Heterogeneity as a Coordination Device*

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Abstract

A model of directed search with a finite number of buyers and sellers is considered, where sellers compete in direct mechanisms. Buyer heterogeneity and Nash equilibrium results in perfect sorting. The restriction to complementary inputs, that the match value function Q is supermodular, in addition coordinates the sellers' strategies. In that case, equilibrium implements positive assortative matching, which is efficient and consistent with the stable (cooperative equilibrium) outcome. This provides a non-cooperative and decentralized solution for the Assignment Game. Conversely, if buyers are identical, no such coordination is possible, and there is a continuum of equilibria, one of which exhibits price posting, another yields competition in auctions.

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1 Introduction

This article considers equilibrium decentralized trade where sellers hold differentiated goods and buyers have idiosyncratic preferences. The Walrasian equivalent of allocating heterogeneous buyers and sellers is solved in what is known as the *assignment game* (see Roth and Sotomayor (1990) for a survey). Assuming the preferences are common knowledge, this literature identifies equilibrium trading prices and allocations which describe an equilibrium.[1] In many contexts of trade however, prices are much more instrumental in the allocation process. Rather than the mere outcome of an optimization process, price is very often the strategic variable. Differentiated goods for example are commonly advertised by announcing the price in addition to the characteristics of the good. Many job announcements in newspapers, or through agencies, include a wage together with the job description. We therefore adopt the *directed search* approach (Montgomery (1991), Peters (1991), Acemoglu and Shimer (1998), Burdett, Shi and Wright (1998)). One side of the market, the sellers, publicly advertise their goods, their location and their price(s). The other side, the buyers, then choose which seller to visit. The central theme in the existing directed search literature is the inefficiency of the equilibrium allocation: buyers do not manage to coordinate their visit strategies. The question this paper addresses is whether sellers are able to solve the coordination problem and whether they can implement efficient allocations. For that purpose, we add two features. 1. Rather than restrict sellers' strategies to posting prices, sellers here are allowed to compete in direct mechanisms. For example they could advertise an auction. The issue we investigate is what mechanisms these competing sellers adopt in equilibrium, and whether the final outcome is efficient. 2. We introduce two-sided heterogeneity, which makes our model isomorphic to the assignment game model. Will the decentralized trade model of the assignment game efficiently allocate differentiated buyers and sellers?

Our central result is that buyer heterogeneity and seller competition in mechanisms results in perfectly directed search, which is efficient. This result is surprising because the set of equilibria is discontinuous in buyer valuations: minimally different buyer valuations are sufficient for ruling out bad coordination equilibria, while

[1] The so-called "stable" outcome has been shown to coincide with the Walrasian equilibrium.

coordination failure cannot be ruled out for identical buyers. Further, if there is also seller heterogeneity and supermodularity in the match value function, equilibrium results in the unique optimal allocation. The directed search model with competing mechanisms provides a decentralized solution for the assignment game.

Heterogeneity is critical for these results. With identical buyers, there exists a coordination problem where no buyer knows which seller the other buyers will visit, so the buyers' visiting strategy is to randomize over sellers. As there is a positive probability that several buyers will choose the same seller, the final outcome is inefficient: goods remain unsold with positive probability. We show that with buyer heterogeneity, each seller's mechanism is allocationally efficient [it allocates the good to the highest valuation buyer should more than one show]: *ex post*, buyers who enter the mechanism truthfully reveal their type. The byproduct of truthful revelation is that it allows sellers to announce sufficiently rich mechanisms *ex ante*. Because sellers know the buyers will truthfully reveal their types, they can choose the mechanism's incentive compatible payoffs in order to affect buyers visit strategy. This perfectly directs buyers to the appropriate sellers and solves the coordination problem.

This paper extends the recent directed search literature in two ways. It assumes there is two sided heterogeneity in the market and that sellers compete by advertising prices/mechanisms. Much of the directed search literature has instead assumed no or one-sided heterogeneity, where the advertisers post a price, and the (identical) searchers choose which advertiser to visit [e.g. Montgomery (1991), Peters (1991), Acemoglu and Shimer (1999), Burdett et al (1998)]. This generates the coordination failure mentioned above, as identical searchers do not know which advertiser each will choose to visit.[2] When the advertisers are heterogeneous, coordination failure remains while price dispersion arises as an equilibrium outcome [Montgomery (1991)]. Otherwise, all advertisers post the same price.

Of course, when buyers are heterogeneous, posting a single price cannot be an optimal mechanism. If several buyers visit the same seller, maximising joint surplus

[2] An exception is Acemoglu and Shimer (1999) who in a labour market context assume workers (searchers) differ in wealth and hence have different attitudes to risk. However, they also assume no worker is unique and so each worker type faces the same type-specific coordination problem.

requires allocating the good to the buyer who values it most. McAfee (1993), Peters (1997), Peters and Severinov (1997), Burguet and Sakovics (1999), Coles (1999) show that when buyers have independent private values, then in equilibrium sellers advertise second price sealed bid auctions and compete on reserve price.

In this article, it is not assumed that buyers have independent private values. Instead, as in the assignment game literature, there is a finite number of buyers where the distribution of preferences is common knowledge. Each buyer knows his own preference type, but that knowledge is private information: buyers are *anonymous*. It is precisely this type of buyer preferences that generate the coordination result. When two buyers turn up at one seller, the seller knows that they have different valuations. The efficient mechanism will always allocate the good to the high valuation buyer. In addition, it will generate coordination as in equilibrium, no two buyers will ever turn up at the same seller. In the announcement stage of the mechanism, the seller can choose the transfers such that only one of the two buyers wants to visit her. If one seller announces terms of trade that attract one buyer, the best response of the other firm is to attract the other buyer. The common value assumption about the preferences that drives the coordination result is a crucial feature of the directed search and assignment game literature. Though absent in most of the optimal competing mechanism literature, recent work by Biais, Martimort and Rochet (1999) and Maskin (1999) introduces a common value environment.

This common value feature of preferences is of course widely adopted in the *common agency* literature (see Bernheim and Whinston (1986a and 1986b), Dixit, Grossman and Helpman (1997) and Bergemann and Välimäki (1999)). Common agency is a multilateral relationship in which several principals simultaneously try to influence the actions of one agent. Though the model we present is fundamentally different, several aspects are common to the models in that literature. We have two principals (the sellers) who simultaneously try to induce two agents (the buyers) to participate in their mechanism. Moreover, the point of departure in the common agency literature, as well as in our model, is the use of Groves-Clarke mechanisms: truthtelling is a dominant strategy. It is then no surprise that even though the mechanisms in our model can coordinate the actions of the buyers, there exist multiple equilibrium transfer prices. These are the result of "non-serious" announcements by sellers even though these announcements do not change the equilibrium allocation. This has led Bernheim and Whinston to concentrate on the refinement called *truthful equilibrium*. A strategy is truthful relative to a given action by the buyers (agents) if it accurately reflects the seller's (the principal) willingness to pay for any other action relative to the given action. We show that a truthful equilibrium yields a unique set of equilibrium prices for the coordinated equilibrium. If the match value function is supermodular, the truthful equilibrium allocation is also unique. This corresponds to the centralized solution of the assignment game, which exhibits postitive assortative matching.

Most of the paper focusses on the two by two case. After describing the framework, section 3 shows there is a continuum of (perfect) Nash equilibria when buyers (and sellers) are identical. Those equilibria are characterized by coordination failure: buyers randomise their visiting strategy. The continuum of bad coordination equilibria is new to the directed search literature. Section 4 then assumes buyers are heterogeneous. The equilibrium seller mechanisms perfectly coordinate the visit strategies of buyers. That implies that the allocation is always efficient. Truthful equilibrium also yields a set of unique prices. Section 5 then considers heterogeneity of both buyers and sellers and shows that for a matching value function strictly supermodular, the allocation is unique. That equilibrium implies positive assortative matching and payoffs which are consistent with a stable outcome. This result is then generalised to the $N \times N$ case in section 6.

2 The Directed Search Model

There are two buyers and two sellers. Each seller has one unit of an indivisible good for sale, and each buyer wishes to purchase one unit. The buyers and sellers are heterogeneous, indexed by a type $x \in \mathcal{X} = \{1, 2\}$ for buyers and $y \in \mathcal{Y} = \{1, 2\}$ for sellers. For any matched pair $(x, y) \in \mathcal{X} \times \mathcal{Y}$, the utility generated is denoted by the match value $Q : \mathcal{X} \times \mathcal{Y} \to \Re$. Although $\mathcal{X}, \mathcal{Y}$ and Q are common knowledge, buyers are anonymous.

Prices and the allocation of goods are determined by competition in seller mechanisms. In the first stage of the game, the sellers simultaneously post advertisements which describe their good (i.e. describe their type y) and a price/allocation mechanism. Each seller posts a direct mechanism - each visiting buyer (if any) simultaneously sends a message $m \in \mathcal{X}$ to the seller, and conditional on those messages, the mechanism determines who gets the good and any sidepayments. Only equilibria with pure seller strategies are considered.

In the second stage of the game, each buyer visits at most one seller. In particular, given the advertized 'mechanisms', buyers simultaneously choose which seller to visit. Given those decisions, and prior to the mechanism being played, each buyer observes whether the other buyer has made the same choice or not. At this stage a buyer can choose to walk away and so realize a payoff of zero. Given any buyers who remain, the seller's advertised mechanism is then played and determines the final payoffs; i.e. the good is allocated and all sidepayments are made.

All buyers and sellers are risk neutral, expected utility maximizers. If buyer x consumes good y and pays price p, the buyer obtains utility $Q(x, y) - p$ and the seller obtains payoff p from the transaction. The seller obtains zero utility by consuming his own good.

The first step is to consider a simple benchmark case - that all buyers and all sellers are identical.

3 The Benchmark Case: Identical Buyers, Identical Sellers

Assume $Q(x, y) = Q$ for all x, y. As the distribution of buyer types is degenerate, the restriction to direct mechanisms implies both buyers obtain the same payoff if they visit the same seller. Suppose both buyers visit the same seller y and their equilibrium expected payoff is u. As each has the option to walk away, this expected payoff satisfies $u \geq 0$. But note all are risk neutral. The seller's optimal direct mechanism simply sells the good with probability one, while providing each buyer with expected payoff u (this maximises joint surplus and so maximises seller profit). One such mechanism chooses either buyer with equal probability and sells the good at price p where $u = \frac{1}{2}[Q - p]$.

Hence the identical buyer case implies a very simple direct mechanism; a seller announces a price pair (p_1, p_2) where the good is sold at price p_1 if only one buyer shows, and it is sold at price p_2 and allocated randomly if two buyers show. The price posting approach in the directed search literature assumes $p_2 = p_1$. Conversely, the second price sealed bid auction literature implies $p_2 = Q$ and p_1 is then interpreted as the reserve price (should only one buyer show).[3] At this stage there are no restrictions on p_2. Obviously announcing $p_1 \leq Q$ is a dominant strategy, otherwise the single buyer walks away. But it follows that announcing $p_2 \leq Q$ is also dominant.[4] Hence there is no loss in generality by restricting prices to $p_1, p_2 \leq Q$.

Given sellers use pure pricing strategies, let (p_1', p_2') denote the price pair announced by seller 1, and (p_1, p_2) the price pair announced by seller 2. Given those price announcements, let $\sigma_i : (p_1, p_2, p_1', p_2') \to [0, 1]$ denote the probability that buyer $i = 1, 2$ chooses to visit seller 1. As the buyers are identical and anonymous, for most of the section we focus on the symmetric case where buyers and sellers use the same strategy $\sigma_i = \sigma$ for $i = 1, 2$.[5]

Definition 1 *A perfect Nash equilibrium requires identifying a quadruple of prices* (p_1, p_2, p_1', p_2') *and a function* σ *where*

(a) given (p_1, p_2, p_1', p_2'), σ *describes the Nash equilibrium in visit strategies for the two buyers, and*

(b) given the subgame visit strategies $\sigma : (p_1, p_2, p_1', p_2') \to [0, 1]$, (p_1, p_2) *and* (p_1', p_2') *describe a Nash equilibrium in pricing strategies for the two sellers.*

[3] Julien, Kennes and King (1998) consider a hybrid case, where in a prior stage, each seller simultaneously commits to using either an auction or a fixed price.

[4] If $p_2 > Q$ and $p_1 \leq Q$ then the buyers will play a war of attrition - each wants the other to walk away. Suppose each walks away with probability π and their corresponding expected payoff is $u \geq 0$. It follows that posting p_2 where $u = \frac{1}{2}[Q - p_2]$ dominates. Both buyers obtain the same expected payoff, but the total surplus generated increases as the good is sold with probability one (rather than probability $(1 - \pi)^2 \leq 1$).

[5] The next section assumes heterogeneous buyers where buyers use different strategies. Equilibrium finds that buyers use pure visit strategies and visit different sellers. This section assumes identical buyers cannot co-ordinate in this way. Nonetheless, those results also apply to the case of identical buyers by setting $Q^H = Q^L = Q$, imposing $u_L = u_H = u$ (anonymity) and invoking a tie breaking assumption which co-ordinates the buyers' visit strategies (the notation is defined in the next section); e.g. if buyer i is indifferent to visiting either seller, that buyer visits the corresponding seller $j = i$. Note this requires that sellers are not anonymous.

The next theorem describes the set of symmetric (perfect) Nash equilibria where the sellers use the same pricing strategies.

Theorem 2 *(Identical Buyers and Identical Sellers). There is a continuum of symmetric Nash equilibria indexed by $\alpha \in [-Q, Q]$. Given any such α, an equilibrium exists where each seller announces $(p_1, p_2) = (p_1', p_2') = (\frac{Q}{2}, \alpha)$ and each buyer visits either seller with equal probability $\sigma = \frac{1}{2}$.*

This result is perhaps surprising as one might have expected that (should both buyers show) holding an auction would be a dominant strategy - it is an efficient way to allocate the good and extracts maximum surplus. But with identical buyers, randomly allocating the good is an equally efficient allocation device. Of course, though the good is always allocated efficiently, $\alpha < Q$ implies that not all surplus is extracted when two buyers visit. But this is not inefficient ex-ante, as a seller first needs to attract a buyer. Seller competition implies each seller offers some surplus to buyers in their advertisements in order to attract them. Posting $p_2 < Q$ is one way to do this. There is a continuum of equilibria, one of which involves price posting $[p_2 = p_1 = \frac{Q}{2}]$ and one of which involves posting an auction $[p_2 = Q]$.

The rest of this section formally establishes this result using standard backward induction arguments. Anticipating that Nash equilibria where $\sigma = 0, 1$ do not exist (one seller would then make zero expected profit and a profitable deviation always exists), we first compute σ assuming mixing by buyers.

Lemma 3 *Given (p_1, p_2, p_1', p_2'), if $\sigma \in (0, 1)$ then it satisfies*

$$\sigma = \frac{Q + p_2 - 2p_1'}{2Q + p_2 + p_2' - 2p_1 - 2p_1'} \tag{1}$$

Proof. An interior solution to σ requires buyer 1 is indifferent between visiting either seller. This requires

$$(Q - p_1')(1 - \sigma_2) + \frac{1}{2}(Q - p_2')\sigma_2 = (Q - p_1)\sigma_2 + \frac{1}{2}(Q - p_2)(1 - \sigma_2)$$

where the left hand side is buyer 1's expected payoff to visiting seller 1 given σ_2 is the probability that buyer 2 also visits seller 1. Solving for σ_2 and noting that symmetry requires $\sigma_2 = \sigma$ implies (1).■

Note that if σ has an interior solution, Lemma (3) implies that both buyers use the same strategy $\sigma_1 = \sigma_2 = \sigma$. This follows from the fact that the mixed strategy requires them to be indifferent between both sellers. This is only true for identical buyers if both use the same strategy. Now if seller 1 announces (p_1', p_2') and σ is the visit probability of each buyer, then seller 1's expected payoff is

$$\pi' = 2\sigma(1-\sigma)p_1' + \sigma^2 p_2', \tag{2}$$

where $2\sigma(1-\sigma)$ is the probability one buyer shows (and trade occurs at price p_1') and σ^2 is the probability that two buyers show.

We now construct seller 1's best response function. Suppose seller 2's strategy is (p_1, p_2). Further suppose that seller 1 announces (p_1', p_2') and $\sigma \in (0,1)$ in the resulting subgame. As Lemma 1 implies a unique solution for σ in this case, we can use (1) to substitute out p_2' in (2) and so obtain a reduced form profit function $\widetilde{\pi}'$ for seller 1

$$\widetilde{\pi}'(\sigma; p_1, p_2) = \sigma[Q + p_2] - \sigma^2[2Q + p_2 - 2p_1], \tag{3}$$

which holds for all $\sigma \in (0,1)$. Note the surprising result - although we only substituted out p_2' using (1) in (2), all the p_1' terms have cancelled out. It is this property of the model which generates the continuum result. It occurs because all agents are risk neutral. For example, suppose given (p_1, p_2) and any $\sigma \in (0,1)$, seller 1 raises p_2' and lowers p_1' while holding σ constant as defined in (1). Such a variation in prices implies that the expected payoff to the buyers, and hence their visit strategies, are unchanged (see the proof of lemma 1). But as all are risk neutral, this means that the seller's expected profit is also unchanged. Perturbations in (p_1', p_2') which hold σ constant do not change expected payoffs.

Lemma 4 shows that given (p_1, p_2), the best response of seller 1 ties down σ, which is denoted $\sigma_1^* = \sigma_1^*(p_1, p_2)$. But seller 1 has a continuum of best responses for (p_1', p_2') which are defined by (1) with $\sigma = \sigma_1^*$.

Lemma 4 *Given (p_1, p_2), the best response σ_1^* of seller 1 is :*

(a) if $2Q+p_2-2p_1 > 0$, then
$$\begin{array}{ll} \sigma_1^* = 0 & \text{if } p_2 \leq -Q \\ \sigma_1^* = \frac{1}{2}\left(\frac{Q+p_2}{2Q+p_2-2p_1}\right) & \text{if } p_2 > -Q, \text{ and } 4p_1 - p_2 < 3Q; \\ \sigma_1^* = 1 & \text{if } p_2 > -Q \text{ and } 4p_1 - p_2 \geq 3Q \end{array}$$

$$\textit{(b) if } 2Q + p_2 - 2p_1 < 0, \textit{ then } \begin{array}{ll} \sigma_1^* = 0 & \textit{if } 2p_1 - Q < 0 \\ \sigma_1^* \in \{0,1\} & \textit{if } 2p_1 - Q = 0 \\ \sigma_1^* = 1 & \textit{if } 2p_1 - Q > 0 \end{array}$$

(c) if $2Q + p_2 - 2p_1 = 0$, *then*

$$\begin{array}{ll} \sigma_1^* = 0 & \textit{if } 2p_1 - Q < 0 \\ \sigma_1^* \in [0,1] & \textit{if } 2p_1 - Q = 0 \\ \sigma_1^* = 1 & \textit{if } 2p_1 - Q > 0 \end{array}$$

Proof. In Appendix.

The same argument describes σ_2^*. Identifying a perfect Nash equilibrium requires finding a $\sigma \in [0,1]$ where $\sigma_1^* = \sigma_2^* = \sigma$ (so that both sellers are playing best responses). It immediately follows that if a perfect Nash equilibrium exists, it implies $\sigma \in (0,1)$.[6]

Lemma 5 *Any solution for* (p_1, p_2, p_1', p_2') *and* $\sigma \in (0,1)$ *which satisfies (1),*

$$\sigma = \frac{1}{2}\left(\frac{Q + p_2}{2Q + p_2 - 2p_1}\right) \tag{4}$$

$$1 - \sigma = \frac{1}{2}\left(\frac{Q + p_2'}{2Q + p_2' - 2p_1'}\right) \tag{5}$$

and the inequalities

$$p_2 > -Q, 4p_1 - p_2 < 3Q, \textit{ and } p_1, p_2 \leq Q \tag{6}$$

$$p_2' > -Q, 4p_1' - p_2' < 3Q \textit{ and } p_1', p_2' \leq Q. \tag{7}$$

describes a perfect Nash equilibrium.

Proof. Lemma (4a) implies that seller 1 is playing a best response if (4) and inequalities (6) hold, where it should be noted that those inequalities guarantee $2Q + p_2 - 2p_1 > 0$. The same argument applies to seller 2, where (5) describes the best response of seller 2 if inequalities (7) hold. As (1) describes the buyers' optimal strategies in the subgame (given $\sigma \in (0,1)$ in an equilibrium), any solution to these conditions describes a perfect Nash equilibrium.■

[6] If $\sigma = 0$, then $\sigma_1^* = 0$ implies that seller 2 must announce $p_2 \leq -Q$ [see Figure **??**]. As seller 2 then gets both buyers, she makes a strict loss and this cannot be an optimal strategy.

There is a continuum of equilibria as we only have 3 equations (1),(4),(5) to tie down 5 unknowns $\{p_1, p_2, p_1', p_2', \sigma\}$, and the inequalities (6),(7) admit a continuum of such solutions. The simplest to characterize are the (seller) symmetric equilibria where $p_1' = p_1$ and $p_2' = p_2$. In that case, (1) implies $\sigma = \frac{1}{2}$ and (4),(5) imply $p_1' = p_1 = \frac{Q}{2}$. But p_2 and p_2' are not tied down. As the inequalities are satisfied for $p_2 = p_2' = \alpha$ where $\alpha \in (-Q, Q]$, symmetric equilibria exist for those values.[7]

This continuum arises because each seller has a continuum of best responses. In any equilibrium (which implies some value of $\sigma \in (0,1)$), any (p_1', p_2') satisfying (1) describes a best response for seller 1. But changing (p_1', p_2') while holding σ constant changes the elasticities of σ with respect to p_1, p_2. This in turn changes firm 2's best response correspondence. A continuum of 'best response intersections' are possible (as described by lemma 3).

In the symmetric equilibria, if seller 2 offers more surplus to the buyers, say by lowering p_2, seller 1's best response is to offer more surplus to remain competitive. Indeed, a best response of seller 1 is to lower p_2' by exactly the same amount, and so a continuum of equilibria are possible.

Lemma (5) also implies there exists a continuum of equilibria where sellers use different pricing strategies. For example an equilibrium exists where $p_1' = 0$ and $p_2' = Q$. Seller 1 offers to give the good away if only one buyer shows, but will sell at the monopoly price if two show. This describes a perfect Nash equilibrium when seller 2 announces $p_1 = \frac{2Q}{3}$, $p_2 = \frac{Q}{3}$ and the corresponding visit strategies imply $\sigma = \frac{2}{3}$. Note that this equilibrium is less efficient than the symmetric equilibrium as the probability that one buyer does not obtain a good has increased. Indeed this is true for all asymmetric equilibria as they necessarily imply $\sigma \neq 0.5$.

4 Heterogeneous Buyers

The continuum result of the previous section is new to the literature. But it also confirms the main existing result on directed search with identical buyers: sellers are unable to coordinate identical buyers' visit strategies and as a result the equilibrium

[7]By inspection of lemma (5c), it follows that $p_1' = p_1 = \frac{Q}{2}$ and $p_2 = p_2' = -Q$ also describes an equilibrium. This is the zero profit equilibrium.

allocation is inefficient. The trading environment is very different with heterogeneous buyers. Unlike the previous section, sellers now find that (in reduced form) they advertise three prices. In particular, the emergence of that third price allows sellers to perfectly direct the visit strategies of buyers. Competition in seller mechanisms results in perfectly directed search.

Suppose sellers $y = 1, 2$ are identical but now $Q(1, y) = Q^H$ denotes the value to buyer 1 of consuming the good, and $Q(2, y) = Q^L$ denotes the value to buyer 2 of consuming the good, where $Q^H > Q^L > 0$. To stress the heterogeneity of buyers, $x \in \{H, L\}$ now indexes the respective buyers. With heterogeneous buyers, the seller's advertised mechanism now plays a double role.

(i) In the second stage, *given* that two buyers have arrived at that seller, the mechanism allocates the good to one of these buyers. The following shows that the firm's optimal direct mechanism implies (a) each buyer truthfully reveals her type and (b) allocates the good to the high valuation type. The resulting payoffs to the buyers are denoted $u_L, u_H \geq 0$.

(ii) Taking into account these surplus maximizing *separating mechanisms*, then in reduced form the seller's pricing strategy advertises a triple (p_1, u_L, u_H) where p_1 is the price charged if only one buyer shows, while if two buyers show the optimal *separating mechanism* provides utility payoffs u_H and u_L to the respective types. Even though the good is always allocated to the H type, the implicit transfer payments u_H, u_L determine the relative attractiveness of being visited.

4.1 An Optimal Direct Mechanism

A direct mechanism corresponds to a message game $\Gamma(\mathcal{X} \times \mathcal{X}, u(\cdot))$, where each participant sends a message $m_x \in \mathcal{X} = \{L, H\}$, and conditional only on those messages, an allocation rule and sidepayments implies an outcome function $u(\cdot) : \mathcal{X} \times \mathcal{X} \to [0, \infty] \times [0, \infty]$.

Suppose the seller wishes to construct a direct mechanism which implements expected payoffs $u_L, u_H \geq 0$ to the respective buyers, should both arrive. Obviously such a mechanism needs to be incentive compatible. But clearly we should be most interested in an efficient direct mechanism - one that allocates the good to the highest valuation buyer. If such a mechanism exists, it must be optimal as by maximising joint surplus, it then maximises the payoff to the seller [given the advertised payoffs u_L, u_H]. The following establishes that such a direct mechanism always exists, for any $u_L, u_H \geq 0$. Given that, the advertising game will then determine u_L, u_H

Fix any $u_L, u_H \geq 0$. (A1)-(A3) below describe a set of (anonymous) allocation rules which induce truth-telling as an iterated dominant strategy equilibrium *and* implement payoffs $u_L, u_H \geq 0$.

$(A1)$ If both buyers report $m_x = H$, the seller gives each buyer sidepayment u_L and allocates the good with equal probability at a price $p = \frac{1}{2}\left(Q^L + Q^H\right)$. In this event, buyer H obtains expected payoff $a = u_L + \frac{1}{4}[Q^H - Q^L] > u_L$ and buyer L obtains expected payoff $b = u_L - \frac{1}{4}[Q^H - Q^L] < u_L$ for buyer L.

$(A2)$ If both buyers report message $m_x = L$, the good is not sold and both buyers obtain a payoff of zero.

$(A3)$ If one buyer reports H, the other L, the seller allocates the good to the buyer reporting H at price $p = Q^H - u_H$ and pays u_L to the buyer reporting L.

These allocation rules imply a message game with the following normal form [where buyer H plays rows (and receives the first number in the pay-off pair) and buyer L plays columns].

		m_L: H	m_L: L
m_H	H	(a, b)	(u_H, u_L)
	L	$(u_L, Q^L - [Q^H - u_H])$	$(0, 0)$

Although these allocation rules respect anonymity, they do not imply symmetric payoffs as the buyers obtain different payoffs by consuming the good.

As allocation rule (A1) implies $a > u_L$, then for $u_H > 0$, buyer $H's$ strict dominant strategy is to report $m_H = H$.[8] Further, as (A1) also implies $b < u_L$, buyer $L's$ (iterated) dominant strategy is to report $m_L = L$.

Allocation rules (A1)-(A3) imply that truth telling is incentive compatible. As rule (A3) allocates the good to the high valuation buyer, this mechanism also maximises

[8]All equilibria described below imply $u_H > 0$. Of course, $u_H = 0$ implies only weak dominance.

joint surplus. Further as rule (A3) also implements the (given) payoffs $u_H, u_L \geq 0$, this mechanism also maximises the seller's own surplus. Hence given any $u_H, u_L \geq 0$, these allocation rules describe an optimal direct mechanism and the seller's corresponding payoff is $Q^H - u_H - u_L$.

From now on assume that both sellers use a direct mechanism of this form and as a result, the only payoff relevant variables when two buyers turn up are the seller's advertised choice of u_H and u_L.

4.2 A Nash Equilibrium in Seller Mechanisms

The previous subsection implies the seller's optimal direct mechanism reduces to a triple (p_1, u_H, u_L) where p_1 is the price should only one buyer visit, and $u_L, u_H \geq 0$ are the respective payoffs of the buyers should both visit.

To solve for a (perfect) Nash equilibrium let (p_1', u_L', u_H') denote the advert posted by seller 1, and (p_1, u_L, u_H) denote the advert posted by seller 2. Given those posted adverts, let σ_x, $x \in \{L, H\}$, denote the probability that buyer x visits seller 1.

But equilibrium is now quite different. To see why, suppose for now that the two sellers announce $p_1, p_1' \leq Q^L$. A little work shows that in any mixed strategy (subgame) equilibrium, σ_L, σ_H satisfy

$$\sigma_H = \frac{u_L + p_1' - Q^L}{u_L' + u_L + p_1' + p_1 - 2Q^L} \tag{8}$$

$$\sigma_L = \frac{u_H + p_1' - Q^H}{u_H' + u_H + p_1' + p_1 - 2Q^H}, \tag{9}$$

and seller 1's payoff function is

$$\pi' = [\sigma_L(1 - \sigma_H) + \sigma_H(1 - \sigma_L)]\, p_1' + \sigma_L \sigma_H \left[Q^H - u_H' - u_L'\right].$$

Following the previous approach, suppose $\sigma_L, \sigma_H \in (0, 1)$ and so use (8) and (9) to substitute out u_L' and u_H' in π'. The p_1' term again cancels out [all are risk neutral] and the reduced form profit function for seller 1 is

$$\widetilde{\pi}'(\sigma_L, \sigma_H) = \sigma_L \left[Q^L - u_L\right] + \sigma_H \left[Q^H - u_H\right] - \sigma_L \sigma_H \left[Q^H + 2Q^L - u_H - u_L - 2p_1\right] \tag{10}$$

But now there is a crucial difference. Previously, visiting strategies for buyers with identical valuations were identical (from Lemma 3); i.e. $\sigma_L = \sigma_H = \sigma$. This also follows immediately from insepction of (8) and (9) for equal valuations Q and equal transfers u. It followed that the profit function (3) was then concave in σ (at the equilibrium). But with heterogeneous buyers, the payoff function described by (10) is not concave in σ_L, σ_H.

Lemma 6 *Given $(\sigma_L, \sigma_H) \in [0,1] \times [0,1]$, then $\widetilde{\pi}'$ defined by (10) is a maximum at one of the corners where $\sigma_L, \sigma_H \in \{0,1\}$.*

Proof : Given any $\sigma^H \in [0,1]$, notice that $\widetilde{\pi}'$ is linear in σ^L. Hence for any such σ^H, $\widetilde{\pi}'$ is a maximum at $\sigma^L = 0$ or 1. Now fix $\sigma^L = 0$ or 1. $\widetilde{\pi}'$ is now a linear function of σ^H and hence a maximum occurs at $\sigma^H = 0$ or 1. This completes the proof of the lemma.■

This is a striking difference. When buyers have different preferences, the seller's optimal separating mechanism not only allocates the good efficiently (should both buyers visit), but also co-ordinates the visit strategies of buyers - there is no randomization by buyers in equilibrium.

Characterising the set of perfect Nash equilibria requires a different approach. For simplicity, restrict attention to equilibria where both sellers make strictly positive expected profit.[9] Anticipating that the visit decisions of the two buyers are polarized in equilibrium, suppose in equilibrium the high valuation buyer visits seller 1 and the low valuation buyer visits seller 2; i.e. $\sigma_H = 1$ and $\sigma_L = 0$. A strictly positive profit equilibrium then requires $p_1' \in (0, Q^H]$ and $p_1 \in (0, Q^L]$ (otherwise the respective buyers walk away).

As before, the choice of (p_1', u_L', u_H') by seller 1 as a best response of seller 1 ties down the visit probabilities σ_H and σ_L. For each continuum of best responses (p_1', u_L', u_H'), there is a corresponding σ_H and σ_L. Let (σ_H', σ_L') denote the best response of seller 1 given seller 2's strategy (p_1, u_L, u_H). Then (σ_H, σ_L) denotes seller 2's best response. The following Lemma identifies necessary and sufficient conditions on (p_1, u_L, u_H) so that $(\sigma_H', \sigma_L') = (1, 0)$.

[9] Lemma (6) and (10) imply a possible equilibrium is $\sigma_x \in \{0,1\}$ and the other $\sigma_{-x} \in (0,1)$ [i.e. at most one buyer mixes]. But this outcome is a (weak) best response for both sellers only if one seller makes zero profit [the one which buyer x never visits]. Restricting attention to strictly positive profit equilibria implies only perfectly polarised equilibria exist; i.e. (σ_H, σ_L) is either $(1,0)$ or $(0,1)$.

Lemma 7 *Given $p_1 \in (0, Q^L]$ and $u_L, u_H \geq 0$, necessary and sufficient conditions on (p_1, u_L, u_H) so that (i) $(\sigma'_H, \sigma'_L) = (1, 0)$ is a best response, and (ii) seller 1 makes strictly positive profits, are*

$(R1)$ $Q^H - u_H > 0$,

$(R2)$ $u_H - u_L \leq Q^H - Q^L$,

$(R3)$ $2p_1 \leq Q^H + Q^L - u_H$.

Further, given $(R1) - (R3)$, seller 1's best pricing response is

$$p'_1 = Q^H - u_H, \; u'_L \leq Q^L - p_1,$$

which generates expected payoff $Q^H - u_H$.

Proof. In Appendix.

Given seller 2's strategy satisfies $(R1) - (R3)$, seller 1's best response induces $(\sigma_H, \sigma_L) = (1, 0)$. In particular, note that price strategy

$$p'_1 = Q^H - u_H - \varepsilon, \; u'_L \leq Q^L - p_1, u'_H \text{ large}$$

where $\varepsilon > 0$ (but small) implies buyer $H's$ strict dominant strategy is to visit seller 1. Further sideoffer $u'_L \leq Q^L - p_1$ then implies $\sigma_L = 0$ is an optimal strategy for buyer L. This price strategy essentially undercuts buyer $H's$ outside option, which is to visit seller 2 and so obtain payoff u_H, by a 'penny'. By perfectly co-ordinating the visit strategies of buyers in this way, seller 1 generates expected payoff $Q^H - u_H - \varepsilon$. Lemma (7) implicitly sidesteps the 'penny' issue by assuming that seller 1 can set $\varepsilon = 0$ and buyer H will still choose $\sigma_H = 1$.[10]

$(R3)$ is a *competition condition.* If $(R3)$ does not hold, seller 1's best response is to poach both customers by offering $u'_H = Q^H - p_1$ and $u'_L = Q^L - p_1$.[11] That would give a payoff of $2p_1 - Q^L$ and $(R3)$ is necessary to ensure this does not exceed equilibrium payoff $Q^H - u_H$. Of course in equilibrium, seller 2 chooses p_1 to satisfy

[10] Formally for equilibrium to be well defined [as prices are continuous and a so-called 'penny' does not exist] we need an appropriate tie breaking assumption. In this equilibrium we would need buyer H visits seller 1 if indifferent to doing so. We shall return to this issue later.

[11] By also setting $p'_1 < Q^L - u_L$, $\sigma^L = 1$ is a dominant strategy for buyer L, and $\sigma^H = 1$ then describes a Nash equilibrium in visit strategies.

$(R3)$ and so prevents seller 1 from poaching both buyers. In this sense $(R3)$ describes price competition; seller 2 sets p_1 sufficiently low so as to attract at least one buyer.

$(R2)$ is a *co-ordination condition.* It guarantees that seller 1 is better off attracting buyer H (with payoff $Q^H - u_H$) rather than buyer L (with payoff $Q^L - u_L$). $(R2)$ determines which buyer seller 1 will choose to attract. When seller 2 announces $u_H - u_L < Q^H - Q^L$, we shall refer to this strategy as playing '**weak**' - it invites seller 1 to attract the high valuation buyer. Conversely announcing $u_H - u_L > Q^H - Q^L$ is called playing '**tough**' - it invites seller 1 to attract the low valuation buyer.

Similar conditions also describe (σ_H, σ_L), the best response of seller 2.

Lemma 8 *Necessary and sufficient conditions on* (p_1', u_L', u_H') *so that* (i) $(\sigma_H, \sigma_L) = (1, 0)$ *is a best response and* (ii) *seller 2 makes strictly positive profits, are*

$(R1')$ $Q^L - u_L' > 0$,

$(R2')$ $u_H' - u_L' \geq Q^H - Q^L$,

$(R3')$ $2p_1' \leq 2Q^L - u_L'$.

Given $(R1') - (R3')$, *seller 2's best pricing response is*

$$p_1 = Q^L - u_L' \text{ and } u_H \leq Q^H - p_1'$$

which generates payoff $Q^L - u_L$.

Proof. In Appendix.

If seller 1 plays tough [i.e. $(R2')$ holds] and announces a sufficiently low price p_1' [satisfying $(R3')$] then seller 2's best response is to attract buyer L using the above price strategy. Using Lemmas 7 and 8, it is now straightforward to describe perfect Nash equilibria with $(\sigma_H, \sigma_L) = (1, 0)$. The main feature is that there is a continuum of such equilibria. We illustrate this with an example.

An equilibrium with $(\sigma_H, \sigma_L) = (1, 0)$ exists where seller 1 posts

$$p_1' = \frac{1}{2}Q^L, u_L' = 0.9Q^L \text{ and any } u_H' > Q^H - 0.1Q^L.$$

As $(\sigma_H, \sigma_L) = (1, 0)$ in this equilibrium, seller 1 obtains payoff $\frac{1}{2}Q^L$. However note seller 1 makes an extravagant sideoffer $u_L' = 0.9Q^L$ to buyer L [should L deviate from the equilibrium] and $u_H' > u_L' + [Q^H - Q^L]$ to imply 'tough'.

As this price strategy satisfies $(R1') - (R3')$, then seller 2's best response implies $p_1 = 0.1Q^L$ (by Lemma 8). Note that the extravagant sideoffers of seller 1 force seller 2 to offer a very low price. But also note seller 1 has chosen p_1' small enough to ensure seller 2 does not poach both buyers (i.e. to satisfy $(R3')$).

Of course seller 1's strategy also has to be a best response. Lemma 7 requires seller 2 posts $u_H = Q^H - \frac{1}{2}Q^L$. Hence as long as u_L is large enough (i.e. seller 2 plays "weak"), then the above strategy for seller 1 and

$$p_1 = 0.1Q^L, u_H = Q^H - \frac{1}{2}Q^L \text{ and } u_L \text{ large enough.}$$

describe a perfect Nash equilibrium with $(\sigma_H, \sigma_L) = (1, 0)$. [12]

This example reveals the source of multiplicity - the sideoffers determine the equilibrium prices p_1, p_1', but the sideoffers themselves are not determined. In particular given $(R1) - (R3)$, seller 1's best response is only

$$p_1' = \frac{1}{2}Q^L, u_L' \leq 0.9Q^L.$$

Seller 1 has a unique optimal choice for p_1', but the optimal choice of sideoffers requires only that u_L' is sufficiently low that he does not attract both buyers. In contrast, equilibrium requires very generous sideoffers - seller 1 not only offers the highest possible value of u_L' consistent with not attracting buyer L, but also posts an even higher sideoffer u_H' to guarantee 'tough'. This seems somewhat inconsistent with $(R3)$; p_1 is sufficiently low that seller 1 has no incentive to attract both buyers. Indeed if buyer L were to deviate and visit seller 1, seller 1 would realise a loss of at least $0.8Q^L$. Such generous equilibrium sideoffers u_L', u_H' are weakly dominated by posting less generous ones.

This source of multiplicity is well known in the game theory literature and the concept of trembling hand perfection is typically used as the appropriate equilibrium refinement [for example see the Bertand pricing game as described in Mas-Colell, Whinston and Green (1994)].[13] In an uncertain world, if there is a small (vanishing) probability that buyer L will deviate (i.e. a tremble is possible), then generous sideoffers are strictly profit reducing; there is a small but positive probability that both

[12] Note that though sellers are identical, they do not necessarily receive the same profits as we have not imposed symmetric pricing strategies.

[13] The relevant Bertrand pricing game has two sellers who have different unit costs $0 < c_L < c_H$, and both announce a price simultaneously. With the right tie breaking assumption to guarantee

buyers will visit. However, as Bernheim and Whinston (1986a) argue, formalizing this notion imposes severe difficulties. Rather than complicating the analysis, we borrow the refinement adopted in the common agency literature: *truthful equilibrium.* This requires that side-offers are serious. Relative to an action that involves sideoffers $u_L, u_H \geq 0$, the seller's profit is required not to be lower should the seller succeed in attracting both buyers.

Definition 9 *(Truthful Nash Equilibrium) An equilibrium* $\{(p_1, u_L, u_H), (p'_1, u'_L, u'_H), (1, 0)\}$ *with corresponding equilibrium payoffs* $\pi'^* = p'_1, \pi^* = p_1,$ *is truthful if and only if it is a Nash equilibrium and the strategies are truthful:*
(T1) $Q^H - u'_L - u'_H \geq \pi'^* = p'_1,$
(T2) $Q^H - u_L - u_H \geq \pi^* = p_1.$

The restriction to no frivolous offers uniquely determines equilibrium payoffs.

Theorem 10 (Heterogeneous Buyers) *Strictly positive profit, perfect Nash equilibria with* $(\sigma_H, \sigma_L) = (1, 0)$ *and (T1)-(T2) exist and imply*

(i) $p'_1 = Q^L, u'_L = 0$ *and* $u'_H = Q^H - Q^L$

(ii) $p_1 = Q^L, u_L = 0,$ *and* $u_H = Q^H - Q^L.$

Proof. In Appendix**.**

With no frivolous sideoffers, equilibrium implies both sellers post second price sealed bid auctions with reserve price Q^L, and both obtain the same profit Q^L. Clearly this outcome does not resolve the coordination problem. The restriction to no frivolous offers implies neither seller plays strictly tough, nor strictly weak. Indeed, the same strategies correspond to a pure strategy Nash equilibrium with $\sigma_L = 1, \sigma_H = 0.$[14]

existence, (i.e. that if a customer is indifferent she goes to the lower cost firm), then there is a continuum of Nash equilibria; both sellers announce the same price $p_L = p_H = p$, where $p \in [c_L, c_H]$, and all customers go to seller L. Selten's trembling-hand argument rules out all equilibria except $p = c_H$. But note that in this surviving equilibrium, if the buyers deviate and visit the high cost seller, that seller's profit is no less than equilibrium profit. This is not true for the eliminated equilibria with $p < c_H$. In this case, restrictions of the form $(NF1), (NF2)$ described below play the same role as trembling hand perfection.

[14] The formal difference being in assumed tie breaking assumptions - see footnote 9.

Further, a third equilibrium seems likely where sellers randomize on prices [e.g. Burguet and Sakovics (1999)]. However seller heterogeneity now resolves this final co-ordination problem.

5 Heterogeneous Buyers, Heterogeneous Sellers

This time denote $Q(1,1) = Q^{HH}, Q(1,2) = Q^{HL}, Q(2,1) = Q^{LH}, Q(2,2) = Q^{LL}$ where the first superscript refers to the buyer, the second to the seller. Seller 1 holds the more valuable good in that both buyers prefer his good; i.e. $Q^{HH} > Q^{HL} > 0$ for buyer H and $Q^{LH} > Q^{LL} > 0$ for buyer L. We shall continue to use $x \in \{L, H\}$ to index the respective buyers, but to limit confusion over notation use $y \in \{1, 2\}$ to index the sellers, where seller 1 has the more valuable good.

Throughout assume

$$Q^{HH} + Q^{LL} > Q^{HL} + Q^{LH}. \tag{11}$$

We again construct perfect Nash equilibrium but assuming (11) and strategies must be truthful. Theorems (11) and (12) imply that such heterogeneity and equilibrium perfectly co-ordinates the sellers' strategies. Equilibrium [in pure seller strategies] is unique and generates positive assortative matching. In that equilibrium, seller 2 plays strictly weak and attracts buyer L. Indeed seller 2's mechanism corresponds to a second price sealed bid auction with reserve price Q^{LL}. Seller 1 attracts buyer H by matching the value of buyer H's outside option, which is to bid in seller 2's auction.

As the argument is (almost) the same as with identical sellers, the following quickly outlines the details. Let σ_H, σ_L denote the visit strategies, where σ_x is the probability that buyer $x = L, H$ visits seller 1, let (p_1', u_L', u_H') denote the mechanism posted by seller 1, and (p_1, u_L, u_H) denote the mechanism posted by seller 2.

As before only consider strictly positive profit equilibria, where a little work establishes that in a pure strategy pricing equilibrium, the visit strategies imply $\sigma_H, \sigma_L \in \{0, 1\}$. The intuition is as before - each seller's best response coordinates the visit strategies of each buyer. Also, only consider non-frivolous side-offers which requires

$(T3)$ $Q^{HH} - u_L' - u_H' \geq \pi_1^* = p_1'$

$(T4)$ $Q^{HL} - u_L - u_H \geq \pi_2^* = p_1,$

where the interpretation is the same as before.

Theorem 11 *Given (11) and (T3),(T4), a Nash equilibrium with* $\sigma_H = 1$, $\sigma_L = 0$ *exists and implies*

$$\begin{aligned} p_1 &= Q^{LL} \\ p_1' &= Q^{LL} + [Q^{HH} - Q^{HL}], \end{aligned}$$

$u_H' = u_H = Q^{HL} - Q^{LL}$, $u_L' = u_L = 0$

Proof. In Appendix.

This result is closely related to that described in Theorem 10. No frivolous side-offers implies seller 2 extracts all the rents from the low valuation buyer. But seller 2 also competes for buyer H by offering surplus $u_H = Q^{HL} - Q^{LL}$. Indeed, seller 2's equilibrium mechanism corresponds to a second price sealed bid auction with reserve price Q^{LL}. Such competition then forces seller 1 to set p_1' as described in the Theorem (to attract buyer H). The restriction to non-frivolous offers implies the equilibrium mechanisms are uniquely determined.

However, unlike Theorem 10, the seller strategies described in Theorem 11 are strictly coordinated. With heterogeneous sellers, the relevant coordination condition ($R2$) is

$$(R2)\ u_H - u_L \leq Q^{HH} - Q^{LH}.$$

Given ($R2$), seller 1 prefers to attract buyer H rather than buyer L. But Theorem (11) implies $u_H - u_L = Q^{HL} - Q^{LL}$, and (11) implies this is a strictly "play weak" strategy. Hence when seller 2 competes with a second priced sealed bid auction, seller 1 strictly prefers to attract buyer H.[15]

Theorem (12) establishes uniqueness, that an equilibrium with negative assortative matching (and no frivolous offers) does not exist.

Theorem 12 *Given (11) and (T3),(T4), a Nash equilibrium with* $\sigma_H = 0$, $\sigma_L = 1$ *does not exist.*

[15] Note that, unlike in theorem (10), $(\sigma_H, \sigma_L) = (1, 0)$ is a still dominant strategy subgame perfect equilibrium should both sellers shave their prices p_1', p_1 by a 'penny'.

Proof. In Appendix.

Together Theorems (11) and (12) imply that (11) perfectly coordinates the sellers' strategies. In the unique equilibrium, seller 2 plays strictly weak. The sellers' corresponding pricing strategies then perfectly direct the buyers' search strategies and implies positive assortative matching. The final section now extends this result to the N seller, N buyer case.

6 The $N \times N$ Case

Clearly this case is much more complicated. We simply describe a candidate equilibrium using the insights provided by Theorem (11) and then prove that it does indeed describe a (perfect) Nash equilibrium.

Suppose there are N buyers with valuations x_i where $x_1 < x_2 < x_3, ... < x_N$ and N sellers with goods of quality y_j where $y_1 < y_2, ... < y_N$. The utility to buyer i by consuming seller $j's$ good is $Q(x_i, y_j)$. Assume $Q(x_1, y_1) > 0$, that Q is strictly increasing in both arguments and strictly supermodular where

$$Q(x_i, y_i) + Q(x_j, y_j) > Q(x_i, y_j) + Q(x_j, y_i) \text{ for all } i, j \neq i.$$

Each seller j simultaneously advertises a direct mechanism. Given those advertisements, each buyer simultaneously chooses which seller to visit. Let σ_{ij} denote the probability that buyer i visits seller j. As before we wish to find a perfect Nash equilibrium to this mechanism game.

Let u_i^* denote the equilibrium payoff to buyer i and π_j^* the equilibrium payoff to seller j in a perfect equilibrium. We suppose positive assortative matching describes the final equilibrium outcome; that $\sigma_{ii} = 1$ for all i. This implies that equilibrium payoffs satisfy

$$\pi_i^* = Q(x_i, y_i) - u_i^* \text{ for all } i. \tag{12}$$

The 2×2 case suggests that payoffs might be determined by competition in second price sealed bid auctions; that is

$$u_i^* = Q(x_i, y_{i-1}) - Q(x_{i-1}, y_{i-1}),$$

where buyer $i's$ outside option is to visit seller $i-1$ and obtain that good in a second price auction. That result does not quite generalise to the $N \times N$ case. As the above equation implies $u_{i-1}^* > 0$ for $i > 2$, the above equation with (12) implies $\pi_{i-1}^* + u_i^* < Q(x_i, y_{i-1})$; a gain to trade would exist between each seller $i-1$ and buyer i for $i > 2$. As the following suggests, seller $i-1$ could construct a deviating mechanism which attracts buyer i [by offering expected payoff "u_H"$= u_i^* + \varepsilon$] and so increase profit.

In equilibrium, each seller i competes for buyer $i+1$ by offering some payoff "u_H" should buyer $i+1$ visit. Of course, in equilibrium seller $i+1$ matches that outside offer. Such Bertrand competition implies buyer $i+1$'s equilibrium payoff satisfies

$$u_{i+1}^* = Q(x_{i+1}, y_i) - \pi_i^*, \tag{13}$$

where seller i is just indifferent to attracting this buyer. For existence of equilibrium we invoke the following tie breaking assumption : if buyer $i+1$ is indifferent to visiting seller $i+1$ or i then she chooses to visit seller $i+1$.[16]

Given starting value $u_1^* = 0$, the candidate equilibrium payoffs are now defined recursively by (12) and (13). Most importantly, these payoffs describe what is defined as a 'stable outcome" in the assignment literature [see Proposition 1 in Cole, Mailath and Postlewaite (1998) for a proof]. This implies

$$\pi_j^* = \max_i [Q(x_i, y_j) - u_i^*]$$

and so there is no further gain to trade between seller j and any other buyer, and of course

$$u_i^* = \max_j [Q(x_i, y_j) - \pi_j^*].$$

Given these candidate equilibrium payoffs, we now construct the equilibrium direct mechanisms.

As each seller receives exactly one visitor in equilibrium, then this candidate equilibrium requires each seller j specifies trading price $p_j = \pi_j^*$ if one buyer shows. If two buyers show, seller j uses the direct mechanism described earlier. In particular, the two buyers are asked to report $m \in \{L, H\}$, and allocations and prices are as described in that section, but with $Q^H \equiv Q(x_{j+1}, y_j)$, $Q^L \equiv Q(x_j, y_j)$, $u_L = 0$ and

[16] This assumption explicitly deals with the 'penny' issues described previously.

$u_H = u^*_{j+1}$. Note this mechanism is consistent with the two buyers being $i = j, j+1$. If the message pair is (L, H) the good is sold to the buyer reporting H at the same price p_j as when only one buyer shows.[17] Conversely if the message pair is (H, H) the good is randomly allocated at price $\frac{1}{2}[Q(x_{j+1}, y_j) + Q(x_j, y_j)] > p_j$. Hence whenever two buyers show, the good is either sold at a price no lower than π^*_j, or is not sold at all [if (L, L)].

Recall we are constructing an equilibrium where $\sigma_{ii} = 1$ for all i describes the equilibrium outcome. Hence any buyer $i \neq j$ who deviates from this equilibrium by visiting seller j,will then expect to compete against buyer j for seller $j's$ good. The above mechanism implies a trading price no lower than π^*_j. As the set of equilibrium payoffs describe a stable outcome, this implies all buyers $i \neq j, j+1$ strictly prefer not to deviate in this way.

Now consider buyer $i = j+1$ and suppose that buyer j and seller j believe that if a second buyer appears, that it is buyer $j+1$. If this buyer deviates by visiting seller j, this mechanism implies buyer $j+1$'s dominant strategy is to report 'H', and buyer $j's$ iterated dominant strategy is then to report 'L' [the price will otherwise be too high]. Hence by deviating to seller j, buyer $j+1$ obtains the good at price π^*_j and (13) implies he obtains his equilibrium payoff u^*_{j+1}. Hence u^*_{j+1} describes the outside option of buyer $j+1$ (as required). Further this mechanism is non-frivolous as seller j also obtains his equilibrium profit π^*_j.

For the purposes of checking for a Nash equilibrium, we consider deviations by one player at the time, so the seller's equilibrium mechanism need not specify what happens when more than two buyers visit.

Theorem 13 *A (perfect) Nash equilibrium exists where each seller j uses the direct mechanism described above, and given those mechanisms, each buyer i chooses $\sigma_{ii} = 1$ and obtains equilibrium payoff u^*_i.*

Proof : Clearly $\sigma_{ii} = 1$ describes a Nash equilibrium in visit strategies, given the posted mechanisms. If any buyer i deviates by visiting any seller $j \neq i$, the direct mechanism [given at least two buyers visit] implies she can obtain the good at a price

[17] The allocation rules specify that the buyer reporting H receives the good at price $p = Q^H - u_H \equiv Q(x_{j+1}, y_j) - u^*_{j+1}$. Equation (13) implies this price equals π^*_jwhich is p_j.

no lower than π_j^* and as these prices are consistent with a stable outcome, such a deviation cannot make her better off.

Now consider the optimal direct mechanism of seller k, given all other sellers $j \neq k$ post the direct mechanisms as described above. In particular, suppose seller k deviates by posting $p_k > \pi_k^*$; she raises her price in the event of only one buyer showing. Then regardless of whatever else she specifies in her mechanism, the corresponding Nash equilibrium in visit strategies by the buyers is (a) $\sigma_{ii} = 1$ for $i > k$, (b) $\sigma_{i,i-1} = 1$ for $i \leq k$ and $i > 1, (c)$ $\sigma_{11} = 1$.

To see why, note that these strategies imply the deviating seller k does not make a sale [$\sigma_{ik} = 0$ for all i] while all other sellers attract at least one buyer. Consider (a) - those buyers $i > k$. By visiting any seller $j \neq i$, the posted seller mechanisms imply buyer i expects to pay price no lower than π_j^* for that good. As the set of equilibrium payoffs describe a stable outcome, it follows that $\sigma_{ii} = 1$ is privately optimal for these buyers.

The interesting case is buyer $i = k$. As seller k has raised price $p_k > \pi_k^*$, buyer k strictly prefers to visit seller $k-1$ and obtain his original equilibrium payoff u_k^*. But now buyer $i = k-1$ realises that she won't obtain the good if she goes to seller $k-1$; instead she visits seller $k-2$ and obtains her original payoff u_{k-1}^*, and so on. All buyers below k step down to the seller below except for buyer 1 who has nowhere else to go. Hence $(a) - (c)$ describe an equilibrium in visit strategies. The critical feature of course is that all buyers obtain their original expected payoff u_i^*.

The posted mechanisms of the other sellers coordinate the buyers' visit strategies. In particular, given this (co-ordinated) subgame response by buyers, each buyer i is guaranteed a payoff of at least u_i^*. Hence to attract at least one buyer with positive probability, seller k must offer $p_k \leq \pi_k^*$ and a mechanism which offers an expected payoff of at least u_i^* for some buyer i. But as these payoffs describe a stable outcome, an optimal mechanism is to attract buyer k and sell at price $p_k = \pi_k^*$. Hence the stated mechanism is an optimal strategy and we have a (perfect) Nash equilibrium.■

7 Conclusion

With a finite number of buyers and sellers, buyer heterogeneity and equilibrium in pure pricing strategies implies buyer search is perfectly directed. But if sellers are identical, they face a second problem - who will play tough and who will play weak? The restriction to complementary inputs, that the match value function Q is supermodular, perfectly coordinates those seller strategies; the seller holding the less valuable good plays weaker. A perfect Nash equilibrium in seller mechanisms implements positive assortative matching [the efficent allocation] and the realised equilibrium payoffs are consistent with a stable outcome as defined in the assignment literature with perfect matching.

Unlike the competing auction literature with independent private values, the equilibrium mechanisms do not correspond to second price auctions. In fact the equilibrium mechanisms correspond closely to Bertrand competition - each seller i's price p_i [when only one buyer shows] matches the outside option of buyer i [which is to visit seller $i-1$], while $i's$ mechanism for two buyers competes for buyer $i+1$. Such Bertand competition generates the competitive outcome, even though the number of sellers is finite, they hold differentiated goods and each acts strategically.

Finally, our coordination result sheds some new light on efficiency in the random (as opposed to directed) matching literature with two-sided heterogeneity (see for example Shimer and Smith (1999), Eeckhout (1999), Burdett and Coles (1999)). With random matching, sellers cannot advertise prices which then direct buyer search. Instead prices are determined ex-post by bilateral bargaining. In contrast to the results obtained here, trade with two sided heterogeneity in the random matching framework is unlikely to be efficient. Ex-post bargaining implies the trading price does not correspond to the shadow market values of the buyer and seller. Burdett and Coles (1999) shows this creates a sorting externality: when two agents trade and exit the market, they do not take into account that they change the composition of the market, which then affects the trading opportunities of the remaining buyers and sellers. But allowing only one-sided heterogeneity removes this sorting externality. Moen (1996) and Acemoglu and Shimer (1998) show that trade with random matching is constrained efficient. Constrained, because even though the search decisions are efficiently directed, there is still friction. We show that for small markets of heterogeneous buyers and with appropriate seller mechanisms, even the search inefficiency as a result of coordination failure disappears.

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8 Appendix

Proof of Lemma 5

Throughout assume (p_1, p_2) fixed. The text has described those pricing strategies (p_1', p_2') which generate mixed visit strategies and corresponding payoffs. But we must also consider those pricing strategies which generate pure visit strategies.

Consider those price strategies (p_1', p_2') which imply $\sigma = 0$. This requires (p_1', p_2') satisfy $Q - p_1' \leq 0.5[Q - p_2]$ so that given $\sigma = 0$, both buyers prefer to visit seller 2 and gamble on price p_2 than pay p_1'. Hence $\sigma = 0$ is a subgame equilibrium if and only if $p_1' \geq 0.5[Q + p_2]$ and generates profit $\pi' = 0$. Clearly if a best response implies $\sigma = 0$, then an optimal pricing strategy implies p_1', $p_2' = Q$ which generates payoff $\pi' = 0$.

Consider those price strategies (p_1', p_2') which imply $\sigma = 1$. This requires (p_1', p_2') satisfy $Q - p_1 \leq 0.5[Q - p_2']$ so that given $\sigma = 1$, both buyers prefer to visit seller 1 and gamble on price p_2' than pay p_1. Hence $\sigma = 1$ is a subgame equilibrium if and only if $p_2' \leq 2p_1 - Q$, which generates profit $\pi' = p_2'$. Obviously in the set of pricing strategies that generate $\sigma = 1$, the optimal strategy is $p_2' = 2p_1 - Q$ [and p_1' small so that no other equilibrium exists]. Hence if $\sigma = 1$ is a best response, the correspondingly optimal price strategy generates profit $2p_1 - Q$.

It now follows that identifying seller 1's best response reduces to maximising the profit function defined by (3) with respect to $\sigma \in [0, 1]$, as the payoffs at the corners $\sigma = 0, 1$ defined by (3) correspond to the payoffs described above. Establishing the lemma is now trivial.

(a) If $2Q + p_2 - 2p_1 > 0$, the profit function (3) is strictly concave in σ. For $p_2 \leq -Q$, the corner solution $\sigma = 0$ is optimal, for $p_2 > -Q$ and $4p_1 - p_2 \geq 3Q$ the corner solution $\sigma = 1$ is optimal, and otherwise we have the interior optimum.

(b) If $2Q+p_2-2p_1 < 0$, the profit function (3) is strictly convex in σ. If $2p_1-Q < 0$ then $\sigma = 0$ is optimal, while $2p_1-Q > 0$ implies $\sigma = 1$ is optimal. When $2p_1-Q = 0$, then both corners $\sigma \in \{0, 1\}$ are optimal and generate zero profit $\pi' = 0$.

(c) If $2Q + p_2 - 2p_1 = 0$, the profit function (3) is linear in σ. If $2p_1 - Q < 0$ then $\sigma = 0$ is optimal, while $2p_1 - Q > 0$ implies $\sigma = 1$ is optimal. When $2p_1 - Q = 0$, then any $\sigma \in [0, 1]$ is optimal and generate zero profit $\pi' = 0$.■

Proof of Lemma 7

Fix (p_1, u_L, u_H) satisfying $p_1 \leq Q^L$ and $u_L, u_H \geq 0$. Given seller 1's choice of (p_1', u_L', u_H'), the visiting strategies $\sigma_L, \sigma_H \in [0,1]$ satisfy

$$\begin{array}{lcl}
\text{if } (1-\sigma_L)\max[Q^H - p_1', 0] + \sigma_L u_H' & > & \sigma_L[Q^H - p_1] + (1-\sigma_L)u_H \text{ then } \sigma_H = 1 \\
\text{if } (1-\sigma_L)\max[Q^H - p_1', 0] + \sigma_L u_H' & = & \sigma_L[Q^H - p_1] + (1-\sigma_L)u_H \text{ then } \sigma_H \in [0,1] \\
\text{if } (1-\sigma_L)\max[Q^H - p_1', 0] + \sigma_L u_H' & < & \sigma_L[Q^H - p_1] + (1-\sigma_L)u_H \text{ then } \sigma_H = 0
\end{array}$$

$$\begin{array}{lcl}
\text{if } (1-\sigma_H)\max[Q^L - p_1', 0] + \sigma_H u_L' & > & \sigma_H[Q^L - p_1] + (1-\sigma_H)u_L \text{ then } \sigma_L = 1 \\
\text{if } (1-\sigma_H)\max[Q^L - p_1', 0] + \sigma_H u_L' & = & \sigma_H[Q^L - p_1] + (1-\sigma_H)u_L \text{ then } \sigma_L \in [0,1] \\
\text{if } (1-\sigma_H)\max[Q^L - p_1', 0] + \sigma_H u_L' & < & \sigma_H[Q^L - p_1] + (1-\sigma_H)u_L \text{ then } \sigma_L = 0
\end{array}$$

where if only one buyer visits seller 1, they walk away if the purchase price p_1' exceeds their valuation of the good [and note $p_1 \leq Q^L$]. We prove $(R1) - (R3)$ are necessary and sufficient conditions in turn.

(i) $(R1) - (R3)$ **are necessary**.

First consider those strategies (p_1', u_L', u_H') which imply $(\sigma_H, \sigma_L) = (1, 0)$. The above conditions with $(\sigma_H, \sigma_L) = (1, 0)$ imply $\max[Q^H - p_1', 0] \geq u_H$, which can be rewritten as $p_1' \leq Q^H - u_H$, and $u_L' \leq [Q^L - p_1]$. Indeed, $(\sigma_H, \sigma_L) = (1, 0)$ is the unique subgame equilibrium if $p_1' = Q^H - u_H - \varepsilon$, $u_L' \leq [Q^L - p_1] - \varepsilon$ and u_H' large, where $\varepsilon > 0$. In that case buyer $H's$ dominant strategy is to visit seller 1 and $L's$ iterated dominant strategy is to visit seller 2. With those pricing strategies, the seller's payoff is $\pi' = p_1' = Q^H - u_H - \varepsilon$. Clearly the optimal strategy that generates $(\sigma_H, \sigma_L) = (1, 0)$ sets $\varepsilon = 0$. [Later in the paper we shall invoke appropriate tie-breaking assumptions which guarantee existence of equilibria.] With $\varepsilon = 0$, the strategy as described in the Lemma generates $(\sigma_H, \sigma_L) = (1, 0)$, obtains payoff $Q^H - u_H$ and $(R1)$ is necessary for this to describe strict positive profit.

Now consider strategy $u_H' = Q^H - p_1 - \varepsilon, u_L' = Q^L - p_1 - \varepsilon$ [and p_1' small] where $\varepsilon > 0$. The above conditions imply $(\sigma_H, \sigma_L) = (1, 1)$ and generates profit $\pi' = Q^H - u_L' - u_H' = 2p_1 - Q^L - 2\varepsilon$. If $(R3)$ does not hold, there exists $\varepsilon > 0$ and small enough that this strategy dominates the one stated in the lemma. Hence $(R3)$ is necessary.

Now consider strategy $p_1' = Q^L - u_L - \varepsilon$, $u_H' \leq [Q^H - p_1]$ and u_L' large, where $\varepsilon > 0$. This implies $(\sigma_H, \sigma_L) = (0, 1)$. The seller's profit is $\pi' = p_1' = Q^L - u_L - \varepsilon$. If $(R2)$ does not hold, there exists $\varepsilon > 0$ and small enough that this strategy dominates the one stated in the lemma. Hence (R2) is necessary.

(ii) $(R1) - (R3)$ **are sufficient.**

By multiplying both sides by σ_H, the conditions determining σ_H described above imply :

$$\sigma_H \sigma_L u_H' \geq \sigma_H \left[\sigma_L[Q^H - p_1] + (1 - \sigma_L) u_H - (1 - \sigma_L) \max[Q^H - p_1', 0].\right] \tag{14}$$

Similarly, the conditions determining σ_L imply

$$\sigma_L \sigma_H u_L' \geq \sigma_L \left[\sigma_H[Q^L - p_1] + (1 - \sigma_H) u_L - (1 - \sigma_H) \max[Q^L - p_1', 0]\right] \tag{15}$$

There are three cases depending on the seller's choice of p_1'.

(a) Price strategies where $p_1' \leq Q^L$. In this case, each buyer will purchase the good at price p_1' if only one buyer shows, and the seller's expected payoff is then

$$\pi_1' = [\sigma_L(1 - \sigma_H) + \sigma_H(1 - \sigma_L)]\, p_1' + \sigma_L \sigma_H \left[Q^H - u_H' - u_L'\right].$$

Using (14),(15) to substitute out u_L', u_H' and rearranging implies

$$\begin{aligned}
\pi_1' \;\; \leq \;\; & [Q^H - u_H] - [1 - \sigma_H - \sigma_L + \sigma_L \sigma_H][Q^H - u_H] \\
& - \sigma_L \sigma_H [Q^L + Q^H - u_H - 2p_1] \\
& - \sigma_L (1 - \sigma_H)[Q^H - u_H - Q^L + u_L].
\end{aligned}$$

Hence $(R1) - (R3)$ and $\sigma_H, \sigma_L \in [0, 1]$ imply $\pi_1' \leq [Q^H - u_H]$ and so all strategies with $p_1' \leq Q^L$ are dominated by the one described in the lemma.

(b) Price strategies where $Q^L < p_1' \leq Q^H$. In this case, buyer L does not purchase the good if the solo visitor, and so the seller's expected payoff is

$$\pi_1' = \sigma_H \left[(1 - \sigma_L) p_1' + \sigma_L (Q^H - u_H' - u_L')\right].$$

Using (14),(15) to substitute out u_L', u_H' and rearranging implies

$$\pi_1' \leq \sigma_H[Q^H - u_H] - \sigma_L\sigma_H\left[Q^H + Q^L - u_H - 2p_1\right] - \sigma_L(1-\sigma_H)u_L$$

Hence $(R1)-(R3)$, $\sigma_H, \sigma_L \in [0,1]$ and $u_L \geq 0$ imply $\pi_1' \leq [Q^H - u_H]$ as required.

(c) Price strategies with $p_1' > Q^H$ imply payoff

$$\pi' = \sigma_L\sigma_H(Q^H - u_H' - u_L').$$

Using (14),(15) to substitute out u_L', u_H' and rearranging implies

$$\pi_1' \leq \sigma_L\sigma_H[Q^H - u_H] - \sigma_L\sigma_H\left[Q^H + Q^L - u_H - 2p_1\right] - \sigma_H(1-\sigma_L)u_H - \sigma_L(1-\sigma_H)u_L$$

Hence (R1)-(R3), $\sigma_H, \sigma_L \in [0,1]$ and $u_L, u_H \geq 0$ imply $\pi_1' \leq [Q^H - u_H]$ as required.

This completes the proof of Lemma 7.■

Proof of Lemma 8

There are two cases depending on whether $p_1' > Q^L$ or $p_1' \leq Q^L$.

(i) If $p_1' \leq Q^L$, the proof which established lemma **7** applies directly and implies $(R1') - (R3')$.

(ii) Suppose instead $p_1' > Q^L$. If $(\sigma_2^H, \sigma_2^L) = (1,0)$ were a best response, seller 2 would achieve profit $Q^L - u_L \leq Q^L$. But there is a dominating strategy - seller 2 posts

$$p_1 < Q^L - u_L', u_L = \varepsilon, u_H = Q^H - p_1' + \varepsilon$$

where $\varepsilon > 0$ (but arbitrarily small). This price strategy attracts both buyers; p_1 small enough implies L's dominant strategy is to visit seller 2, and H is then also better off visiting seller 2. Seller 2's payoff is now $p_1' - 2\varepsilon > Q^L$ for ε small enough, and hence is a dominating price strategy.

Hence $(\sigma_H, \sigma_L) = (1,0)$ is a best response only for case (i), and note that the corresponding condition $(R3')$ guarantees $p_1' \leq Q^L$ as required.■

Proof of Theorem 10

Any positive profit Nash equilibrium with $(\sigma^H, \sigma^L) = (1,0)$ requires that both sellers are playing best responses. By lemmas 5,6 those best responses imply

$$p_1' = Q^H - u_H,\ p_1 = Q^L - u_L'. \tag{16}$$

Using (16) to substitute out p_1, p_1' in $(R1)-(R3), (R1')-(R3')$ which describe necessary and sufficient conditions that these define best responses, we obtain equilibrium constraints

$$(R1): u_H < Q^H, (R2): u_H - u_L \leq Q^H - Q^L, (R3): u_H - 2u_L' \leq Q^H - Q^L$$

$$(R1'): u_L' < Q^L, (R2'): u_H' - u_L' \geq Q^H - Q^L, (R3'): 2u_H - u_L' \geq 2[Q^H - Q^L].$$

where $u_H', u_L', u_H, u_L \geq 0$.

We also use (16) to substitute out π'^*, π^* in $(T1), (T2)$ [where $\pi'^* = p_1'$ and $\pi^* = p_1$ with $(\sigma_H, \sigma_L) = (1, 0)$] to obtain

$$u_H' - u_H \leq -u_L' \tag{17}$$

$$u_L + u_H - u_L' \leq Q^H - Q^L. \tag{18}$$

We now solve these conditions. Note (R2) and(R2′) imply $u_H' - u_H \geq u_L' - u_L$. With (17) this implies $u_L' - u_L \leq -u_L'$, and so $u_L \geq 2u_L'$.11

Subtracting (R3) from (R3′) implies $u_H + u_L' \geq Q^H - Q^L$. With (18) this implies $u_L + u_H - u_L' \leq u_H + u_L'$, and so $u_L \leq 2u_L'$. Hence $u_L = 2u_L'$.

We can now substitute out u_L. (R2) becomes $u_H - 2u_L' \leq Q^H - Q^L$, and (18) becomes $u_H + u_L' \leq Q^H - Q^L$. Adding these two inequalities implies $2u_H - u_L' \leq 2[Q^H - Q^L]$, and with $(R3')$ this now implies

$$2u_H = u_L' + 2[Q^H - Q^L].$$

Using these solutions to substitute out u_L, u_H in (18) now implies $u_L' \leq 0$. Hence $u_L' = 0$ and so $u_L = 0$ and $u_H = Q^H - Q^L$. (17) and $(R2')$ now imply $u_H' = Q^H - Q^L$. Direct inspection shows that these values satisfy all the above conditions, which completes the proof of the Theorem.■

Proof of Theorem 11

As the structure of the proof is identical to the proof of Theorem 10 we only sketch details. First we must obtain the conditions analogous to lemmas 7 and 8.

Lemma A2 : Given $p_1 \in (0, Q^{LL}]$, necessary and sufficient conditions on (p_1, u_L, u_H) so that (i) a best response by seller 1 implies $\sigma_H = 1, \sigma_L = 0$ and (ii) seller 1 makes strictly positive profits, are:

$(R1)$ $Q^{HH} - u_H > 0$
$(R2)$ $u_H - u_L \leq Q^{HH} - Q^{LH}$
$(R3)$ $2p_1 \leq Q^{HL} + Q^{LL} - u_H$
Seller 1's best response implies

$$p'_1 = Q^{HH} - u_H \text{ and } u'_L \leq Q^{LL} - p_1.$$

Proof. The argument used to prove lemma 7 applies directly.■

Lemma A3 : Given $p'_1 \in (0, Q^{HH}]$, necessary and sufficient conditions on (p'_1, u'_L, u'_H) so that (i) a best response by seller 2 implies $\sigma_H = 1, \sigma_L = 0$ and (ii) seller 2 makes strictly positive profits, are:
(R1′) $Q^{LL} - u'_L > 0$
(R2′) $u'_H - u'_L \geq Q^{HL} - Q^{LL}$,
(R3′) $p'_1 - \max[Q^{LH} - p'_1, 0] + u'_L \leq Q^{HH} + Q^{LL} - Q^{HL}$.
Seller 2's best response implies

$$p_1 = Q^{LL} - u'_L \text{ and } u_H \leq Q^{HH} - p'_1.$$

Proof. Is straightforward by adapting the proof of lemmas 5,6. $(R3')$ arises because seller 2 can attract both buyers (i.e $(\sigma_H, \sigma_L) = (0, 0)$) by posting $u_H = Q^{HH} - p'_1$ and $u_L = \max[Q^{LH} - p'_1, 0]$ (where buyer L does not buy from seller 1 if $p'_1 > Q^{LH}$). That such a strategy is not optimal requires $Q^{LL} - u'_L \geq Q^{HL} - u_H - u_L$, which implies $(R3')$■

Any positive profit Nash equilibrium with $(\sigma_H, \sigma_L) = (1, 0)$ requires that both sellers are playing best responses. Lemmas A2,A3 imply

$$p'_1 = Q^{HH} - u_H, \; p_1 = Q^{LL} - u'_L. \tag{19}$$

Using (19) to substitute out p_1, p'_1 in $(R1) - (R3), (R1') - (R3')$ implies equilibrium constraints

$$(R1) : u_H < Q^{HH}, (R2) : u_H - u_L \leq Q^{HH} - Q^{LH}, (R3) : u_H - 2u'_L \leq Q^{HL} - Q^{LL}$$

$$(R1') : u'_L < Q^{LL}, (R2') : u'_H - u'_L \geq Q^{HL} - Q^{LL},$$

$$(R3') : u_H + \max[u_H - Q^{HH} + Q^{LH}, 0] - u'_L \geq Q^{HL} - Q^{LL},$$

where $u'_H, u'_L, u_H, u_L \geq 0$. Also use (19) to substitute out π'^*, π^* in $(T3), (T4)$ to obtain

$$u_H - u'_L - u'_H \geq 0. \tag{20}$$

$$u_L + u_H - u'_L \leq Q^{HL} - Q^{LL}. \tag{21}$$

The problem is to solve $(R1) - (R3'), (20)$ and (21) for $u_L, u'_L, u_H, u'_H \geq 0$.

Lemma A4 : A solution does not exist if $u_H \geq Q^{HH} - Q^{LH}$.

Proof : By contradiction. Suppose $u_H \geq Q^{HH} - Q^{LH}$, and so $(R3')$ reduces to

$$(R3') : 2u_H - u'_L \geq Q^{HH} + Q^{HL} - Q^{LH} - Q^{LL}. \tag{22}$$

Subtracting $(R2')$ from $(R2)$ and using (20) implies $2u'_L - u_L \leq Q^{HH} - Q^{LH} - Q^{HL} + Q^{LL}$.

Subtracting $(R3)$ from (22) implies $u_H + u'_L \geq Q^{HH} - Q^{LH}$. With (21) it follows that $2u'_L - u_L \geq Q^{HH} - Q^{LH} - Q^{HL} + Q^{LL}$ and so by the previous paragraph

$$u_L = 2u'_L - [Q^{HH} - Q^{LH} - Q^{HL} + Q^{LL}].$$

Adding $(R2)$ and (21) gives $2u_H - u'_L \leq Q^{HH} + Q^{HL} - Q^{LH} - Q^{LL}$. (22) now implies $2u_H - u'_L = Q^{HH} + Q^{HL} - Q^{LH} - Q^{LL}$.

Substituting out u_L, u_H in (21) implies $u'_L \leq \frac{1}{3}[Q^{HH} - Q^{HL} - Q^{LH} - Q^{LL}]$. But the above solution for u_L now implies $u_L \leq -\frac{1}{3}[Q^{HH} - Q^{HL} - Q^{LH} + Q^{LL}]$ and (11) implies the required contradiction. This completes the proof of Lemma A4.■

Hence if a solution exists, it implies $u_H < Q^{HH} - Q^{LH}$ and $(R3')$ implies $u_H - u'_L \geq Q^{HL} - Q^{LL}$. (21) immediately implies $u_L = 0$ and also that $u_H = u'_L + Q^{HL} - Q^{LL}$. Adding $(R2')$ and (20) imply $u_H - 2u'_L \geq Q^{HL} - Q^{LL}$. Substituting out u_H, using the solution given, now implies $u'_L \leq 0$. Hence $u'_L = 0$. Finally (20) and $(R2')$ now imply $u'_H = Q^{HL} - Q^{LL}$. Given (11), direct inspection shows that these values satisfy all of the above conditions, which completes the proof of the Theorem.■

Proof of Theorem 12

The methodology is identical to the proofs of Theorems 10 and 11. We sketch the essential points.

Assuming an equilibrium with $(\sigma_H, \sigma_L) = (0, 1)$ exists, then the usual argument implies this consistent as a best response for seller 1 if and only if : $(R1) : Q^{LH} - u_L > 0$, $(R2) : u_H - u_L \geq Q^{HH} - Q^{LH}$, and $(R3) : Q^{HH} - \max[Q^{HL} - p_1, 0] - \max[Q^{LL} - p_1, 0] \leq Q^{LH} - u_L$. In that case the seller posts $p_1' = Q^{LH} - u_L = \pi'^*$.

This outcome is also a best response for seller 2 if and only if $(R1')$: $Q^{HL} - u_H' > 0$, $(R2') : u_H' - u_L' \leq Q^{HL} - Q^{LL}$, and $(R3')$: $Q^{HL} - \max[Q^{HH} - p_1', 0] - \max[Q^{LH} - p_1', 0] \leq Q^{HL} - u_H'$. In that case, seller 2 posts $p_1 = Q^{HL} - u_H' = \pi^*$.

Given $p_1 = Q^{HL} - u_H'$, a contradiction argument using $(R3)$ and (11) implies $u_H' \geq Q^{HL} - Q^{LL}$. Substituting out p_1 in $(R3)$ now implies

$$2u_H' - u_L \geq Q^{HH} + Q^{HL} - Q^{LH} - Q^{LL}. \tag{23}$$

Similarly, substituting out $p_1' = Q^{LH} - u_L$ in $(R3')$ implies

$$u_H' - 2u_L \leq Q^{HH} - Q^{LH}. \tag{24}$$

Also no frivolous offers requires

$$u_H' + u_L' - u_L \leq Q^{HH} - Q^{LH}, \tag{25}$$

$$u_H + u_L - u_H' \leq 0. \tag{26}$$

The Theorem is established by proving no solution exists to (23)-(26), $(R1), (R2), (R1'), (R2')$ with $u_L, u_L', u_H, u_H' \geq 0$.

Subtracting (24) from (23) implies

$$u_H' + u_L \geq Q^{HL} - Q^{LL} \tag{27}$$

and subtracting (27) from (25) gives

$$u_L' - 2u_L \leq Q^{HH} - Q^{LH} - Q^{HL} + Q^{LL}. \tag{28}$$

Now $(R2), (R2')$ imply

$$u_H - u_L - u_H' + u_L' \geq Q^{HH} - Q^{HL} - Q^{LH} + Q^{LL}. \tag{29}$$

and subtracting (26) from (29) gives

$$u'_L - 2u_L \geq Q^{HH} - Q^{HL} - Q^{LH} + Q^{LL} \tag{30}$$

Hence (28) and (30) imply

$$u'_L = 2u_L + Q^{HH} - Q^{HL} - Q^{LH} + Q^{LL} \tag{31}$$

Substituting out u'_L in (29) using (31) gives

$$u_H + u_L - u'_H \geq 0 \tag{32}$$

Hence (32) and (26) imply

$$u'_H = u_H + u_L \tag{33}$$

Substituting out u'_H in (27) using (33) gives $u_H + 2u_L \geq Q^{HL} - Q^{LL}$while substituting out u'_L, u'_H in (25) using (31) and (33) implies $u_H + 2u_L \leq Q^{HL} - Q^{LL}$. Hence $u_H + 2u_L = Q^{HL} - Q^{LL}$. Using (33) to substitute out u'_H, and this latter condition to substitute out u_H, (23) now implies

$$u_L \leq -\frac{1}{3}[Q^{HH} - Q^{LH} - Q^{HL} + Q^{LL}]$$

and (11) implies a solution with $u_L \geq 0$ cannot exist. This completes the proof of Theorem 12.■