Competing Teams[*]

Hector Chade[†] and Jan Eeckhout[‡]

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Abstract

In numerous economic applications of matching, the teams that form compete later. For instance, pharmaceuticals assemble R&D teams to develop new drugs and then compete for patents. Such post-match competition in the output market introduces interdependence at the matching stage, as the value of a match depends on the composition of R&D teams in the other firms. This paper develops a matching model with externalities and transferable utility and analyzes the optimal and equilibrium sorting patterns that ensue. We show that results substantially differ from the standard model without externalities (Becker (1973)): there can be multiple equilibria; both optimal and equilibrium matching can involve randomization; equilibrium can be inefficient with a matching that can drastically deviate from the optimal one; and match complementarities are no longer exclusively related to sorting. We analyze several canonical applications, such as spillovers, patent races, and oligopoly. Finally, we shed light on policies that can implement the planner's matching.

Keywords. Matching with Externalities. Sorting. Strategic Interaction. Spillovers. Patent Race. Auctions. Oligopoly. Balanced Competition.

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[†]Arizona State University, `hector.chade@asu.edu`

[‡]University College London and Barcelona GSE-UPF, `j.eeckhout@ucl.ac.uk`.

1 Introduction

The success of a firm, a team or a partnership often depends on how it fares relative to its competitors. For instance, in a patent race between pharmaceutical companies, where the game is zero sum and the winner takes all the benefit, a competing team affects the other teams' performance negatively. If the competitor discovers the blockbuster drug, the rest gets nothing. Competing teams may also affect the outcome positively whenever the performance of the competitor generates knowledge spillovers that boost the own firm's performance. The success in the discovery of the structure of DNA for which James Watson and Francis Crick at Cambridge gained credit would not have been possible without the spillovers from the team led by Rosalind Franklin and Maurice Wilkins at Kings College.[1]

It has long been recognized that externalities between firms have important implications for the provision of effort and therefore for the efficient allocation of resources.[2] In this paper, we build further on the insights from this literature, but focus on the effects of these externalities on *team composition*, rather than on effort provision. While effort is an important component, it is clear that team composition is an equally crucial determinant of performance. If a pharmaceutical firm gets the best scientists, they are more likely to make new discoveries and the firm is more likely to obtain a patent. The fact that firms spend so much time and resources carefully choosing their skilled workers – often poaching them away from competitors – is direct evidence that team composition is an important strategic tool in the competition with other firms. Consulting firms, banks and law firms try to hire the best young talent; university departments constantly attempt to attract the most productive academics; and research divisions in technology companies lure the best engineers.

To motivate the importance of team composition further, consider the discovery of evidence of the Higgs Boson in the Large Hadron Collider near Geneva in 2012.[3] This was simultaneously confirmed by two independent teams, ATLAS and CMS, each consisting of thousands of collaborators. Both teams are said to be comparable in the composition of the talent of their researchers with some superstars and some less established researchers. Would the discovery process have been more efficient had the best physicists been allocated to one team and the worst ones to the other team? Maybe the finding would have been made faster by a superstar team. However, the second team may not have been able to corroborate the findings or in the case of knowledge spillovers, it may not have been able to provide the first team with sufficient findings to stimulate the fast discovery.

This paper sheds light on robust qualitative features of the effects of externalities on matching. We develop a simple model with a large number of heterogenous agents, just like the standard matching model (Becker (1973)), but now in the presence of externalities. We show that externalities have profound implications and derive insights that have no counterpart in the standard model. For instance, we show that despite the presence of complementarities the optimal matching may be 'interior,' i.e., the planner matches a fraction of the population in a positively assortative way (PAM) and the rest negatively assortative (NAM). We also provide conditions under which the planner chooses between just PAM or NAM: intuitively, the choice depends on the trade off between complementarities and the differential effect of the externality under each matching.

Progress in characterizing the equilibrium allocation has been hindered by problems of existence of competitive

[1] Watson and Crick received the Nobel Prize for the celebrated model of DNA as a double helix, fruit of their brilliant intuition and the meeting of different but complementary minds. Yet, Watson was motivated by a talk by Wilkins on the molecular structure of DNA in 1951, and together with Crick, before coming up with their model, they had access to Franklin's X-ray photographs that documented the helical structure.

[2] Most notably, the literature on tournaments, contests, and patent races has extensively focused on important aspects such as long-term, repeated interaction (Che and Yoo (2001)) and the optimal provision of effort (Che and Gale (2003)).

[3] The finding corroborated the Standard Model of Physics, a theory by Peter Higgs that posits that the boson is the agent that gave mass and energy to matter after the Big Bang.

equilibrium as well as by the combinatorial complexity of the matching problem.[4] In our tractable setup, we sidestep some of these problems and show that a competitive equilibrium always exists but there could be multiple ones and also cases with mixing. As expected, equilibria can be inefficient due to the externalities, and we provide interpretable conditions under which this arises. An interesting simple feature that we underscore and that might be useful in empirical work is that externalities drive a wedge between complementarities and sorting, terms that are 'synonyms' in the standard model (e.g., PAM if and only if supermodular match output). For instance, complementarities of agents' attributes in production can be consistent with negatively sorted matched partners.

All these insights are robust and cannot arise without externalities. More importantly, they have profound implications for the optimal labor force composition in firms. A striking feature of our results is the fact that whenever the equilibrium is inefficient, the optimal team composition tends to look dramatically different from the equilibrium allocation. For example, the outlook of the market is very different if the planner's allocation is NAM (diversity within teams, homogeneity between teams) while the equilibrium allocation is PAM (homogeneity within teams, diversity between teams). These different outcomes can arise even for infinitesimal changes in the technology, for example as the strength of the differential externality becomes slightly stronger. The discontinuity of the equilibrium allocation in the properties of the technology is of course well-known from the assignment game without externalities (Becker (1973)): as the cross-partial of the match surplus switches from positive to negative, the allocation discontinuously jumps from PAM to NAM. The novelty here is the inefficiency. When a policy maker has reasons to intervene on efficiency grounds, this intervention in the competitive market will lead to a *maximal reallocation* of workers to teams, moving the economy from PAM to NAM or vice versa.

We apply the general setting of our model to four economically relevant applications: knowledge spillovers, patent races, auctions between teams, and oligopoly. These settings provide a micro foundation for the externalities in the output market that is realistic and concrete. In conjunction with the supermodularity in the private benefits, we obtain a tangible interpretation of the interplay between complementarities and externalities. In each application, we shed light on the main properties of the market inefficiency and the primitives that drive it.

We provide new insights regarding the effects of policy interventions at the matching stage that can help achieve an efficient allocation of resources. Because intervention in the output market is often difficult, our approach sheds light on interesting policy alternatives in markets with externalities. To make the policy discussion more concrete, we use a stylized version of our model that can be taken as a metaphor for matching in professional sports teams, and we study the impact of three distinct policy measures: taxes and subsidies, salary caps and a 'rookie draft.' These policy measures are visible as actual interventions in one form or another in some markets for sports teams. In most European professional sports leagues such as soccer or basketball, there is little intervention in the team composition. The result is that in most leagues, a few teams get all the top players and the lion share of audiovisual and commercial revenues. Many of these teams are at the top of the competitions year after year.[5] This is in sharp contrast with most sports leagues in the US where radical policy measures ensure what is known as *balanced competition*: the difference between teams is limited, and all teams are composed of both superstars as well as more modest players.[6] These policy measures motivated by sports applications provide us

[4]See the seminal paper by Koopmans and Beckmann (1957). Their "quadratic assignment problem," which need not have a competitive equilibrium, has generated a huge literature in Operations Research and Combinatorial Optimization. Despite its apparent simplicity, it is considered to be one of the most difficult NP-Hard problems (in the sense that unless one proves that P=NP, one cannot even obtain f-approximation algorithms for any constant f (see the recent survey Loiola, de Abreu, Bonaventura-Netto, Hahn, and Querido (2007)). This speaks to the overall difficulty of the topic of matching with externalities.

[5]In the last ten years, Real Madrid has ended in the top two spots of the league every single year, and FC Barcelona nine times. And in the English Premier League, Manchester United won the League eleven times in the last twenty years.

[6]While those policies do not completely eradicate the existence of leading teams such as the Lakers in the NBA, the Yankees and Red Sox in MLB and the Patriots in the NFL, the record of the winning teams provides evidence that those policies are designed to

with insights into the role of policy intervention that is important in many other environments such as markets for pharmaceutical research, academia, and R&D intensive industries.

As mentioned, this paper contributes to the analysis of matching with externalities, an important topic that has received scant attention in the matching literature, despite the pervasiveness of externalities in economic applications. Its importance was recognized in the seminal matching paper by Koopmans and Beckmann (1957), who analyze a variation of their matching problem between locations and plants in the presence of transportation costs between locations, which generate externalities in the optimal assignment. They show that in their model a competitive equilibrium does not exist, and left the problem open for future research. Sasaki and Toda (1996) provided a suitable concept of stability in matching with externalities, and analyzed its implications for the marriage model and assignment games. A recent paper by Pycia and Yenmez (2014) generalizes the analysis of stable matchings to many-to-many and many-to-one matching problems, and show several properties of core allocations, including some comparative statics. What distinguishes our paper from the rest of the literature is our focus on sorting patterns and the conditions for PAM and NAM in both the planner and the market problems. We study both the optimal matching problem from a planner's perspective and a decentralized version using a standard notion of market equilibrium with externalities. Our parsimonious model and the equilibrium notion affords a complete solution to the problem and an explicit comparison between the equilibrium allocation and the planner's solution. Moreover, we shed light on the intuition underlying the inefficiencies that we derive.

The rest of the paper proceeds as follows. The next section describes the model. Section 3 contains the main results regarding sorting patterns and the inefficiency that the externalities can generate as well as an extension to environments with uncertainty. In Section 4 we flesh out four economic applications that provide a microfoundation for the payoff function of each team and the externalities assumed in the general model, and we analyze the effects of alternative policies aimed at correcting the inefficiency. Section 5 concludes. The Appendix contains all the proofs omitted from the text plus some additional discussion.

2 The Model

We analyze the simplest possible model that allows us to address the main issues related to sorting in the presence of externalities in the output market. Consider an economy with a large labor market where skilled workers form teams. Absent externalities, and assuming transferable utility, this would be a standard matching problem (e.g., as in Becker (1973)). In our model, however, after the matching stage there is competition in an output market, where teams compete against each other. To capture the effects of the externalities without unnecessary technical complications, we assume that teams compete in the output market in a pairwise fashion to be specified below. The crucial assumption is that a team's output depends on that of the other team, thus capturing the strategic interaction between them, or production externalities. For example, in a winner-take-all patent race, the winner's output is simply the maximum of each of the teams' research outputs and the loser gets zero.

As a result, competition in the output market feeds back into the sorting that takes place in the competitive labor market, since the payoff of each team is, via the externalities in the output market, dependent on both its composition and that of the other team it ends up competing against. Because of the interdependence, the efficient assignment of workers to teams maximizes the sum of payoffs of all the teams, where each term now is

correct a market outcome that tends towards PAM in the direction of NAM, which is essentially diametrically opposed. It is worth noting that policies agreed upon by team owners might also be aimed at extracting rents from players, but it is not clear why this would lead to a change in the sorting pattern.

affected by both the composition of the team and the matching itself. Hence, pinning down the conditions for PAM and NAM in the planner's problem is more complex than in the absence of externalities. More importantly, the competitive equilibrium solution is typically not efficient, unlike in the standard case (Becker (1973)).

Population and Preferences. There is a continuum of agents or workers of total measure equal to two. Workers are indexed by a characteristic or 'type' $x \in \mathcal{X}$. We focus on the case where $\mathcal{X} = \{\underline{x}, \overline{x}\}$, where $\overline{x} > \underline{x}$ and where exactly half of the agents are of type $\overline{x}$ and half are of type $\underline{x}$.[7] Following the standard assumption in most matching models, every worker is allocated to a team, each with two members. Agents have linear preferences over money, i.e., this is a transferable utility model.[8]

Allocations. An allocation of workers to firms (a matching) is a partition of the set of workers into subsets of size two. For notational convenience, we represent the composition of firm i's team by $X_i = (x_{i1}, x_{i2})$. Since there are only three possible team configurations, we will use the following shorthand notation: $\overline{X}$ for a team consisting of two agents with type $\overline{x}$, $\underline{X}$ for a team with two low type $\underline{x}$ members, and $\hat{X}$ for a mixed team with one member of each type. In the entire economy, there is potentially a continuum of different allocations, all consisting of combinations of teams $\overline{X}, \hat{X}, \underline{X}$. Using the standard vector order, it is clear that $\underline{X} < \hat{X} < \overline{X}$. There is a unit measure of teams, and thus we can, with some abuse of notation, summarize a matching by the fraction α of those teams that are matched PAM (i.e., $\alpha/2$ teams are $\overline{X}$ and $\alpha/2$ are $\underline{X}$), while $1-\alpha$ are matched NAM (i.e., these are all teams $\hat{X}$). By varying α we span all the possible matchings in this economy. The 'extreme' cases or monotone matchings are $\alpha = 1$ or PAM, denoted by μ_+, where half of the teams are $\overline{X}$ teams and the other half are $\underline{X}$ teams, and $\alpha = 0$ or NAM, denoted by μ_-, where all teams are $\hat{X}$.

Technology and Matching. The output produced by a given team depends on its composition, which is standard in these allocation problems. In addition, it depends on the composition of the *competing* team, the novelty of our analysis. This captures externalities or strategic interaction in the output market and will be made explicit once we study some economic applications and introduce micro foundations. The value of output produced by a team that consists of members X_i and competes against a team with composition X_j is denoted by $V(X_i|X_j)$, with $V : [a, b]^4 \to \mathbb{R}$, where $[a, b]$ contains all possible values that $\underline{x}$ and $\overline{x}$ can assume. Notice that this notation implicitly embeds the assumption that the value of output is symmetric in the types of the two members of the team, and similarly symmetric in the types of the two members of the competing team.[9]

In this large economy, we assume that, after they form, teams are randomly matched in pairs, and then they compete against each other. We make the pairwise assumption to avoid the complications that ensue when the output of a team depends on the composition of many other teams, while still capturing the essence of the externalities that downstream competition impose on matching. In turn, the random matching assumption ensures symmetry, which also affords tractability. We will see below that the details of this competition depend on the application at hand; at this point all we need to know is that such competition entails externalities. Since after the team is formed it is randomly paired with another team against which it competes, it follows that

[7] The equal number of high and low types is made for convenience. Otherwise in the case of negative sorting one needs to keep track of the measure of agents on the long side who match among themselves once cross matches are exhausted.

[8] Admittedly, assuming only two types is restrictive. In the Appendix, we illustrate how we can analyze an economy with pairwise matching of a continuum of types. Also, we refrain from introducing a general Pareto frontier with non-transferable utility. For an exhaustive treatment of matching without externalities and with non-linear frontiers, see Legros and Newman (2007).

[9] This assumption is standard in one-sided matching problems without externalities and rules out the task specific productivity, e.g., where the same worker is more productive if assigned to task 1 (say manager) than to task 2 (mechanic). See Kremer and Maskin (1996) for the analysis of matching without the symmetry assumption. We also abstract from the important issue of private information in matching problems, as explored in, for instance, Liu, Mailath, Postlewaite, and Samuelson (2014).

the team payoff at the matching stage is an expected payoff, for it depends on the potential composition of the competing team. Crucially, who the expected opponent is depends on the equilibrium allocation. For instance, under PAM the composition of the competing team is equal to $\overline{X}$ or $\underline{X}$ with probability 1/2, whereas under NAM it is $\hat{X}$ with probability one. Thus, the expected value of forming a match under PAM is

$$\mathcal{V}(X_i|\mu_+) = \mathbb{E}_{\mu_+} V(X_i|\tilde{X}_j) = \frac{1}{2} V(X_i|\overline{X}) + \frac{1}{2} V(X_i|\underline{X}),$$

and under NAM

$$\mathcal{V}(X_i|\mu_-) = \mathbb{E}_{\mu_-} V(X_i|\tilde{X}_j) = V(X_i|\hat{X}).$$

Hence for any matching α we have that

$$\mathcal{V}(X_i|\alpha) = \alpha \mathcal{V}(X_i|\mu_+) + (1-\alpha)\mathcal{V}(X_i|\mu_-).$$

The Planner's Problem. The planner takes the structure of competition after matching as given, and her objective is to choose the matching $\alpha \in [0,1]$ that maximizes the total expected output of the economy. Formally

$$\begin{aligned}\max_{\alpha\in[0,1]} \quad & \frac{\alpha}{2}\left(\alpha\mathcal{V}(\overline{X}|\mu_+) + (1-\alpha)\mathcal{V}(\overline{X}|\mu_-)\right) + \frac{\alpha}{2}\left(\alpha\mathcal{V}(\underline{X}|\mu_+) + (1-\alpha)\mathcal{V}(\underline{X}|\mu_-)\right) \\ & +(1-\alpha)\left(\alpha\mathcal{V}(\hat{X}|\mu_+) + (1-\alpha)\mathcal{V}(\hat{X}|\mu_-)\right),\end{aligned}$$

which can be written as

$$\max_{\alpha\in[0,1]} \frac{\alpha^2}{2}A + \frac{\alpha}{2}B + C, \tag{1}$$

where $C = \mathcal{V}(\hat{X}|\mu_-)$ and

$$A = \left(\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) - 2\mathcal{V}(\hat{X}|\mu_+)\right) - \left(\mathcal{V}(\overline{X}|\mu_-) + \mathcal{V}(\underline{X}|\mu_-) - 2\mathcal{V}(\hat{X}|\mu_-)\right) \tag{2}$$

$$B = \left(\mathcal{V}(\overline{X}|\mu_-) + \mathcal{V}(\underline{X}|\mu_-) - 2\mathcal{V}(\hat{X}|\mu_-)\right) + 2\left(\mathcal{V}(\hat{X}|\mu_+) - \mathcal{V}(\hat{X}|\mu_-)\right). \tag{3}$$

Equilibrium. Our competitive equilibrium concept for the labor market is fairly standard and based on the textbook definition of competitive equilibrium in the presence of externalities (see for example Mas-Colell, Whinston, and Green (1995), and the seminal Arrow and Hahn (1971) chapter 6). When choosing the composition of workers, firms take as given both market wages (i.e., they are price takers) as well as the externalities. This implies that each firm behaves as if its own choice does not affect the candidate equilibrium allocation, a conjecture that is consistent with our large economy environment. This is the most natural equilibrium notion in this setup with a large labor market.[10] Moreover, the competitive equilibrium in the labor market provides us with a benchmark that is distorted exclusively by the strategic interaction from the output market.

There are two interpretations to competitive equilibrium in this model. One is that any given worker hires another coworker at the Walrasian auctioneer's wage rate, where identical types must be indifferent between hiring a coworker and keeping the output net of the wage, or being hired and receiving a wage. The alternative

[10] In the Appendix we discuss the small economy case with a finite number of agents.

interpretation is that there is a large number of identical firms, and each of them makes wage offers to two workers, who will then split the entire surplus due to a zero profit condition. In both cases, identical workers get paid identical wages, and we denote the wage of a worker with type x under the allocation α by $w(x, \alpha)$. When the matching α is clear from the context, we will often use the convention $\overline{w} = w(\overline{x}, \alpha)$, $\underline{w} = w(\underline{x}, \alpha)$. For most of the paper we will focus on the first interpretation, where a worker hires a coworker to form a team.

Accordingly, a *competitive equilibrium* of the matching problem is a triple $(\underline{w}, \overline{w}, \alpha)$, such that, each worker hires the coworker that maximizes his payoff, taking wages and the allocation as given, and the allocation is consistent with all the workers choices of coworkers.

For instance, given wages and a PAM equilibrium allocation, it must be the case that:

$$\mathcal{V}\left(\overline{X}|\mu_+\right) - \overline{w} \geq \mathcal{V}(\hat{X}|\mu_+) - \underline{w} \tag{4}$$

$$\mathcal{V}\left(\underline{X}|\mu_+\right) - \underline{w} \geq \mathcal{V}(\hat{X}|\mu_+) - \overline{w}. \tag{5}$$

Adding both constraints reveals that a necessary condition for a PAM equilibrium is that $\mathcal{V}(\cdot|\mu_+)$ be supermodular in the vector X, i.e., $\mathcal{V}\left(\overline{X}|\mu_+\right) + V\left(\underline{X}|\mu_+\right) - 2V(\hat{X}|\mu_+) \geq 0$.[11] The equivalent conditions for NAM are

$$\mathcal{V}\left(\overline{X}|\mu_-\right) - \overline{w} \leq \mathcal{V}(\hat{X}|\mu_-) - \underline{w} \tag{6}$$

$$\mathcal{V}\left(\underline{X}|\mu_-\right) - \underline{w} \leq \mathcal{V}(\hat{X}|\mu_-) - \overline{w}, \tag{7}$$

where in addition to the inequality also the allocation changes. In this case a necessary condition is that $\mathcal{V}(\cdot|\mu_-)$ be submodular in X. And for $\alpha \in (0, 1)$, the inequalities, conditioned now on α, must hold with equality to make agents indifferent between hiring a low or a high type. That is,

$$\mathcal{V}\left(\overline{X}|\alpha\right) - \overline{w} = \mathcal{V}(\hat{X}|\alpha) - \underline{w} \tag{8}$$

$$\mathcal{V}\left(\underline{X}|\alpha\right) - \underline{w} = \mathcal{V}(\hat{X}|\alpha) - \overline{w}. \tag{9}$$

Then the assumption is that high/medium/low teams will form according to the proportions determined by α. A necessary condition for an interior matching equilibrium to exist is that $\mathcal{V}(\cdot|\alpha)$ be modular in X.

3 Characterization Results

3.1 Efficient Matching

The following proposition characterizes the solution to the planner's problem in terms of A and B as defined above. It follows easily from the quadratic nature of the planner's objective function:

Proposition 1 *Assume that either $A \neq 0$ or $B \neq 0$.*[12] *The optimal matching is as follows:*

(i) If $A \geq 0$, then the planner chooses PAM if

$$\mathcal{V}\left(\overline{X}|\mu_+\right) + \mathcal{V}\left(\underline{X}|\mu_+\right) \geq 2\mathcal{V}(\hat{X}|\mu_-), \tag{10}$$

[11] Recall that a function f defined on a lattice of $\mathbb{R}^2$ is supermodular if given any two points (x, y) and (x', y'), $f(x \vee x', y \vee y') + f(x \wedge x', y \wedge y') \geq f(x, y) + f(x', y')$; it is submodular if the inequality is reversed; and it is modular if it holds with equality.

[12] This is just to avoid the nongeneric case where any matching is optimal.

and NAM if the inequality is reversed;

(ii) If $A < 0$ and $B \leq 0$, then the planner chooses NAM;

(iii) If $A < 0$ and $B + 2A \geq 0$, then the planner chooses PAM;

(iv) If $A < 0$, $B > 0$, and $B + 2A < 0$, then the planner chooses $\alpha = -B/2A \in (0, 1)$.

Proof. (i) Since the planner's objective function is quadratic, it is convex if $A \geq 0$. If strictly convex (or linear but with $B \neq 0$), then the optimal solution is at a corner, $\alpha \in \{0, 1\}$, and which corner depends on whether $0.5(A + B)$ is bigger or smaller than C, which is equivalent to inequality (10).

(ii)–(iii) A necessary condition for an interior solution is that $A < 0$, so the planner's objective is strictly concave. But this is not sufficient since the solution can still be at a corner. If $B \leq 0$, then the planner's objective peaks at $\alpha = 0$ and NAM is optimal, while if $B + 2A \geq 0$ then it peaks at $\alpha = 1$ and PAM is optimal.

(iv) If $A < 0$, $B > 0$, and $B + 2A < 0$, then the planner's objective function is strictly concave and peaks at the interior $\alpha = -B/2A$. Hence, an interior matching is optimal. □

For future reference, notice that (10) for PAM can be rewritten in the more intuitive way

$$\mathcal{V}\left(\overline{X}|\mu_+\right) + \mathcal{V}\left(\underline{X}|\mu_+\right) - 2\mathcal{V}(\hat{X}|\mu_+) \geq 2\left(\mathcal{V}(\hat{X}|\mu_-) - \mathcal{V}(\hat{X}|\mu_+)\right), \tag{11}$$

so PAM ensues if the complementarities in $\mathcal{V}$ are strong enough to offset the differential effect of the externality. The *complementarity v. externality* is at the heart of the tension in the planner's optimal matching.

As intuition would suggest, notice that the planner's solution trivializes when there are no externalities, for then $A = 0$ and $B = \mathcal{V}(\overline{X}) + \mathcal{V}(\underline{X}) - 2\mathcal{V}(\hat{X})$. Hence, if $\mathcal{V}$ is supermodular, the optimal matching is PAM ($\alpha = 1$), and it is NAM ($\alpha = 0$) if submodular, which is the standard sorting result in matching theory (Becker (1973)). Unlike the standard case, when externalities are present an interior matching can be optimal. Moreover, in the standard case a marginal change in complementarities that changes $\mathcal{V}$ from being supermodular to submodular switches the optimal matching from PAM to NAM, i.e., from one corner to the other one. With externalities such a change does not have the same impact, for now the sorting pattern depends in a more complex way on the properties of $\mathcal{V}$ embedded in the coefficients A and B given by expressions (2)–(3).

Since the optimal matching depends on the properties of A and B, it is instructive to understand these expressions better. Compared to the case without externalities, now $A \neq 0$ and there is an additional term in B besides the conditioning on the matching. Regarding the coefficient A, notice that it is an 'increasing difference' condition: $A \geq 0$ if, loosely, complementarities (measured as the sum of values of the payoffs of a high and low team minus twice the value of a mixed team) *increase* when we move from NAM to PAM. The opposite is true for $A < 0$. Regarding B, it is a sum of the complementarities under NAM and the change in the externality effect on a mixed team when moving from NAM to PAM.

Example: Multiplicative Case. As an illustration, let $V(X_i|X_j) = k(X_i)\ell(X_j)$. Then it is easy to verify that

$$A = \left(k(\overline{X}) + k(\underline{X}) - 2k(\hat{X})\right)\left(\frac{1}{2}\ell(\overline{X}) + \frac{1}{2}\ell(\underline{X}) - \ell(\hat{X})\right) = \left(k(\overline{X}) + k(\underline{X}) - 2k(\hat{X})\right)\left(\ell(\mu_+) - \ell(\mu_-)\right),$$

where $\ell(\mu_+) = 0.5(\ell(\overline{X}) + \ell(\underline{X}))$ and $\ell(\mu_-) = \ell(\hat{X})$. Notice that $A \geq 0$ if k is supermodular in X and the externality is larger under PAM than under NAM, or if k is submodular in X and the externality is larger for

NAM. In particular, $A \geq 0$ in the symmetric case in which $k = \ell$. In *all* these instances the optimal matching is either PAM or NAM depending on the inequality (10), and $\alpha \in (0, 1)$ is never socially optimal. (The expression for A is the same if $V(X_i|X_j) = g(X_i) + h(X_j) + k(X_i)\ell(X_j)$, which commonly arises in applications.)

Let us now use part (iv) of Proposition 1 to construct an interior optimal matching with the multiplicative payoff function. The expressions for B and $B + 2A$ are given by

$$\begin{aligned} B &= 2k(\hat{X})(\ell(\mu_+) - \ell(\mu_-)) + \ell(\mu_-)\left(k(\overline{X}) + k(\underline{X}) - 2k(\hat{X})\right) \\ B + 2A &= 2(\ell(\mu_+) - \ell(\mu_-))\left(k(\overline{X}) + k(\underline{X}) - k(\hat{X})\right) + \ell(\mu_-)\left(k(\overline{X}) + k(\underline{X}) - 2k(\hat{X})\right) \end{aligned}$$

We are ready to find conditions under which an interior solution is optimal. Notice that $A < 0$ if, for instance, the externality effect is larger under PAM and k is submodular in X. Similarly, $B > 0$ if the externality under PAM is large enough to offset the submodularity of k. Finally, a sufficient condition for $B + 2A < 0$ is that k be 'single peaked' and $\underline{x}$, $\overline{x}$ such that k satisfies $k(\overline{X}) + k(\underline{X}) \leq k(\hat{X})$. For a simple numerical example, suppose that $k(\overline{X}) = 0.5$, $k(\underline{X}) = 0.5$, and $k(\hat{X}) = 1$. If $\ell(\mu_+) > 2\ell(\mu_-)$, then the optimal matching $\alpha \in (0, 1)$.

3.2 Competitive Equilibrium

Our model permits a sweeping description of the set of competitive equilibria and their sorting properties, as the following results demonstrate.

Proposition 2 *(i) The equilibrium allocation exhibits PAM if and only if* $\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) - 2\mathcal{V}(\hat{X}|\mu_+) \geq 0$. *The wages that support it are* $\overline{w} = 0.5\mathcal{V}\left(\overline{X}|\mu_+\right)$ *and* $\underline{w} = 0.5\mathcal{V}\left(\underline{X}|\mu_+\right)$*;*

(ii) The equilibrium allocation exhibits PAM if and only if $\mathcal{V}(\overline{X}|\mu_-) + \mathcal{V}(\underline{X}|\mu_-) - 2\mathcal{V}(\hat{X}|\mu_-) \leq 0$. *The wages that support it are any* $\underline{w} \in \left[0.5\mathcal{V}\left(\underline{X}|\mu_-\right), \mathcal{V}(\hat{X}|\mu_-) - 0.5\mathcal{V}\left(\overline{X}|\mu_-\right)\right]$ *and* $\overline{w} = \mathcal{V}(\hat{X}|\mu_-) - \underline{w}$*;*

(iii) There is a competitive equilibrium with allocation $\alpha \in (0, 1)$ *if and only if* $\mathcal{V}(\cdot|\alpha)$ *is modular in* X. *The wages that support it are* $\overline{w} = 0.5\mathcal{V}(\overline{X}|\alpha)$ *and* $\underline{w} = 0.5\mathcal{V}(\underline{X}|\alpha)$.

Proof. (i) Supermodularity of $\mathcal{V}(\cdot|\mu_+)$ and inequalities (4)–(5) reveal that for the PAM allocation to be part of a competitive equilibrium, we need to find $\overline{w}$ and $\underline{w}$ that satisfy the inequalities. Now, since two agents with the same type must obtain the same wage, and output is fully distributed between the team members, it follows that $\overline{w} = 0.5\mathcal{V}\left(\overline{X}|\mu_+\right)$ and $\underline{w} = 0.5\mathcal{V}\left(\underline{X}|\mu_+\right)$. Routine calculations reveal that these wages satisfy the above inequalities, and hence we have constructed the equilibrium with PAM.

(ii) Submodularity of $\mathcal{V}(\cdot|\mu_-)$, inequalities (6)–(7), and $\overline{w} + \underline{w} = \mathcal{V}(\hat{X}|\mu_-)$, reveal that any $\underline{w}$ such that (a) $0.5\mathcal{V}\left(\underline{X}|\mu_-\right) \leq \underline{w} \leq \mathcal{V}(\hat{X}|\mu_-) - 0.5\mathcal{V}\left(\overline{X}|\mu_-\right)$ and (b) $\overline{w} = \mathcal{V}(\hat{X}|\mu_-) - \underline{w}$, are equilibrium wages under NAM. Thus, any pair of wages that satisfy these conditions, along with μ_-, is a competitive equilibrium with NAM.

(iii) Finally, for an interior equilibrium with $\alpha \in (0, 1)$, modularity of $\mathcal{V}(\cdot|\alpha)$ and equations (8)–(9) reveal that it is necessary that $\overline{w} - \underline{w} = \mathcal{V}(\overline{X}|\alpha) - \mathcal{V}(\hat{X}|\alpha) = \mathcal{V}(\hat{X}|\alpha) - \mathcal{V}(\underline{X}|\alpha)$. Since wages must also exhaust output for each team, it easily follows that they must be given by $\overline{w} = 0.5\mathcal{V}(\overline{X}|\alpha)$ and $\underline{w} = 0.5\mathcal{V}(\underline{X}|\alpha)$. Hence, these wages along with the interior matching α constitute a competitive equilibrium. □

Notice that, except for the conditioning on the matching (PAM, NAM, or interior), the equilibrium sorting patterns are very similar to those of the case without externalities, i.e., they are driven by the complementarities (or lack thereof) in the match output function. Also, the construction of an interior equilibrium requires a very special match output function, indifference on all agents, and choices of low and high types according to the mixed matching α. This equilibrium always exists when $\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) - 2\mathcal{V}(\hat{X}|\mu_+) < 0$ and $\mathcal{V}(\overline{X}|\mu_-) + \mathcal{V}(\underline{X}|\mu_-) - 2\mathcal{V}(\hat{X}|\mu_-) > 0$, since in this case there is a unique value of α that makes $\mathcal{V}(\cdot|\alpha)$ modular in X. The following corollary is now immediate:

Corollary 1 *A competitive equilibrium exists for all specifications of $\underline{x}$, $\overline{x}$, and V.*

There are examples in the matching literature of nonexistence of a competitive equilibrium in the presence of externalities (e.g., see the quadratic example in Koopmans and Beckmann (1957)). Interestingly, the particular setup of our model ensures existence, as we have just shown.

Proposition 2 only shows existence of an equilibrium (with the asserted properties). It is natural to inquire when this is the unique equilibrium. Even with the conventional concept of competitive equilibrium with externalities, there could be multiple equilibria due to the strength of the externalities. In the standard matching model without externalities, there is a unique equilibrium whenever the match surplus function is strictly supermodular or submodular. Instead, when it is modular, there are multiple allocations – in fact, a continuum – but all induce the same payoffs, thus rendering the indeterminacy irrelevant in payoffs. With externalities, multiple equilibria may arise when the impact of the externality is significantly different under different allocations. In particular, for given primitives V, $\underline{x}$, $\overline{x}$, can a PAM and a NAM equilibrium coexist? The answer is yes if and only if the externality changes complementarities from positive to negative when changing the allocation, that is,

$$\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) - 2\mathcal{V}(\hat{X}|\mu_+) \geq 0 \geq \mathcal{V}(\overline{X}|\mu_-) + \mathcal{V}(\underline{X}|\mu_-) - 2\mathcal{V}(\hat{X}|\mu_-). \tag{12}$$

Similarly, for multiple interior equilibria to exist, we need $\mathcal{V}(\cdot|\alpha)$ to be modular for multiple values of α, and it is easy to check that this can happen only when (12) holds with equality.

Example: Multiplicative Case. As an illustration, suppose that $V(X_i|X_j) = k(X_i)\ell(X_j)$, so the externality effect is multiplicative. Then using the definition of $\mathcal{V}$ and substituting V in (12) yields

$$\left(k(\overline{X}) + k(\underline{X}) - 2k(\hat{X})\right)\ell(\mu_+) \geq 0 \geq \left(k(\overline{X}) + k(\underline{X}) - 2k(\hat{X})\right)\ell(\mu_-).$$

So if, say, k is supermodular in X, then this inequalities hold if ℓ, $\underline{x}$, and $\overline{x}$, are such that $\ell(\mu_+) \geq 0$ and $\ell(\mu_-) \leq 0$. If k is submodular in X, then the inequalities are reversed for multiple equilibria to exist. And if $\ell(\mu_+) > \ell(\mu_-)$ and both have the same sign, then the equilibrium is unique.

3.3 Sorting Patterns and Inefficiency

In the presence of externalities (or strategic interaction in the output market), it is intuitive to expect that equilibrium will be inefficient. Our stylized model fully captures the inefficiency in the sorting pattern of workers into teams. While in reality these externalities may also have repercussions for agents' investment in ability or for their decision to enter the market, for the purpose of this paper we abstract from these other consequences.

The next result, which follows straightforwardly from Proposition 1, characterizes the inefficiency driven by the externality when the competitive equilibrium is unique.

Proposition 3 *Assume that either both $\mathcal{V}(\cdot|\mu_+)$ and $\mathcal{V}(\cdot|\mu_-)$ are supermodular or both are submodular, so that there is a unique competitive equilibrium with PAM or NAM allocations, respectively.*

(i) If the competitive equilibrium allocation exhibits PAM, then it is inefficient if and only if $A \geq 0$ and $\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) < 2\mathcal{V}(\hat{X}|\mu_-)$, or $A < 0$ and $B \leq 0$, or $A < 0$, $B > 0$, and $B + 2A < 0$;

(ii) If the competitive equilibrium allocation exhibits NAM, then it is inefficient if and only if $A \geq 0$ and $\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) > 2\mathcal{V}(\hat{X}|\mu_-)$, or $A < 0$ and $B + 2A \geq 0$, or $A < 0$, $B > 0$, and $B + 2A < 0$.

The intuition is easy to grasp. If the equilibrium exhibits PAM then it will be inefficient if either the planner chooses an extreme matching that exhibits NAM – which occurs if the differential externality effect $2(\mathcal{V}(\hat{X}|\mu_-) - \mathcal{V}(\hat{X}|\mu_+))$ offsets the complementarities in $\mathcal{V}(\cdot|\mu_+)$ – or the planner chooses an interior matching $\alpha \in (0,1)$. Notice that the planner in one case wants to alter the equilibrium matching and move it to an *interior* one, which can be close to it, and in the other case wants to switch to the *opposite* extreme matching. This cannot happen in the standard case without externalities. A similar analysis applies if the equilibrium instead exhibits NAM.

What if the premise of a unique equilibrium is relaxed? Then the proposition provides conditions for existence of *an* inefficient equilibrium, in which again the planner wants to alter the matching, perhaps in a drastic way.

Example: Multiplicative Case. Continuing with our running example $V(X_i|X_j) = k(X_i)\ell(X_j)$, assume $k \geq 0$ and supermodular in X, $\ell(\mu_+) > 0$, and $\ell(\mu_-) > 0$. Then there is a unique equilibrium with PAM, and it is inefficient if

$$k(\overline{X}) + k(\underline{X}) - 2k(\hat{X}) < 2k(\hat{X})\left(\frac{\ell(\mu_-)}{\ell(\mu_+)} - 1\right),$$

which requires the externality effect under NAM to be sufficiently larger than under PAM.

Example: Additive Case. For another simple example, consider the additive case $V(X_i|X_j) = k(X_i) + \ell(X_j)$, with k supermodular and $\ell(\mu_+)\ell(\mu_-) > 0$. Then there is a unique equilibrium with PAM since complementarities are independent of ℓ. Morover, $A = 0$, and hence the planner wlog can restrict attention to a corner solution. Thus, the equilibrium is efficient if and only if

$$k(\overline{X}) + k(\underline{X}) - 2k(\hat{X}) < 2\left(\ell(\mu_-) - \ell(\mu_+)\right).$$

A nice feature of the additive case is that the condition for the planner reduces to the super- or submodularity of the function $k + \ell$. Both the multiplicative and additive cases are ubiquitous in economic applications.

When is the equilibrium *efficient*? Obviously, it is efficient if it is unique and the necessary and sufficient conditions in Proposition 3 fail. The following is a simple class where efficiency can easily be established.

Example: Sum of Types Case. A simple and commonly used class of problems where equilibrium is efficient is when V depends on the *sum* of types, i.e., $V(X_i|X_j) = V(x_{i1} + x_{2i}|x_{j1} + x_{2j})$, and it is either concave or convex in its domain. In this case, if equilibrium is unique and the planner chooses a corner, i.e., PAM or NAM, then the PAM equilibrium is efficient if V is convex, and the NAM equilibrium is efficient if V is concave. To see this, notice that $\overline{X} = 2\overline{x}$, $\underline{X} = 2\underline{x}$, and $\hat{X} = 0.5(2\overline{x} + 2\underline{x}) = \overline{x} + \underline{x}$. Then it is easy to verify that $\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) \gtreqless 2\mathcal{V}(\hat{X}|\mu_-)$ is equivalent to the following expression

$$\left(\frac{1}{2}V(2\overline{x}|2\overline{x}) + \frac{1}{2}V(2\underline{x}|2\underline{x}) - V(\overline{x} + \underline{x}|\overline{x} + \underline{x})\right) + \left(\frac{1}{2}V(2\overline{x}|2\underline{x}) + \frac{1}{2}V(2\underline{x}|2\overline{x}) - V(\overline{x} + \underline{x}|\ \overline{x} + \underline{x})\right) \gtreqless 0,$$

and this is nonnegative (nonpositive) if V is convex (concave). So the planner chooses PAM if V is convex and NAM if concave. Regarding equilibrium, if V is convex then both $V(\cdot|2\overline{x})$ and $V(\cdot|2\underline{x})$ are convex as well. But it is easy to verify that $\mathcal{V}(\cdot|\mu_+)$ supermodular is equivalent to

$$\left(\frac{1}{2}V(2\overline{x}|2\overline{x})+\frac{1}{2}V(2\underline{x}|2\overline{x})-V\left(\overline{x}+\underline{x}|2\overline{x}\right)\right)+\left(\frac{1}{2}V(2\overline{x}|2\underline{x})+\frac{1}{2}V\left(2\underline{x}|2\underline{x}\right)-V\left(\overline{x}+\underline{x}|\ 2\underline{x}\right)\right)\geq 0,$$

and this follows by convexity of V in its first argument. A similar analysis applies to NAM.

We close this section with a simple yet important implication of our analysis that can be potentially relevant for empirical work on sorting. Without externalities, complemetarities between types in the match output is synonymous with PAM, and similarly with substitutabilities and NAM. Notice that the presence of externalities can dilute the relationship between complementarities between types and sorting patterns.

Indeed, consider a general matching α. Then the joint probability distribution over matched partners is $p(\underline{x},\underline{x})=\alpha/2$, $p(\underline{x},\overline{x})=(1-\alpha)/2$, $p(\underline{x},\overline{x})=(1-\alpha)/2$, $p(\overline{x},\overline{x})=\alpha/2$. If we call x_1 the first partner to a match and x_2 the second one, then

$$\begin{aligned} Cov\{x_1,x_2\} &= \left(\frac{\alpha}{2}\underline{x}^2+(1-\alpha)\underline{x}\overline{x}+\frac{\alpha}{2}\overline{x}^2\right)-\frac{(\underline{x}+\overline{x})^2}{4} \\ Var[x_i] &= \frac{(\underline{x}^2+\overline{x}^2)}{2}-\frac{(\underline{x}+\overline{x})^2}{4} \quad i=1,2. \end{aligned}$$

Recalling that the coefficient of correlation between x_1 and x_2 is $\rho=Cov\{x_1,x_2\}/(Var[x_1]Var[x_2])^{0.5}$, we obtain after simple algebra that $\rho=2(\alpha-0.5)$, which can range from -1 if $\alpha=0$ to 1 if $\alpha=1$. Hence, if externalities are strong enough, complementarities between types are consistent with $\rho=-1$. And although this is based on the planner's solution, the same would hold in the market if there is intervention to correct the inefficiency.

3.4 Uncertainty

Thus far the notion of type and the function V have largely remained unspecified. These primitives will adopt specific functional forms in the economic applications below. But before analyzing them, we derive a general functional form for V that is relevant in a class of problems with externalities that involve payoff uncertainty.

In many economic environments, there will be uncertainty about the outcome produced by a team. As above, firms compete in the labor market to optimally design their firm composition given wages, anticipating that the distribution of outcomes depends on the worker composition within the firm. And because of the externalities, the firm expected payoff also depends on the composition of the potential competing teams in the output market.

The match output V in this case entails three components: the first one is the team composition, X_i in firm i and X_j in firm j; the second is firm i's research output, $v_i\in[0,1]$, drawn from the distribution $F(\cdot|X_i)$; and the third one is the 'market value,' the value of output $z(v_i,v_j)$ based on the realization of the own outcome v_i and that of the competitor v_j, which captures the externality or strategic interaction. In our patent race application, v_i indicates for example the quality of the discovery (or the speed with which a new drug is discovered), $z(v_i,v_j)=\max\{v_i,v_j\}$: there is one patent and the superior discovery takes all the market.

In some applications below, we will assume that v_i is drawn from a stochastically "better" distribution in first-order stochastic dominance (FOSD) sense if the team is better. That is, $F(v_i|\cdot)$ is decreasing in X_i.

For any team i with composition X_i facing a team X_j, we can write the expected value of output by:

$$V(X_i|X_j)=\int_0^1\int_0^1 z(v_i,v_j)dF(v_i|X_i)dF(v_j|X_j) \tag{13}$$

The expected value of the team is the expectation of output z over the own realization v_i and that of the competitor v_j, each drawn independently from a distribution that is contingent on the team composition X_i and X_j. Observe that the strategic interaction or external effects act exclusively through output z.

The ex ante payoff in the presence of random matching is then:

$$
\begin{aligned}
\mathcal{V}(X_i|\mu_+) &= \frac{1}{2}V(X_i|\overline{X}) + \frac{1}{2}V(X_i|\underline{X}) \\
&= \frac{1}{2}\int_0^1\int_0^1 z(v_i,v_j)dF(v_i|X_i)dF(v_j|\overline{X}) + \frac{1}{2}\int_0^1\int_0^1 z(v_i,v_j)dF(v_i|X_i)dF(v_j|\underline{X}) \\
\mathcal{V}(X_i|\mu_-) &= V(X_i|\hat{X}) = \int_0^1\int_0^1 z(v_i,v_j)dF(v_i|X_i)dF(v_j|\hat{X})
\end{aligned}
$$

The next result shows that the expected payoff $V(X_i|X_j)$ can be decomposed into three terms, i.e., $V(X_i|X_j) = g(X_i) + \ell(X_j) + k(X_i,X_j)$. To simplify the notation, we denote by $S_i(\cdot)$ the survivor function $1 - F(\cdot|X_i)$.

Proposition 4 *The expected value $V(X_i|X_j)$ can be written as follows:*

$$
\begin{aligned}
V(X_i|X_j) &= \underbrace{z(0,0) + \int_0^1 \frac{\partial z(v_i,0)}{\partial i}S_i(v_i)dv_i + \int_0^1 2\frac{\partial z(0,v_j)}{\partial j}S_i(v_j)dv_j}_{g(X_i)} \\
&\quad + \underbrace{\int_0^1 \frac{\partial z(0,v_j)}{\partial j}S_j(v_j)dv_j}_{\ell(X_j)} + \underbrace{\int_0^1\int_0^1 \frac{\partial^2 z(v_i,v_j)}{\partial i \partial j}S_i(v_i)S_j(v_j)dv_i dv_j}_{k(X_i,X_j)}.
\end{aligned}
$$

We can therefore write the firm's value additively separable in the direct effect $g(X_i)$ and the external effect $\ell(X_j) + k(X_i,X_j)$ which consists of two components. The first component depends on the competitor only, and the second component captures the interaction between the composition of the own team and the competitor's.

Several of the applications we analyze below involve uncertainty: knowledge spillovers, patent race, auctions between competing teams, as well as in the policy discussion. Those applications often rely on a simplified functional form for z, namely, $z(v_i,v_j) = av_i + bv_j + cv_iv_j$ for all (v_i,v_j), where a,b,c are constants. In this case, it follows from (13) that V is given by (we set $m(X_p) \equiv \mathbb{E}[v_p|X_p]$, $p = i,j$)

$$
V(X_i|X_j) = (a+2b)m(X_i) + bm(X_j) + cm(X_i)m(X_j). \tag{14}
$$

From the definition of $\mathcal{V}$, this immediately implies that

$$
\begin{aligned}
\mathcal{V}(X_i|\mu_+) &= (a+2b)m(X_i) + \frac{1}{2}(b + cm(X_i))\left(m(\overline{X}) + m(\underline{X})\right) \\
\mathcal{V}(X_i|\mu_-) &= (a+2b)m(X_i) + \frac{1}{2}(b + cm(X_i))m(\hat{X})
\end{aligned}
$$

Sorting patterns in our applications with uncertainty will crucially depend on the monotonicity and complementarity properties of $m(\cdot)$, so we comment on them here. If $F(v|\cdot)$ satisfies FOSD, i.e., $F(v|\overline{X}) < F(v|\hat{X}) < F(v|\underline{X})$ for all v, then the expected value of v given X will be increasing in $X : m(\underline{X}) < m(\hat{X}) < m(\overline{X})$. As before, supermodularity/submodularity of m simply requires that $m(\overline{X}) + m(\underline{X}) \gtreqless 2m(\hat{X})$. In turn, log-supermodularity/log-submodularity means that $\log m$ is supermodular/submodular or $m(\overline{X})m(\underline{X}) \gtreqless m^2(\hat{X})$ (this is stronger/weaker than supermodularity/submodularity for functions positive and monotone.

4 Economic Applications

4.1 Spillovers

Consider a market where a firm's value is affected by the output generated by the competitor. The development of the early Apple Macintosh operating system inspired Microsoft to switch from MS-DOS to windows, and later on, the widespread availability of office software let Apple easily get into the market again with a new line of products and operating systems. Ever since Marshall (1890) has it been recognized that in some sectors, the inventions and developments of one firm positively affect those in other firms. Clearly, the opposite is true when the development at one firm adversely affects the prospects of the competitor, i.e., when spillovers are negative.

To capture those spillovers in the presence of uncertainty in a tractable yet rich manner, we use the bilinear z function above with $c = 0$ and $a \geq 0$, i.e., $z(v_i, v_j) = av_i + bv_j$. We can write the value of a team X_i conditional on matching with team X_j as $V(X_i|X_j) = (a + 2b)m(X_i) + bm(X_j)$.[13] From the analysis of the additive case example, we know that there is a unique equilibrium and also that the planner can wlog focus on choosing between PAM or NAM (these insights are independent of the sign of b). Now we can establish the following result:

Proposition 5 *Let $z = av_i + bv_j$, with $a \geq 0$.*

1. *If $b \notin \left(-\frac{a}{2}, -\frac{a}{3}\right)$, the equilibrium allocation is efficient;*
2. *If $b \in \left(-\frac{a}{2}, -\frac{a}{3}\right)$, the equilibrium is inefficient: if m is supermodular (submodular), the equilibrium exhibits PAM (NAM), while the planner's solution exhibits NAM (PAM).*

The intuition of this result is as follows. When $b > 0$, the spillover effect is positive and both complementarities and externality go in the same direction and reinforce each other and equilibrium is efficient. If $b < 0$ is negative enough, then both terms in V are negative and once again both effects reinforce each other. In turn, when $b < 0$ but 'close' to zero, there is conflict between the two but complementarities dominate and thus efficiency ensues. A real conflict emerges when $b < 0$ but not sufficiently so; in this case the externality effect is strong enough to offset the benefits from complementarities, and the equilibrium (either PAM or NAM) is inefficient. So despite the presence of spillover effects, efficiency is only compromised in a small parameter range.

Additive spillovers are not the only ones considered in the literature. For a simple example with multiplicative spillovers, consider the following set up inspired by the seminal analysis of knowledge spillovers in the endogenous growth literature pioneered by Romer (1986) and Lucas (1988). We borrow the functional form for their production technology, but now in a simple static context focusing on the matching problem. Let $V(X_i|X_j) = S(\mathbb{E}[k|\mu])k(X_i)$ where k is a knowledge function and S is a positive function that summarizes the knowledge spillover effect, which depends on the average knowledge in the economy given the matching.[14]For instance, under PAM $S(\mathbb{E}[k|\mu_+]) = S(0.5(k(\overline{X}) + k(\underline{X})))$, and under NAM $S(\mathbb{E}[k|\mu_-]) = S(k(\hat{X}))$. Therefore, in this setup the spillover S is multiplicatively separable *and* common to all firm types.[15] For simplicity, we will assume that only corner solutions are relevant and ignore the possibility of an interior matching $\alpha \in (0, 1)$. As usual, there is PAM in equilibrium if k is supermodular and NAM if k is submodular in X. In turn, the planner

[13]Notice that this expected value need not be positive. An easy fix is to add a constant term $v_0 > 0$ large enough. Since this is irrelevant for the results we simply omit it.

[14]In the literature, S is referred to as Total Factor Productivity (TFP).

[15]This takes us outside our pairwise random matching downstream competition in the sense that the externality only works through a common factor determined by the matching and not in particular by the composition of the competing firm's team.

chooses NAM over PAM if

$$k(\overline{X}) + k(\underline{X}) - 2k(\hat{X}) \leq 2k(\hat{X}) \left(\frac{S(k(\hat{X}))}{S\left(\frac{k(\overline{X})+k(\underline{X})}{2}\right)} - 1 \right),$$

and will choose PAM if the inequality is reversed. Notice that this expression rearranges to

$$S\left(\frac{k(\overline{X}) + k(\underline{X})}{2}\right)\left(\frac{k(\overline{X}) + k(\underline{X})}{2}\right) \leq S(k(\hat{X}))k(\hat{X}),$$

which is simply $S(\mathbb{E}[k|\mu_+])\mathbb{E}[k|\mu_+] \leq S(\mathbb{E}[k|\mu_-])\mathbb{E}[k|\mu_-]$. Thus, if k is supermodular (submodular) then the PAM (NAM) equilibrium is inefficient if $S(z)z$ is decreasing (increasing) in z, which holds if the elasticity of S is uniformly less (bigger) than -1. For example, if $S(z) = z^{-\beta}$, $\beta > 0$, then the elasticity of S is simply $-\beta$, and the inefficiencies are pinned down by its magnitude.[16]

4.2 Patent Race

A patent race is a well-studied and important economic application with negative spillovers. Because the pure patent game with only one discovery to make is zero sum, each team exerts a negative externality on the other. In this section we present a simple patent race model inspired by Loury (1980) seminal paper. Unlike his model, ours is static, which is not a significant difference as in his case the race proceeds mechanically after firms invest in R&D. Also, instead of an irreversible amount spent on R&D that determines the firm's probability of success, which in Loury (1980) can be interpreted as effort, we assume that the probability of success is determined by the composition of the R&D team of the firm, which is assembled at the matching stage.[17] This introduces an asymmetry in the model, as the effect on the probability of success of a high or low type member added to the team depends on whether the other member is of high or low type. Finally, another obvious difference is that a firm competes with a randomly chosen opponent after teams are formed.

Consider a market in the pharmaceutical industry where firms first hire their R&D teams and then compete pairwise against a randomly chosen firm. The quality of an innovation by firm i is uncertain and denoted by $v_i \in \{0, v\}$, where $v_i = 0$ is failure and $v_i = v$ success. The probability that a firm succeeds is p, which is increasing in X: $p(\overline{X}) > p(\hat{X}) > p(\underline{X})$. The firm that finds the best solution will obtain the patent and serve the entire market, which has a value equal to the value of its invention. Therefore we model $z(v_i, v_j) = \max\{v_i, v_j\}$. The loser gets a zero payoff. And if both firms succeed, so that both have an innovation of the same quality, then we assume that they obtain the patent with equal probability.

The payoff to team i with composition X_i facing a competitor with composition X_j is:

$$V_i(X_i|X_j) = p(X_i)p(X_j)\frac{v}{2} + p(X_i)(1 - p(X_j))v = vp(X_i)\left(1 - \frac{p(X_j)}{2}\right). \tag{15}$$

With probability $p(X_i)p(X_j)$ they both draw v, in which case they both get the patent with equal probability. With probability $p(X_i)(1 - p(X_j))$ firm i succeeds and j fails, and hence i obtains the patent at value v. When i

[16] One can construct a variation of this example based on Eeckhout and Jovanovic (2002) that allows for type dependent externality by assuming $S(\mathbb{E}[k|X > X_i, \mu])k(X_i)$, where a firm 'copies' only from better firms. For instance, under PAM only $\underline{X}$ copies from $\overline{X}$, under NAM there is no spillovers, and under α there is copying by both $\underline{X}$ and $\hat{X}$.

[17] We abstract from the important issue of endogenous timing of different stages in optimal patent design (e.g., Riis and Shi (2012)).

draws $v_i = 0$ which happens with probability $1-p(X_i)$, it either loses to firm j and gets zero, or also firm j draws 0 in which case they both obtain zero. Alternatively, i obtains v with probability $p(X_i)$ times the probability that j does not succeed long with i and obtains the innovation. Notice that, as expected, V_i is increasing in the success probability of i and decreasing in that of j, given the zero-sum game nature of the patent race. It is also submodular in (X_i, X_j), which stems from the submodularity of the max function z in (v_i, v_j).

From the analysis of the multiplicative case above, we know that in this case equilibrium is unique. Regarding the planner, notice that the total value of any two given teams X_i, X_j is $[1-(1-p(X_i))(1-p(X_j))]v$. Given random matching in a large market, under PAM there is one half probability of matching with $p_j = p(\overline{X})$ and one half with $p_j = p(\underline{X})$, while under NAM, $p_j = p(\hat{X})$ with probability one. From this it is fairly easy to write down the planner's objective function (see Appendix).

We now investigate the efficiency properties of the market equilibrium in the patent race application with matching. We abstract from other potential sources of inefficiencies (for example, those related to entry or to effort provision), and focus on the efficiency properties of sorting. We have the following result:

Proposition 6 *Equilibrium is efficient. The allocation has PAM/NAM if p is supermodular/submodular in X.*

It is easy to see from (15) that the equilibrium exhibits PAM (NAM) if and only if p is supermodular (submodular) in X. For V is multiplicatively separable in $p(X_i)$ and $p(X_i)$, and hence so is $\mathcal{V}(\cdot|\cdot)$ in $p(X_i)$ and the externality effect. It follows that $p(X_i)$ then determines the supermodularity (submodularity) of $\mathcal{V}$ in X_i.

It is somewhat surprising that the equilibrium is always efficient given the negative externality imposed by other teams. The intuition follows from the multiplicative structure of the payoff function. One can check that the difference of the effect of the externality under PAM and under NAM is negative. Thus, if p_i is submodular, then the complementarity and externality effects reinforce each other and NAM is efficient. So the only possibility for an inefficient equilibrium is when it exhibits PAM. But the negative differential externality effect is a fraction of the complementarity effect, and hence it is not enough to overturn the choice of PAM by the planner.[18]

4.3 Auctions between Teams

Consider an auction with two firms or teams bidding for one object, say, a license to be the sole producer of a good in a certain market. Intuitive, how large is the valuation of a firm or team for the license depends on its composition. For example, better engineers will make more efficient use of the bandwidth that a mobile operator can acquire in a spectrum auction. Or, better geologists will be able to get a more precise estimate of the value to extract timber from a plot of mountainous land. Clearly, the firm composition is endogenous, and it is only natural to add a pre-auction stage where it is determined in a matching environment.

The setup is as follows. First, all teams X_i are formed. Second, given the team composition, the value for the object for firm i, $v_i \in [0, \overline{v}]$, is drawn from a distribution $F(v_i|X_i)$. We assume for simplicity that valuations are independently distributed across firms, but we make no assumptions about the dependence of the distribution function F on X. Finally, each team X_i is paired with a team X_j and a second price auction is held. Teams submit their bids b_i and b_j given the realization of their valuations v_i and v_j respectively. The winning bid obtains the object and pays the loser's bid as a price. This setup fits the general setup with uncertainty with the payoff to firm i when its valuation is v_i and that of the competitor is v_j given by $z(v_i, v_j) = \max\{v_i - v_j, 0\}$. Starting from

[18] This argument relies on the random matching assumption that we have made. If instead one considers a 'small market' where each teams knows which other team they will compete against in the downstream stage, then equilibrium can be inefficient.

the initial equation (13) for firm value, we show in the Appendix that $V(X_i|X_j) = \int_0^{\overline{v}} F(v|X_j)(1-F(v|X_i)dv$. Intuitively, this expression asserts that the value of the auction for bidder i equals the expected probability of winning it. For any v, the integrand is $\mathbb{P}[v_j < v|X_j]\mathbb{P}[v_i > v|X_i]$, and the ex-ante payoff for bidder i is the integral of this product across all values of v.

In this application, our large market setting with pairwise auctions can be interpreted as one in with anonymous participants such as telephone bidders at Christie's or online auctions on eBay.[19] Since the opponent is unknown, the team's payoff incorporates the expectation of the composition over all randomly selected teams. But the distribution of teams varies with the equilibrium, and thus so does $\mathcal{V}$. Indeed, when the matching exhibits PAM, then with probability one half, a team X_i competes against a team $\overline{X}$ in which case its expected payoff is $\int_0^{\overline{v}} F(v|\overline{X})(1-F(v|X_i))dv$ and with probability one half it competes against a team $\underline{X}$ with payoff $\int_0^{\overline{v}} F(v|\underline{X})(1-F(v|X_i))dv$. Therefore

$$\mathcal{V}(X_i|\mu_+) = \int_0^{\overline{v}} \frac{F(v|\overline{X}) + F(v|\underline{X})}{2}(1-F(v|X_i))dv.$$

It is easy to verify that $\mathcal{V}(\cdot|\mu_+)$ is supermodular in X, and thus the equilibrium exhibits PAM, if and only if

$$\int_0^{\overline{v}} \frac{F(v|\overline{X}) + F(v|\underline{X})}{2}\left(2F(v|\hat{X}) - F(v|\overline{X}) - F(v|\underline{X})\right) dv \geq 0,$$

and this holds if $F(v|\cdot)$ is submodular in X for all v.

Similarly, if the matching exhibits NAM we obtain

$$\mathcal{V}(X_i|\mu_-) = \int_0^{\overline{v}} F(v|\hat{X})(1-F(v|X_i))dv.$$

and $\mathcal{V}(\cdot|\mu_-)$ is submodular in X, which implies that the equilibrium exhibits NAM, if and only if

$$\int_0^{\overline{v}} F(v|\hat{X})\left(F(v|\overline{X}) + F(v|\underline{X}) - 2F(v|\hat{X})\right) dv \geq 0,$$

which holds if $F(v|\cdot)$ is supermodular in X for all v.

Albeit restrictive, there are simple sufficient conditions on F that ensure the equilibrium either PAM or NAM. Moreover, under these conditions equilibrium is unique (e.g., if $F(v|\cdot)$ is submodular in X, then there cannot be an equilibrium with NAM, and similarly for PAM and submodularity).

For the planner's solution, we can verify that

$$\begin{aligned} A &= -\frac{1}{2}\int_0^{\overline{v}} \left(2F(v|\hat{X}) - F(v|\overline{X}) - F(v|\underline{X})\right)^2, \\ B &= \int_0^{\overline{v}} \left(2F(v|\hat{X}) - 1\right)\left(2F(v|\hat{X}) - F(v|\overline{X}) - F(v|\underline{X})\right) dv, \\ B + 2A &= \int_0^{\overline{v}} \left(F(v|\overline{X}) + F(v|\underline{X}) - 1\right)\left(2F(v|\hat{X}) - F(v|\overline{X}) - F(v|\underline{X})\right) dv. \end{aligned}$$

Notice that $A < 0$ (unless F is independent of X), and hence the planner's objective is strictly concave in α.

[19] Knowledge of the teams at the bidding stage is irrelevant for strategic behavior since in the second price auction it is a dominant strategy to bid the true valuation.

The next result is a straightforward implication of the analysis above and Proposition 3.

Proposition 7 *(i) If $F(v|\cdot)$ is submodular in X for all v, then the equilibrium exhibits PAM. It is inefficient if* $\int_0^{\overline{v}} \left(2F(v|\hat{X}) - 1\right)\left(2F(v|\hat{X}) - F(v|\overline{X}) - F(v|\underline{X})\right) dv \leq 0$*;*

(ii) If $F(v|\cdot)$ is supermodular in X for all v, then the equilibrium exhibits NAM. It is inefficient if $\int_0^{\overline{v}} \left(F(v|\overline{X}) + F(v|\underline{X}) - 1\right)\left(2F(v|\hat{X}) - F(v|\overline{X}) - F(v|\underline{X})\right) dv \geq 0$.

Unfortunately it is not easy to pin down a general class of F that will yield an inefficient equilibrium. But it is straightforward to construct examples. For instance, assume that X is the sum of the types of the team, $\overline{x} = 1$, $\underline{x} \in (0, 1)$, $F(v|X) = v^X$, and $v \in [0, 1]$. Since F is strictly convex in X, it follows that $F(v|\cdot)$ is supermodular in the types of the members of the team. It is also strictly decreasing in X, so it exhibits the natural property that better teams yield stochastically better valuations. Supermodularity of F implies that the equilibrium exhibits NAM. The proposition then asserts that it will be inefficient if $B + 2A \geq 0$. Tedious integration yields

$$B + 2A = \frac{2}{15} - \frac{1}{1+4a} - \frac{2}{2+a} + \frac{2}{4+a} + \frac{1}{1+2a} - \frac{2}{3+2a} + \frac{2}{2+3a},$$

and this expression is positive for $a > 0.0322$. Thus, there is a range of values for which equilibrium is inefficient. The planner would like a drastic change in the equilibrium allocation, from one extreme matching to the other.

4.4 Oligopolistic Competition

Another obvious form of strategic interaction in the output market is oligopolistic competition. Firms hire workers, and the composition of the firm's labor force determines the cost of production. Then they compete pairwise in the output market. We will use the canonical example with a linear demand given by $P = a - b(q_1 + q_2)$, and where the firm's marginal cost is constant for all units produced, but its level is determined by the team composition: that is, $C_i = c(x_1, x_2)q_i$. As before, we adopt the convention for $c(\overline{X}), c(\hat{X}), c(\underline{X})$, noting that here high types have low costs: $c(\overline{X}) < c(\hat{X}) < c(\underline{X})$. We consider Bertrand competition since it provides a nice illustration of the main results. Since the model assumes homogeneous products, we have:

$$\begin{aligned} V(X_i|X_j) &= (c(X_j) - c(X_i))\left(\frac{a - c(X_j)}{b}\right) \\ V(X_j|X_i) &= 0, \end{aligned}$$

whenever $c(X_i) \leq c(X_j)$. The firm with the lowest cost gets the entire market at a price equal to marginal cost of the high cost firm. Then $V(\underline{X}|\overline{X}) = V(\overline{X}|\overline{X}) = V(\underline{X}|\underline{X}) = V(\hat{X}|\overline{X}) = V(\hat{X}|\hat{X}) = 0$, $V(\overline{X}|\underline{X}) = (c(\underline{X}) - c(\overline{X}))\left((a - c(\underline{X}))/b\right)$, $V(\overline{X}|\hat{X}) = (c(\hat{X}) - c(\overline{X}))\left((a - c(\hat{X}))/b\right)$, and $V(\hat{X}|\underline{X}) = (c(\underline{X}) - c(\hat{X}))\left((a - c(\underline{X}))/b\right)$.

Consider first equilibrium behavior. The condition for PAM is that $\mathcal{V}(\cdot|\mu_+)$ be supermodular in X, or

$$\frac{1}{2}(c(\underline{X}) - c(\overline{X}))\left(\frac{a - c(\underline{X})}{b}\right) \geq (c(\underline{X}) - c(\hat{X}))\left(\frac{a - c(\underline{X})}{b}\right),$$

and this holds if and only if $c(\overline{X}) + c(\underline{X}) - 2c(\hat{X}) \leq 0$, or c submodular in X.

The condition for NAM, however, never holds, for $\mathcal{V}(\cdot|\mu_-)$ submodular in X is tantamount to

$$0 \geq (c(\hat{X}) - c(\overline{X}))\left(\frac{a - c(\hat{X})}{b}\right),$$

and the right side is positive. Notice however, that an interior equilibrium $\alpha \in (0,1)$ exists when c is supermodular, since there is a value of α that makes $\mathcal{V}(\cdot|\alpha)$ modular in X. In fact, easy algebra shows that

$$\mathcal{V}(\overline{X}|\alpha) + \mathcal{V}(\underline{X}|\alpha) = 2\mathcal{V}(\hat{X}|\alpha) \;\Leftrightarrow\; \alpha = \frac{(a - c(\hat{X}))(c(\hat{X}) - c(\overline{X}))}{(a - c(\underline{X}))(c(\overline{X}) + c(\underline{X}) - 2c(\hat{X})) + (a - c(\hat{X}))(c(\hat{X}) - c(\overline{X}))},$$

which is clearly between zero and one when c is supermodular in X. And since $c(\overline{X}) + c(\underline{X}) - 2c(\hat{X})$ is either positive or negative, it follows that an equilibrium always exists.

Regarding the planner's problem, A and B are given by

$$\begin{aligned} A &= -\left(c(\overline{X}) + c(\underline{X}) - 2c(\hat{X})\right)\left(\frac{a - c(\underline{X})}{2b}\right) - \left(c(\hat{X}) - c(\overline{X})\right)\left(\frac{a - c(\hat{X})}{b}\right) \\ B &= \left(c(\hat{X}) - c(\overline{X})\right)\left(\frac{a - c(\hat{X})}{b}\right) + \left(c(\underline{X}) - c(\hat{X})\right)\left(\frac{a - c(\underline{X})}{b}\right). \end{aligned}$$

If c is submodular in X, so the equilibrium exhibits PAM, then it is easy to check that $A < 0$, so the planner's objective function is strictly concave in α. From Proposition 3, equilibrium will be inefficient if either $B \leq 0$ or if $B > 0$ and $B + 2A < 0$. Now, by inspection $B > 0$, while simple algebra reveals that $B + 2A$ is given by

$$\begin{aligned} B + 2A &= \left(c(\underline{X}) - c(\hat{X})\right)\left(\frac{a - c(\underline{X})}{b}\right) - \left(c(\overline{X}) + c(\underline{X}) - 2c(\hat{X})\right)\left(\frac{a - c(\underline{X})}{b}\right) - \left(c(\hat{X}) - c(\overline{X})\right)\left(\frac{a - c(\hat{X})}{b}\right) \\ &= \frac{\left(c(\hat{X}) - c(\overline{X})\right)\left(c(\hat{X}) - c(\underline{X})\right)}{b} < 0. \end{aligned}$$

Thus, the optimal matching is interior and given by $\alpha = -B/2A$.

If c is supermodular in X, it is evident that $A < 0$ and once again the optimal matching is interior and given by $\alpha = -B/2A$. Notice that the interior value for the optimal matching is different from that of the equilibrium.

We have thus proved the following result:

Proposition 8 *(i) If c is submodular in X, the equilibrium exhibits PAM and it is inefficient;*
(ii) If c is supermodular in X, there is the equilibrium matching is interior and inefficient.

The intuition for an optimal interior matching is clear in this setting. Under NAM, all the firms obtain zero profits, while under PAM only a firm with $\overline{X}$ competing against a firm with $\underline{X}$ yields positive profits. In the mixing case $\alpha \in (0,1)$, matches between $\overline{X}$ and either $\hat{X}$ or $\underline{X}$, or between $\hat{X}$ and $\underline{X}$ yield positive profits, thus increasing the overall payoff of the planner. The inefficient is now evident when the equilibrium exhibits PAM, and it is also intuitive when the equilibrium matching is interior, as firms do not internalize the externality.

4.5 Policy Implications and Balanced Competition

In settings with externalities, it is a truism to wonder about policy remedies. To avoid revisiting each of the above applications and in order to make the following discussion more vivid, we illustrate some policy prescriptions by means of a simple example, inspired by sports competitions.[20]

[20]See Palomino and Sákovics (2004) for the explicit analysis of team composition in sports competitions with competitive balance.

We analyze policy starting from the benchmark model with uncertainty, where the team's realized performance v_i is stochastic, but drawn from a distribution that depends on the team composition X_i. Consider a very simple setup where the team's outcome in a game is $z_i = a_0 + av_i + bv_j$ with $a_0, a \geq 0$. This is the setup we analyzed in the section on spillovers, where we showed that equilibrium is unique and the planner can wlog choose between PAM or NAM. We will focus on the case of $b < 0$: better performance of the opponent generates negative spillovers. We can write $V(X_i|X_j) = a_0 + (a+2b)m(X_i) + bm(X_j)$ and therefore

$$\begin{aligned}
\mathcal{V}(X_i|\mu_+) &= a_0 + (a+2b)m(X_i) + b\,\frac{m(\overline{X}) + m(\underline{X})}{2} \\
\mathcal{V}(X_i|\mu_-) &= a_0 + (a+2b)m(X_i) + b\,m(\hat{X})
\end{aligned}$$

Then building immediately on the derivations from Section 4.1, we can write the conditions for PAM in equilibrium $(a+2b)\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right) > 0$ and NAM by the planner $(a+3b)\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right) < 0$, which are satisfied provided $b \in \left[-\frac{a}{2}, -\frac{a}{3}\right]$ and m is supermodular in X.

In this simple context we analyze three different policies that are applied in actual top sports competitions: Taxes, a Salary Cap and a Rookie Draft. These policies should be illustrative of their potential in markets beyond these sports competitions. To streamline the presentation, we place in the Appendix the formal analysis of the assertions we make below, and we limit ourselves to discussing the findings.

Taxes. If the planner wants to implement the NAM allocation, she can do so by taxing $\overline{X}$ and $\underline{X}$ allocations and by subsidizing $\hat{X}$ allocations. Let $\bar{t}$ be the tax on a team with one high type that hires another hight type and $\bar{s}$ the subsidy if it hires a low type. Likewise, let $\underline{t}$ be the tax on the firm with a low type that hires a second low type and $\underline{s}$ the subsidy if it hires a high type. We will consider budget balancing policies.

In the Appendix we show that with taxes sufficiently high, we can implement the NAM allocation. Observe that we need to check two types of conditions: (i) given a PAM allocation and taxes, all workers are willing to switch to the NAM allocation; (ii) given the desired NAM allocation, no worker is willing to deviate to PAM. Those conditions induce different payoffs because they involve different beliefs about the equilibrium allocation.

It turns out that tax of the $\overline{X}$ firm happens to be the same as the tax on the $\underline{X}$ firm, which is an artifact of the simple problem that we are considering here. In general these taxes will be different. What the taxes achieve is to drive a wedge between the private value of PAM and that of NAM. Not surprisingly therefore, the tax is proportional to the private surplus of a match under PAM relative to NAM.

Salary Cap. Suppose the planner imposes cap on the highest wage paid that teams cannot exceed. Denote the cap on the highest wage by C. To be interesting, we require $C \in [\underline{w}, \overline{w})$. Consider a team that has to decide which two player types to hire. In a PAM equilibrium, where a team has two high types, a salary cap will not dissuade the team from hiring high types. If anything, it can get the high type agent at a lower wage. At most, the team with two low types might now find it attractive to bid for a high type agent. In the Appendix we show that the Salary Cap as described here will not change the inefficient equilibrium allocation.

It should be noted though that what is known as a salary cap in the US sports competitions (NBA for example), is effectively a tax and subsidy scheme. Teams can if they want spend above a threshold total amount on salaries for the players, but for each dollar spent above the wage cap, an additional dollar must be paid to NBA who distributes it amongst all the teams. Effectively that means we have a policy in place with taxes (see above), with the caveat that in case where a tax is effectively paid, it is then distributed amongst the other teams.

Rookie Draft. Now consider the policy where the weakest teams can first pick the best players at a given, exogenously set wage rate. This set wage should be below the equilibrium wage to have any effect. Also, there must be a seniority difference amongst players. To address this issue in our model, suppose that there are two subgroups of players, seniors and rookies. For each subgroup, assume an equal number of high and a low types.

We can interpret the problem as each senior hiring one rookie. Since teams in our model are very stylized, we assume that low type seniors have the first pick in the rookie draft, given some exogenously set wages. Each low type senior chooses between $\underline{x}$ and $\overline{x}$, taking as given wages $\underline{w}_D$ and $\overline{w}_D$ (the D stands for draft).

In the Appendix, for this sequential variation of the model, we look for an equilibrium outcome where low type seniors choose a high type in the rookie draft; thus, high type seniors have no choice but to pick a low type rookie. Moreover, we show that wages $\underline{w}_D$ and $\overline{w}_D$ can be chosen in a way that leaves both senior types better off under the NAM than under the original PAM equilibrium.[21] In other words, the rookie draft can implement the efficient NAM allocation even if the equilibrium outcome is PAM.

5 Conclusion

Complementarities – or peer group effects – and their identification are arguably one of the most challenging problems in economics. In a competitive environment without externalities where the composition of groups (teams or firms) is governed by market prices, there are however no efficiency grounds for intervention. The complementarities or within team externalities are correctly priced at the allocation stage, thus resulting in a Pareto efficient allocation. While there may be equity reasons for intervention, there are no efficiency arguments.

In this paper we have argued that in many market settings, the presumption that teams operate in isolated output markets is tenuous. Often there is strategic interaction between teams once their composition is determined. Or alternatively, there may be direct technological spillovers. We have shown that the equilibrium features in a matching model with externalities differ in several dimensions from the standard matching model: there can be multiple equilibria; both optimal and equilibrium matching can involve randomization; equilibrium can be inefficient with a matching that can drastically deviate from the optimal one; and match complementarities are no longer exclusively related to sorting. We have elaborated on several economic applications that are tangible: knowledge spillovers, patent races, auctions between competing teams, and oligopolistic market competition. Our setup should also be useful to study a diverse range of further issues such as tracking in education (should the best children be put on a different track or should they be mixed with all other kids), or executive compensation.

The setup is amenable for *estimation*. In particular, the version of the model with uncertainty is easily adaptable to account for heterogeneity in outcomes in the product market competition stage. The heterogeneity is part of the model and determines agents' equilibrium choices and as a result equilibrium wages. We can see at least two directions for empirical work. First, with sufficiently detailed data on team composition (research teams, sports teams, class rooms, etc.) and individual performance, one could identify the nature of the externalities in conjunction with the nature of the complementarities. Clearly, if a market setting is estimated with a model without externalities, the obtained estimates for complementarities will be biased. Second, the model can be estimated using wage data. Wages reflect the allocation and if the allocation is inefficient, this will be evident in the wage distribution. Even if the allocation is efficient (say PAM in equilibrium as well as PAM by the planner), wages nonetheless will incorporate the inefficiency and will not be set at private marginal product. Data on

[21] In real world applications such as the NBA draft, rookie wages are set by the NBA and not by the market. So by checking that both senior types prefer the draft to the original PAM allocation, we are ensuring that no senior type will vote against the change.

markups in output markets for example will therefore give an indication of the extent of the externality, and as a consequence of the extent to which wages are set inefficiently.

Much work remains to be done. Our stylized set up permits a complete analysis of the issues at hand, and the insights we derived do not seem to depend on our assumptions. It would be interesting to explore extensions of the model with a large number of types, or with larger size teams, or the case with a finite number of agents (as in the literature discussed in the Introduction). In the Appendix, we illustrate via examples some of the features of these extensions; a thorough analysis is left for future research.

Appendix

Proof of Proposition 4

Integrating by parts V_i twice, we obtain (where $S_i = 1 - F_i$):

$$
\begin{aligned}
V(X_i|X_j) &= z(1,1) - \int_0^1 z_i(v_i,1)F_i dv_i - \int_0^1 z_j(1,v_j)F_j dv_j + \int_0^1\int_0^1 z_{ij}(v_i,v_j)F_iF_j dv_i dv_j \\
&= z(1,1) - z(1,1) + \int_0^1 z(v_i,1)dF_i - z(1,1) + \int_0^1 z(1,v_j)dF_j + \int_0^1\int_0^1 z_{ij}(v_i,v_j)(1-S_i)(1-S_j)dv_i dv_j \\
&= \int_0^1 z(v_i,1)dF_i + \int_0^1 z(1,v_j)dF_j - z(1,1) + \int_0^1\int_0^1 z_{ij}(v_i,v_j)(F_i+F_j)dv_i dv_j - \int_0^1\int_0^1 z_{ij}(v_i,v_j)dv_i dv_j \\
&\quad + \int_0^1\int_0^1 z_{ij}(v_i,v_j)S_iS_i dv_i dv_j \\
&= \int_0^1 z(v_i,1)dF_i + \int_0^1 z(1,v_j)dF_j - z(1,1) + z(1,1) - z(1,0) - \int_0^1 \left[z(v_i,1) - z(v_i,0)\right] dF_i + z(1,1) \\
&\quad - z(0,1) - \int_0^1 \left[z(1,v_j) - z(0,v_j)\right] dF_j - z(1,1) + z(1,0) + z(0,1) - z(0,0) + \int_0^1\int_0^1 z_{ij}(v_i,v_j)S_iS_i dv_i dv_j \\
&= -z(0,0) + \int_0^1\int_0^1 z_{ij}(v_i,v_j)S_iS_j dv_i dv_j + z(1,0) + z(0,1) - \int_0^1 z_i(v_i,0)F_i dv_i - \int_0^1 z_j(0,v_j)F_j dv_j \\
&= -z(0,0) + z(1,0) + z(0,1) + \int_0^1\int_0^1 z_{ij}(v_i,v_j)S_iS_j dv_i dv_j + \int_0^1 z_i(v_i,0)S_i dv_i - \int_0^1 z_i(v_i,0)dv_i \\
&\quad + \int_0^1 z_j(0,v_j)S_j dv_j - \int_0^1 z_j(0,v_j)dv_j \\
&= -z(0,0) + z(1,0) + z(0,1) + \int_0^1\int_0^1 z_{ij}(v_i,v_j)S_iS_j dv_i dv_j + \int_0^1 z_i(v_i,0)S_i dv_i - z(1,0) + z(0,0) \\
&\quad + \int_0^1 z_j(0,v_j)S_j dv_j - z(0,1) + z(0,0) \\
&= z(0,0) + \int_0^1 z_i(v_i,0)S_i dv_i + 2\int_0^1 z_j(0,v_j)S_i dv_j + \int_0^1 z_j(0,v_j)S_j dv_j + \int_0^1\int_0^1 z_{ij}(v_i,v_j)S_iS_j dv_i dv_j \\
&= g(X_i) + h(X_j) + k(X_i,X_j),
\end{aligned}
$$

which completes the proof of the proposition. □

Proof of Proposition 5

To simplify the notation, we set $m(X_i) \equiv m_i$, $\underline{m} \equiv m(\underline{X})$, $\overline{m} \equiv m(\overline{X})$, and $\hat{m} \equiv m(\hat{X})$. Then

$$
\begin{aligned}
\mathcal{V}(X_i|\mu_+) &= (a+2b)m_i + b\left(\frac{\overline{m} + \underline{m}}{2}\right) \\
&= (a+2b)m_i + bm(\mu_+) \\
\mathcal{V}(X_i|\mu_-) &= (a+2b)m_i + b\hat{m} \\
&= (a+2b)m_i + bm(\mu_-).
\end{aligned}
$$

Since the complementarities in X are independent of the matching, all the results from the additive case apply. Thus, the equilibrium sorting patter depends on the sign of $(a+2b)(\overline{m}+\underline{m}-2\hat{m})$, while the planner's optimal choice depends on the sign of $(a+2b)(\overline{m}+\underline{m}-2\hat{m})-2b(m(\mu_-)-m(\mu_+))$, which, using the expressions above, can be rewritten as $(a+3b)(\overline{m}+\underline{m}-2\hat{m})$. It is now clear that equilibrium is inefficient if and only if the signs of $(a+2b)$ and $(a+3b)$ differ, and this can only occur if $b\in(-a/2,-a/3)$. If m supermodular (submodular) and b is in this range, then the equilibrium exhibits PAM but the planner chooses NAM (PAM). □

Proof of Proposition 6

To simplify the notation, we set $p_i \equiv p(X_i)$, $p_j \equiv p(X_j)$, $\underline{p} \equiv p(\underline{X})$, $\overline{p} \equiv p(\overline{X})$, and $\hat{p} \equiv p(\hat{X})$.

Consider first the equilibrium analysis of this market. It exhibits PAM provided that $\mathcal{V}(\cdot|\mu_+)$ is supermodular in X. Since $V(X_i|X_j) = vp_i(1-p_j/2)$, it follows that

$$\mathcal{V}(X_i|\mu_+) = \frac{vp_i}{2}\left(2-\frac{\overline{p}+\underline{p}}{2}\right),$$

from which it is clear that $\mathcal{V}(\cdot|\mu_+)$ is supermodular in X_i if and only if p_i is supermodular in X_i.

A similar analysis holds for NAM, since

$$\mathcal{V}(X_i|\mu_-) = vp_i(1-\frac{\hat{p}}{2}),$$

and hence it is submodular in X_i if and only if p_i is submodular in X_i.

Consider now the planner's problem. Using the formulae above for $\mathcal{V}$ one can show that the coefficient A in the planner's objective function (see (2) is given by

$$A = -\frac{v}{4}(\overline{p}+\underline{p}-2\hat{p})^2 \leq 0,$$

with strict inequality if if the parenthesis is not equal to zero, the generic case. Hence, the planner's objective function is strictly concave in the matching α. To use Proposition 1, we first compute B, given by

$$\begin{aligned} B &= v\left(\overline{p}+\underline{p}-2\hat{p}\right)\left(1-\frac{\hat{p}}{2}\right)+2\left(\frac{v\hat{p}}{2}\left(2-\frac{\overline{p}+\underline{p}}{2}\right)-v\hat{p}\left(1-\frac{\hat{p}}{2}\right)\right) \\ &= v\left(\overline{p}+\underline{p}-2\hat{p}\right)(1-\hat{p}). \end{aligned}$$

Notice that if p_i is submodular in X_i, then the planner chooses NAM (see Proposition 1–(*ii*)) and hence the equilibrium is efficient, thereby proving the efficiency of NAM stated in the proposition.

To complete the proof we compute $B+2A$, given by

$$\begin{aligned} B+2A &= v\left(\overline{p}+\underline{p}-2\hat{p}\right)(1-\hat{p})-\frac{v}{4}(\overline{p}+\underline{p}-2\hat{p})^2 \\ &= \frac{v\left(\overline{p}+\underline{p}-2\hat{p}\right)}{4}\left(4-\left(\overline{p}+\underline{p}+2\hat{p}\right)\right). \end{aligned}$$

Since $B+2A\geq 0$ when p_i is supermodular in X_i, equilibrium is efficient whenever it exhibits PAM. □

Derivation of $V(X_i|X_j)$ in Auctions between Teams

$$\begin{aligned}
V(X_i|X_j) &= \int_0^{\overline{v}}\int_0^{\overline{v}} \max\{v_i - v_j, 0\} dF(v_i|X_i)dF(v_j|X_j) \\
&= \int_0^{\overline{v}}\int_{v_j}^{\overline{v}} (v_i - v_j) dF(v_i|X_i)dF(v_j|X_j) \\
&= \int_0^{\overline{v}} \left(1 - v_j F(v_j|X_i) - \int_{v_j}^{\overline{v}} F(v_i|X_i)dv_i - v_j(1 - F(v_j|X_i))\right) dF(v_j|X_j) \\
&= \int_0^{\overline{v}} \left(\int_{v_j}^{\overline{v}} (1 - F(v_i|X_i))dv_i\right) dF(v_j|X_j) \\
&= \left(\int_{v_j}^{\overline{v}} (1 - F(v_i|X_i))dv_i\right) F(v_j|X_j)\Bigg|_0^{\overline{v}} - \int_0^{\overline{v}} F(v_j|X_j)\left(-(1 - F(v_j|X_i))\right) dv_j \\
&= \int_0^{\overline{v}} F(v_j|X_j)(1 - F(v_j|X_i))dv_j,
\end{aligned}$$

which is the expression in the text. □

Policy Implications and Balanced Competition: Formal Analysis

First we derive the equilibrium wages. Given zero profits, the wage of a match with identical agents splits the surplus in half. This is consistent with the property rights interpretation where any worker can hire another co-worker. In our large market setup, the outside option to taking a wage offer from an identical type x is to make offer to another identical type x. Under PAM, Proposition 2 shows that $w(X_i|\mu) = \frac{1}{2}\mathcal{V}(X_i|\mu)$ and hence

$$\begin{aligned}
\overline{w} &= \frac{1}{2}\left[a_0 + (a + 2b)m(\overline{X}) + b\,\frac{m(\overline{X}) + m(\underline{X})}{2}\right] \\
\underline{w} &= \frac{1}{2}\left[a_0 + (a + 2b)m(\underline{X}) + b\,\frac{m(\overline{X}) + m(\underline{X})}{2}\right].
\end{aligned}$$

Similarly, the following are wages that support a NAM equilibrium:

$$\begin{aligned}
\overline{w} &= \frac{1}{2}\left[\mathcal{V}(\hat{X}|\mu_-) + \frac{1}{2}\left[\mathcal{V}(\overline{X}|\mu_-) - \mathcal{V}(\underline{X}|\mu_-)\right]\right] = \frac{1}{2}\left[a_0 + (a + 3b)m(\hat{X}) + \frac{1}{2}(a + 2b)(m(\overline{X}) - m(\underline{X}))\right] \\
\underline{w} &= \frac{1}{2}\left[\mathcal{V}(\hat{X}|\mu_-) - \frac{1}{2}\left[\mathcal{V}(\overline{X}|\mu_-) - \mathcal{V}(\underline{X}|\mu_-)\right]\right] = \frac{1}{2}\left[a_0 + (a + 3b)m(\hat{X}) - \frac{1}{2}(a + 2b)(m(\overline{X}) - m(\underline{X}))\right].
\end{aligned}$$

We now analyze the three policies in turn: Taxes, Salary Cap and Rookie Draft.

Taxes. The results described in the main text are formalized in the next proposition:

Proposition 9 *Suppose the planner's allocation exhibits NAM. A pair of taxes $\overline{t}, \underline{t}$ on high type firms $\overline{X}$ and low type firms $\underline{X}$, along with $\overline{s} = \underline{s} = 0$, implement the optimal allocation provided that*

$$\overline{t} > \frac{1}{2}(a + 2b)\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right) \quad \text{and} \quad \underline{t} > \frac{1}{2}(a + 2b)\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right).$$

Proof. Consider first an equilibrium allocation that is PAM. Given taxes and subsidies, both types of agents will prefer to hire a different type provided that

$$(a+2bm(\overline{X}))+b\frac{m(\overline{X})+m(\underline{X})}{2}-\overline{w}-\overline{t}<(a+2b)m(\hat{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}-\underline{w}+\overline{s}$$
$$(a+2b)m(\underline{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}-\underline{w}-\underline{t}<(a+2b)m(\hat{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}-\overline{w}+\underline{s}.$$

After substituting for the equilibrium wages we obtain

$$\overline{t}>\frac{1}{2}(a+2b)\left(m(\overline{X})+m(\underline{X})-2m(\hat{X})\right)$$
$$\underline{t}>\frac{1}{2}(a+2b)\left(m(\overline{X})+m(\underline{X})-2m(\hat{X})\right)$$

where we have set $\overline{s}=\underline{s}=0$ to ensure budget balance, in which case when these conditions are satisfied, there will be no taxes actually paid, and the threat of the tax will deter the PAM allocation.

Second, we also need to verify that under a NAM equilibrium, there is no incentive to deviate to PAM. In that case, the taxes (again with $\overline{s}=\underline{s}=0$) must satisfy:

$$(a+2b)m(\overline{X})+bm(\hat{X})-\overline{w}-\overline{t}<(a+2b)m(\hat{X})+bm(\hat{X})-\underline{w}$$
$$(a+2b)m(\underline{X})+bm(\hat{X})-\underline{w}-\underline{t}<(a+2b)m(\hat{X})+bm(\hat{X})-\overline{w}$$

which after substituting for the NAM equilibrium wages we obtain:

$$\overline{t}>\frac{1}{2}(a-2b)(m(\overline{X})+m(\underline{X})-2m(\hat{X}))$$
$$\underline{t}>\frac{1}{2}(a-2b)(m(\overline{X})+m(\underline{X})-2m(\hat{X})).$$

Comparing these constraints on the taxes under both equilibrium beliefs, we need to satisfy the maximum of the two conditions on $\overline{t}$ and the maximum on the two conditions on $\underline{t}$, which are identical in both cases. □

Salary Cap. Under PAM and without imposing the salary cap, we have

$$(a+2b)m(\overline{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}+2\overline{w}>(a+2b)m(\hat{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}-\overline{w}-\underline{w}$$
$$(a+2b)m(\underline{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}+2\underline{w}>(a+2b)m(\hat{X})+b\frac{m(\overline{X})+m(\underline{X})}{2}-\overline{w}-\underline{w}$$

If the salary cap $C<\overline{w}$ binds, these conditions become

$$(a+2b)m(\overline{X})-2C>(a+2b)m(\hat{X})-C-\underline{w}$$
$$(a+2b)m(\underline{X})-2\underline{w}>(a+2b)m(\hat{X})-C-\underline{w}.$$

As mentioned, it is clear that the cap will not have any effect on the team consisting of two high types, for now they have to pay them a lower salary and thus profits are even higher than before. At best, the salary cap can affect the incentives of the team with two low types to hire one high type instead and have a more balanced composition. In short, the salary cap will not be effective in moving the equilibrium from PAM to NAM.

But if the unconstrained wages exceed the constraint, then PAM profits under the constraint will be even larger. This therefore does not affect the equilibrium allocation.

Proposition 10 *A salary cap* $C \in [\underline{w}, \overline{w})$ *cannot switch the allocation from PAM to NAM.*

Rookie Draft. Consider a low type senior who conjectures that all the other low types will choose a high type rookie. He will choose a high type rookie as well so long as

$$(a+2b)m(\hat{X}) + bm(\hat{X}) - \overline{w}_D \geq (a+2b)m(\underline{X}) + bm(\hat{X}) - \underline{w}_D,$$

which reduces to

$$(a+2b)\left(m(\hat{X}) - m(\underline{X})\right) \geq \overline{w}_D - \underline{w}_D. \tag{16}$$

Notice that, since the original PAM equilibrium condition was

$$\overline{w} - \underline{w} \geq (a+2b)\left(m(\hat{X}) - m(\underline{X})\right),$$

it follows that under the draft $\overline{w}_D - \underline{w}_D \leq \overline{w} - \underline{w}$, so salaries are *more compressed* or *less spread out*.

A low type senior prefers NAM that ensues under (16) to the PAM allocation in the original equilibrium if

$$(a+2b)m(\hat{X}) + bm(\hat{X}) - \overline{w}_D \geq (a+2b)m(\underline{X}) + b\frac{m(\overline{X}) + m(\underline{X})}{2} - \underline{w},$$

which reduces to

$$\overline{w}_D \leq \underline{w} + (a+2b)\left(m(\hat{X}) - m(\underline{X})\right) - \frac{b}{2}\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right). \tag{17}$$

Similarly, a high type senior prefers the new NAM allocation to the original PAM equilibrium allocation if

$$(a+2b)m(\hat{X}) + bm(\hat{X}) - \underline{w}_D \geq (a+2b)m(\overline{X}) + b\frac{m(\overline{X}) + m(\underline{X})}{2} - \overline{w},$$

or, equivalently,

$$\overline{w}_D \leq \overline{w} - (a+2b)\left(m(\overline{X}) - m(\hat{X})\right) - \frac{b}{2}\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right). \tag{18}$$

It is easy to find nonnegative wages $\overline{w}_D$ and $\underline{w}_D$ such that (16), (17), and (18) are satisfied. For example, if one sets (16) with equality, then inserting it in (17) and using (18) we obtain

$$\overline{w}_D \leq \min\left\{\underline{w}, \overline{w} - (a+2b)\left(m(\overline{X}) - m(\hat{X})\right)\right\} - \frac{b}{2}\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right).$$

But the PAM equilibrium condition for the high type reveals that the second term inside the min is the smallest one, and hence

$$\overline{w}_D \leq \overline{w} - (a+2b)\left(m(\overline{X}) - m(\hat{X})\right) - \frac{b}{2}\left(m(\overline{X}) + m(\underline{X}) - 2m(\hat{X})\right).$$

Therefore, if we set (16) and (18) with equality, the resulting $\overline{w}_D$ and $\underline{w}_D$ induce all the low type seniors to pick a high type rookie, and both the low and high type seniors are better off under the new NAM than under the original PAM equilibrium allocation. We have thus shown the following result:

Proposition 11 *Suppose the equilibrium is PAM and the efficient allocation is NAM. Then a rookie draft with exogenously set wages can implement the efficient allocation.*

Variations of the Model

We discuss three potential variations of our setup, namely, the case with a continuum of types, the case with teams larger than two, and finally the case with a small number of agents. Our intention is not to derive anew all the insights but to shed light on the similarities and differences of the results one might obtain in these variations.

A Continuum of Types

By means of an example, we show that the mechanics of our analysis is robust beyond the two type distribution we have analyzed. Consider a population of workers with measure one, indexed by type x, uniformly distributed over $[0, 1]$. They form teams of two denoted by $\{x_1, x_2\}$, where, with some abuse of the notation, we will use X_i to denote the product of types x_1x_2 and X_j to denote the sum of types $x_1 + x_2$. Teams are formed in a competitive market, and there is a measure 1/2 of teams. In the second stage, teams randomly draw an opponent team against which output V is generated. The value conditional on the opponent type X_j of the team is

$$V(X_i|X_j) = ax_1x_2 + bX_j^\beta.$$

To simplify the analysis, we just focus on the extreme matchings exhibiting PAM and NAM, ignoring the possibility of intermediate cases. Under PAM and NAM, the expected value of forming a match X_i is, respectively,

$$\begin{aligned}\mathcal{V}(X_i|\mu_+) &= ax^2 + b\int_0^1 (2x)^\beta dx = ax^2 + \frac{b2^\beta}{1+\beta},\\ \mathcal{V}(X_i|\mu_-) &= ax(1-x) + b\int_0^1 (x+1-x)^\beta dx = ax(1-x) + b.\end{aligned}$$

As before, the characteristics of an equilibrium allocation are independent of the external effect. When $a > 0$, there are complementarities in output and there is PAM. When $a < 0$, there is NAM. Instead, the planner does take into account the externalities. The planner's value under PAM is

$$\frac{1}{2}\left(\int_0^1 ax^2 dx + \frac{b2^\beta}{1+\beta}\right) = \frac{a}{6} + \frac{b2^{\beta-1}}{1+\beta}$$

while under NAM the planner's value is

$$\frac{1}{2}\left(\int_0^1 ax(1-x)dx + b\right) = \frac{a}{12} + \frac{b}{2}.$$

Therefore, provided $a > 0$, the equilibrium will be PAM and the planner will choose NAM when $\frac{a}{12} < b\left(\frac{1}{2} - \frac{2^{\beta-1}}{1+\beta}\right)$, and there is inefficiency. For an example, set $b = 1, \beta = 1/2$ and a less than $1/3$. Intuitively, the existence of an inefficient equilibrium depends on the complementarity parameter a and the externality parameters b and β.

Large Teams

We can do a similar exercise and illustrate by example how to handle teams larger than teams of two. Once again we focus on extreme matchings. While PAM and NAM are typically defined in a setting with two-sided matching, we refer to Chade and Eeckhout (2014) for the treatment of sorting when teams are large.

Let there be N agents in each team, and a large population with random matching. Half of the agents are high type $\overline{x}$, half are a low type $\underline{x}$. Let $X = \sum_{i=1}^{N} x_i$. Then let the payoff function of a match be:

$$V(X_i|X_j) = aX_i^\alpha - bX_j^\alpha + c$$

i.e., $g(X_i) = aX_i^\alpha + c$ and $h(X_j) = -bX_j^\alpha$, with $a, b, c > 0$. Therefore

$$\begin{aligned}
\mathcal{V}(\overline{X}|\mu_+) &= a\overline{X}^\alpha - \frac{b}{2}\left(\overline{X}^\alpha + \underline{X}^\alpha\right) + c \\
\mathcal{V}(\underline{X}|\mu_+) &= a\underline{X}^\alpha - \frac{b}{2}\left(\overline{X}^\alpha + \underline{X}^\alpha\right) + c \\
\mathcal{V}(\hat{X}|\mu_-) &= a\hat{X}^\alpha - b\hat{X}^\alpha + c,
\end{aligned}$$

where $\overline{X} = \sum_{i=1}^{N} \overline{x}$ and $\underline{X} = \sum_{i=1}^{N} \underline{x}$.

Then provided that $\alpha > 1$, there is supermodularity of g and we obtain an equilibrium with PAM. Yet, the condition for NAM by the planner is $g(\overline{X}) + g(\underline{X}) - 2g(\hat{X}) < 2h(\mu_-) - 2h(\mu_+)$, or equivalently

$$a\left(\overline{X}^\alpha + \underline{X}^\alpha - 2\hat{X}^\alpha\right) < b\left(\overline{X}^\alpha + \underline{X}^\alpha - 2\hat{X}^\alpha\right)$$

which is satisfied whenever $a < b$. Therefore, with $\alpha > 1$ and $a < b$ there is PAM in equilibrium, NAM in the planner's solution, when teams are large.

Small Markets

We have focused on a large market set up, but analogous results obtain when the market is small, although the sorting conditions need to be suitably modified. If the market is small, this means that at the matching stage, the allocation and wages are determined under circumstances that may not necessarily justify price taking behavior. But just as in the two agent Edgeworth box, the price taking behavior can be rationalized by a replica economy, and one could redo the analysis of the paper in this set up. Alternatively, we could specify an extensive form game where the allocation and transfers are determined strategically. The properties of such an equilibrium will depend on the details of the extensive form game, and the allocation will in general be inefficient.[22] The formal analysis of the small market case follows along the same lines as the one in the text and it is omitted.

[22] There are good reasons to consider the strategic determination of payoffs in settings with externalities. See amongst others Jehiel and Moldovanu (2000) or Ponsatí and Sákovics (1996)

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