}-(a+2b)\left(m(\overline{x},\overline{x})-m(\underline{x},\overline{x})\right)\right\}-\frac{b}{2}\left(m(\overline{x},\overline{x})+m(\underline{x},\underline{x})-2m(\underline{x},\overline{x})\right).$$

But the incentive constraint for an agent with $\overline{x}$ under PAM reveals that the second term inside the min is the smallest one, and hence

$$\overline{w}_d \leq \overline{w}-(a+2b)\left(m(\overline{x},\overline{x})-m(\underline{x},\overline{x})\right)-\frac{b}{2}\left(m(\overline{x},\overline{x})+m(\underline{x},\underline{x})-2m(\underline{x},\overline{x})\right).$$

Therefore, if we set (57) and (59) with equality, the resulting $\overline{w}_d$ and $\underline{w}_d$ induce all the seniors with $\underline{x}$ to pick a rookie with $\overline{x}$, and both types of seniors are better off under the new NAM than under the original PAM equilibrium allocation. This shows that if the competitive equilibrium exhibits PAM and the efficient allocation is NAM, then a rookie draft with exogenously set wages can implement the efficient allocation.

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