Local Supermodularity and Unique Assortative Matching*

Jan Eeckhout†

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– *Preliminary Draft* –

Abstract

Likes tend to match with likes. Supermodularity or increasing differences is a sufficient condition for unique assortative matching. This paper derives a necessary and sufficient condition, Local Supermodularity, which is satisfied for a broad class of economies. Establishing conditions for uniqueness is desirable for the purpose of incentive compatibility in the design of markets, and it guarantees a well-identified problem when estimating a matching model. Moreover, no information is needed on the equilibrium prices or transfers between partners.

Keywords. Local Supermodularity. Two-Sided Matching. Stability. Increasing Differences. Uniqueness.

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†Department of Economics, University of Pennsylvania, eeckhout@ssc.upenn.edu, and Department of Economics, NYU Stern (visiting 06/07).

1 Introduction

Many market environments such as marriage markets and labor markets are adequately represented by two-sided matching markets. In recent years, an increasing number of market design questions are being addressed using the tools and results from the two-sided matching models. Successful applications include on-line dating, school choice, the market for medical residents and hospitals,... Typically, these environments operate like markets in the sense that there is competition between a large number of trading agents.

For those environments, two-sided matching is in some sense the most accurate representation. In the presence of indivisibilities and a wide heterogeneity of preferences and characteristics, an agent's optimizing decision is not so much about "how much to consume" as in Walrasian markets, but rather "whom to match with". That does not mean that the division of the surplus in a match is always established ex ante. In a typical matching market, there are transfers between the matched partners that are determined in equilibrium. And even if those prices are not necessarily observed and are match-specific, for example between household members in a family, the implicit equilibrium prices will certainly affect the final allocation. Agents act as price-takers when considering a partner in an alternative match because they take into account the transfer to be paid in order to induce the partner to switch.

Within those environments, we address the issue of assortative matching. In many two-sided matching markets there is a strong tendency for likes to match with likes. Married partners for example sort on education, wealth, height, ethnic origin,... Understanding how the sorting is affected by the underlying primitives – preferences and technology – and how such sorting propagates through the pricing system, is necessary for adequately characterizing those markets. This in turn is crucial for successfully designing market mechanisms or for performing policy experiments. Since Becker (1973), it is well known that supermodularity and assortative matching are closely linked: if the match value function is supermodular, then the equilibrium allocation will be characterized by positive sorting. It turns out that even if the match value function is not supermodular, there can still be positive assortative matching. In this paper, we derive the necessary and sufficient condition for assortative matching and refer to it as *local supermodularity*. Loosely speaking, it only requires the match value function to be supermodular along the equilibrium allocation, and not too submodular away from the equilibrium.

The starting point for the model in this paper is recent work by Legros and Newman (2002). Two-sided matching models have been extensively studied and successfully applied, building on the two main versions of the two-sided matching model: Gale and Shapley's (1963) marriage model without any transfers, and the Assignment Game which has linear, perfectly transferable utility (Koopmans and Beckmann (1957) and Shapley and Shubik (1972)). Legros and Newman (2002) offer a comprehensive general framework that spans those two and that allows for *limited* transfers between matched partners. Most economic environments are characterized by limited transfers. There are some indivisibilities, hence these markets are not well described by the "standard" Walrasian model with convexity assumptions, but at the same time some limited transfers exist and the division of the surplus is not given ex ante, but is derived in equilibrium. Examples of such matching markets where limited transfers are natural include the division of public goods within the family (Chiappori e.a. (2002)), risk-sharing (Rosenzweig and Stark (1989)) and moral hazard, for example matching Principals to Agents in medieval agricultural production (Ackerberg and Botticini (2002)). Even in the National Resident Matching Program (NRMP) of residents to hospitals (for an overview, see Roth and Sotomayor (1990)) where salaries are set ex-ante and are non-negotiable, hospitals can sweeten deals with attractive rotations and hours, offers of future fellowships, employment for spouses,... The same is true for other labor markets where wage bargaining is centralized, for example the allocation of teachers to public schools[1] (Boyd e.a. (2006)). In a world with limited transfers, Legros and Newman (2002) derive the Generalized Increasing Difference condition, which is the generalization of supermodularity (or increasing differences) from matching models with fully transferable utility.[2] The importance of their result is that it extends the well-known supermodularity conditions to any "realistic" economic environment, which can now be tested and verified.

In this paper, we build further on Legros and Newman (2002). We use the general framework with limited transfers and investigate under which conditions the equilibrium allocation exhibits unique assortative matching even if supermodularity fails. Motivated by recent empirical work on on-line dating (Hortaçsu e.a. (2006)), it becomes apparent that in certain economic environments, supermodularity is likely not to be satisfied. To see why, observe that on-line daters are characterized by a vector that constitutes a large number of physical and personal characteristics. Even if there is complementarity (supermodularity) of any given individual characteristic between the partners, say there is complementarity on education and complementarity on race, it is unlikely that there is perfect correlation between each of characteristics. Education and race are unlikely to be perfectly correlated. As a result, somewhere in the matrix of match values there will be a violation of supermodularity.[3]

Local supermodularity establishes when there will be positive assortative matching even if supermodularity fails. The condition requires supermodularity along the equilibrium allocation, but not otherwise, as long there is not too much submodularity. Suppose for example that high school drop-outs match with high-school dropouts and college students with college students. Local supermodularity requires supermodularity between those matched, but not away from the equilibrium allocation, i.e. there need not be supermodularity in match values between college students and high-school dropouts. In this example, it is clear that the property is also well-defined in a problem with unidimensional

[1] In the US, 3.1 million workers (8.5% of the work force aged 25-65) are employed as teachers. Wages are often determined by collective bargaining; yet the allocation will imply sorting of better teachers into better schools, with lower teacher to student ratios, with better students,...

[2] Becker (1973) attributes the first non-mathematical discussion of complementarities (or supermodularity, or increasing differences) in marriage markets to Winch (1958).

[3] In the next section, we illustrate this with a simple example.

characteristics. However, it is easy to see that with multi-dimensional characteristics, it is unlikely that supermodularity will be satisfied even if education is everywhere supermodular, i.e., also between college students and dropouts. For example, suppose that in addition to the preferences for education, the blue-eyed have a strong enough preference for blue-eyed and the brown-eyed for brown-eyed. Unless eye color and educational attainment are perfectly correlated, the supermodularity requirement is bound to be violated for some unmatched high school-dropout pairs.

For a matching economy, the local supermodularity property is defined for a given set of preferences and characteristics of the agents. Only when requiring *any* arbitrary distribution of characteristics will local supermodudularity break down for some distribution.[4] Imposing supermodularity everywhere is a strong assumption that is easily violated, so we relax it, and local supermodularity is much weaker. For practical purposes, when estimating a model or designing a market, one is typically faced with a given distribution or a class of distributions. Local supermodularity allows one to determine whether there is unique assortative matching, and can establish boundaries on the classes of distributions for which this is satisfied.

The characterization of unique assortative matching is of use in two different strands of applications: market design exercises, and the estimation of matching models in order to perform policy analysis. Economists who use two-sided matching models to tackle market design questions are immediately confronted with the incentives participants in those markets have to misrepresent their preferences. Roth and Peranson (1999) for example analyze the resident-hospital matching mechanism and deduce both that there are little incentives for misrepresentation, and that for the announced preferences, the set of stable allocations is very small. There is a growing literature establishing a close link between the uniqueness of an allocation and the absence of incentives for strategic misrepresentation[5]. The main finding is that market mechanisms for those matching models induce truth-telling if and only if the stable matching is unique. The relevance of our result therefore is that a condition is provided that verifies uniqueness and can therefore guarantee incentive-compatibility of the mechanism. For example, if a government authority wants to allocate a number of licenses (say in telecommunications), one to each of a set of bidders, it will know under which technological and bidder characteristics the bidding mechanism will be incentive-compatible.[6]

In order to perform out of sample policy experiments, knowledge of the underlying preferences is crucial for making policy prescriptions. For example, using on-line dating data, Hortaçsu e.a. (2006) show that positive sorting on education is driven by individuals' preference for a partner with a *similar* level of education, rather by than a preference for a partner with the highest education possible. When

[4]The reason is that local supermodularity specifies supermodularity along the equilibrium allocation. Requiring any distribution implies any allocation can be the equilibrium allocation, and therefore, supermodularity must be satisfied everywhere.

[5]Sönmez (1999) analyzes strategy-proofness, and Ehlers and Massó (2006) consider the incentive compatibility of a Bayesian Game.

[6]For recent work on the application of matching models to multi-unit auctions, see Hatfield and Milgrom (2005).

estimating a two-sided matching model, establishing whether the equilibrium allocation is unique helps the econometrician avoid tackling thorny identification issues. Or at least, if the condition is not satisfied, the econometrician is not left in the dark and she knows that multiple equilibria exist. In addition to the uniqueness issue, an interesting by-product of these matching models is that one can estimate essentially competitive models without any information on prices.[7] No data is needed on prices and transfers to derive the equilibrium outcome. In many applications of two-sided matching, those prices are not available in the data. In the marriage market, the transfers between husbands and wives are rarely recorded. Often, those transfers are non-monetary, like the substitution between home production and market work.

In this paper, as in Becker (1973), the sorting is entirely driven by the preferences and characteristics of the matched partners. Based on those preferences, in equilibrium agents are sorted positively on some characteristics (education, wealth, intelligence, religion, race, age, ethnic origin, geographical proximity,...) while simultaneously they are negatively sorted on others (attitudes to risk, certain psychological traits,...). It is important to stress that in this paper the market mechanism considered is frictionless. In reality, explanations for assortative matching must also take into account market frictions. Market frictions can either help or hinder sorting. Consider for example the case of education. Everything else equal, frictions can induce matchings between agents with certain educational characteristics both to be more or less assorted. For example, due to market frictions, marriage partners are more likely to meet in the same educational establishments than outside, leading to positive sorting. Conversely, large frictions may also lead to less sorting. In a small farming community for example, anyone with some education may well be likely to marry someone without education due to a thin matching market. There are different ways to model frictions and its effect on assortative matching. Shimer and Smith (2000) and Atakan (2006) propose models with search frictions and Anderson and Smith (2004) consider a model with information frictions.

2 Some Preliminary Examples

In this section, we present three examples to illustrate local supermodularity. They are all done for a particular class of matching games, i.e. the assignment game which has perfectly transferable utility. The main intuition behind several of the results can thus be illustrated without first introducing the full model. In the assignment game, a payoff matrix $\mathbf{F}$ determines the surplus for all possible matches between agents of both sides of the market. A stable matching is an allocation and payoffs such that no pair is better off under an alternative allocation given those payoff than under the stable matching. A formal definition is provided in the next section.

Example 1. Multi-dimensional types: wealth and ethnicity

[7] See also Fox (2005).

Consider a marriage market with 3 women and 3 men. Both women (type x) and men (type y) have two characteristics α, β. The first is a commonly ranked characteristic x^α and $y^\alpha \in \{3,2,1\}$ (education, wealth, looks,...); the second is an idiosyncratic component x^β and $y^\beta \in \{A,B,C\}$ (ethnicity, language, piercing,...) which gives extra utility X when partners are "similar". Consider the following ordered set of types $\mathcal{X} = \{(3,A),(2,B),(1,C)\}$ and $\mathcal{Y} = \{(3,C),(2,B),(1,A)\}$ and suppose that the match surplus function is given by $f(x,y) = x^\alpha \cdot y^\alpha + X \cdot \mathbb{I}_{x^\beta = y^\beta}$ which gives rise to the match value matrix $\mathbf{F}$

$$\mathbf{F} = \begin{pmatrix} 9 & 6 & 3+X \\ 6 & 4+X & 2 \\ 3+X & 2 & 1 \end{pmatrix}.$$

The equilibrium notion is stability: it is an allocation and corresponding payoffs to each player such that no pair is better of deviating. When $X = 0$, the match value function is supermodular and the unique equilibrium allocation is positively assorted on wealth. Supermodularity requires that for any 2×2 minor of this matrix, the sum of the diagonal is larger than the sum of the anti-diagonal. When $X = 1$, the match value matrix is

$$\mathbf{F} = \begin{pmatrix} \mathbf{9} & 6 & 4 \\ 6 & \mathbf{5} & 2 \\ 4 & 2 & \mathbf{1} \end{pmatrix}$$

and the unique allocation is still positively assorted on wealth. Now however, relative to the given order (which coincides with the wealth ranking), the match value function is no longer supermodular. There is a double violation of supermodularity (in the 2×2 minors at the top-right and bottom-left: $6 + 2 < 4 + 5$: $f(2,1) + f(3,2) < f(3,1) + f(2,2)$ and $f(1,2) + f(2,3) < f(2,2) + f(1,3)$).[8] For X large enough, positive assortative matching fails. For any $X > 2$ (for example, $X = 3$), the matching is negatively assorted

$$\mathbf{F} = \begin{pmatrix} 9 & 6 & \mathbf{6} \\ 6 & \mathbf{7} & 2 \\ \mathbf{6} & 2 & 1 \end{pmatrix}$$

Example 2. Unidimensional Characteristics

This example confirms that the result is not driven by the multi-dimensionality of agents' characteristics. Let the set of agents be $(x,y) \in \{3,2,1\}^2$ and the surplus generating function $f(x,y) = 4xy - x^2 y$, then the output matrix is

$$\mathbf{F} = \begin{pmatrix} \mathbf{18} & 6 & 3 \\ 12 & \mathbf{8} & 4 \\ 9 & 6 & \mathbf{3} \end{pmatrix}$$

and is not supermodular. The minor in the top-right corner violates supermodularity. This is further confirmed by inspection of the cross-partial derivative $f_{xy} = 4 - 2x$, which is positive for $x = 1$, zero at

[8] Observe that there is monotonicity everywhere: $f(x,y) > f(x,y')$ for all $y > y'$ and $f(x,y) > f(x',y)$ for all $x > x'$.

2 and negative at 3. It is immediate that $f(1,2)+f(2,3)<f(3,2)+f(1,3)$. Nonetheless, the unique matching is positively assorted.

Example 3. Geographic Advantage

Consider 3 Telecommunications firms Atlantic, MidWest and Pacific and 3 distribution networks East, Central, West. Firms have competitive advantages in the type of market (urban or rural) and the closer the market is to their current geographic location. A competitive bidding process will determine the allocation of each of the bidders to the distribution networks. Let the type of the firms $x = (x^{\alpha}, x^{\beta})$ be a vector that consists of an urban and a geographic component where $x^{\alpha} = 1$ if the firms is specialized in urban distribution and 0 otherwise, and $x^{\beta} = 1, 2, 3$ if the location is East,Central, West. Firm types are Atlantic $x = (1,1)$, MidWest $x = (0,2)$ and Pacific $x = (1,3)$. Likewise for $y = (y^{\alpha}, y^{\beta})$ with markets East $(1,1)$, Central $(0,2)$ and West $(1,3)$. Suppose that the value to is given by $f(x,y) = [x^{\alpha}y^{\alpha} + (1-x^{\alpha})(1-y^{\alpha})] - d\cdot\left(x^{\beta}-y^{\beta}\right)^2$. When the urban component matters (for any small d), there is a unique allocation Atlantic-East, MidWest-Central, Pacific-West and supermodularity is violated twice ($2-4d > -2d$ for small d)

$$\mathbf{F} = \begin{pmatrix} 1 & 0-d & 1-4d \\ 0-d & 1 & 0-d \\ 1-4d & 0-d & 1 \end{pmatrix}.$$

3 The Two-Sided Matching Model

The set up of our model follows closely the two-sided matching model with limited transfers by Legros and Newman (2002).[9] Consider two finite and disjoint sets of agents $\mathcal{X}, \mathcal{Y}$ with generic elements x_i, y_j. When referring to the marriage model, $\mathcal{X}$ is the set of women and $\mathcal{Y}$ is the set of men. Without loss of generality, we assume each set is of equal size and the total (even) number of agents is $\#(\mathcal{X}\cup\mathcal{Y}) = 2n$. When referring to ordered sets, we denote $\mathcal{X} = (x_i)$ and $\mathcal{Y} = (y_i), i \in I$, where I is the ordered set $(1, 2, ..., n)$ with the order relation $\geq$ a linear order.

All agents' preferences over the members of the other set of agents are determined by the value of the surplus that such a match generates. The primitive of any pairwise matching x, y is the Pairwise Pareto Frontier. Denote $u = \phi(x, y, v)$ the highest possible utility that x can attain in a match with y while guaranteeing a utility v to y. The Pairwise Pareto Frontier ϕ is assumed to be strictly monotone, continuous, and convex in v. The Pairwise Pareto Frontier is formed by the utility vectors $(u, v) \in U$ for

[9] There are obviously other models with limited transfers in the cooperative games literature. More specifically for two-sided matching, Demange and Gale (1985) allow for common monotonic transformations in the assignment game. More recently, Eriksson and Karlander study hybrid matching models where some agents (firms) can perfectly transfer utility whereas others (workers) cannot transfer utility at all, and Mailath, Postlewaite and Samuelson (2006) consider pricing strategies when property rights determine the ex ante divion of the surplus.

which there is no other $(u', v') \in U$ with $u' \geq u, v' \geq v$ and with at least one strict inequality. Without loss of generality, we assume that ϕ is non-negative and that the utility of being single is zero.

The pairwise utility possibility set between any two agents x, y is denoted by

$$U(x,y) = \left\{(u,v) \in \mathbb{R}^2 : u \leq \phi(x,y,v'), v \leq \psi(y,x,u') \text{ for some } u', v'\right\}.$$

We will assume that all agents always choose outcomes in $\phi(x, y, v)$. We denote the inverse of ϕ with respect to v by $\psi(y, x, u) = \phi^{-1}(x, y, v)$. By construction therefore,

$$\phi(x, y, \psi(y, x, u)) = u.$$

The profile of Pareto Frontiers is the set of all Pairwise Pareto Frontiers and is denoted by Φ. All agents have perfect information about the preferences and there are no frictions in the market (i.e. any pair can be formed at not cost).

An economy is a one-to-one matching problem and is denoted by the three-tuple $(\mathcal{X}, \mathcal{Y}, \Phi)$ that consists of a set of agents on both sides of the market, and a profile of Pairwise Pareto frontiers. An allocation is a matching such that each agent is paired with at most one agent of the other set. We now present some definitions.

Definition (Feasible payoff). *A payoff u_i is feasible for an allocation μ if $u_i \in [0, \phi(i, \mu_i, 0)]$. Likewise for v_j.*

Definition. *An allocation μ for $(\mathcal{X}, \mathcal{Y}, \Phi)$ is a $n \times n$ permutation matrix, i.e. μ_{ij} satisfies: $\sum_i \mu_{ij} \leq 1, \sum_j \mu_{ij} \leq 1, \mu_{ij} \in \{0, 1\}$.*

In line with the tradition of the matching literature, we will also denote by μ the mapping that associates a type x to a type y under this allocation. Thus $\mu(x_i) = y_j$ indicates that x_i is matched to y_j under μ. We can now define an equilibrium in terms of stability. A candidate equilibrium allocation and payoffs is stable if there exists no pair of agents that can be made better off matching with each other rather than each matching with the partner prescribed in the candidate equilibrium. Formally:

Definition. *A stable matching equilibrium (μ, u, v) is an allocation μ and a pair of vectors $u, v \in \mathbb{R}^n$ such that: 1. $u(x) = \phi(x, \mu(x), v(\mu(x)))$; and 2. there is no blocking pair, i.e. for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$*

$$\begin{aligned} u(x) &\geq \phi(x, y_j, v(y_j)), \forall j \\ v(y) &\geq \psi(x_i, y, u(x_i)), \forall i \end{aligned}$$

Notation. In what follows, let μ^* be the candidate equilibrium permutation matrix and corresponding to the identity matrix $\mathbb{I}_n$, i.e. with all entries on the diagonal. In other words, we normalize the order of $\mathcal{X}$ relative to $\mathcal{Y}$ to correspond to that of the equilibrium allocation.[10] Let μ be any alternative permutation matrix. Denote the payoff $u(x_i) = u_i$ and let $\mu(x_i) = \mu_i$. The objective is to compare payoffs that players can obtain when changing partners. A player x_i will compare her payoff from being matched with $\mu^*(x_i)$ to that from being matched with her partner under alternative match $\mu(x_i)$. That payoff in turn depends on what $\mu(x_i)$ obtains under the original match with $\mu^*(\mu(x_i))$. For notational convenience, we will exploit the fact that μ^* is normalized to the diagonal. As a result, the index of y in I of $\mu^*[\mu(x_i)] = \mu^*[\mu_i]$ is the same as that of μ_i, and therefore, we write $\phi(\mu_i, \mu_i, v)$ to denote $\phi(\mu(x_i), \mu^*[\mu(x_i)], v)$. This is without loss of generality as long as it is taken into account that μ_i denotes both an element in $\mathcal{X}$ and one in $\mathcal{Y}$. Which one it is can be derived from whether it is the first or the second argument in ϕ. In general $\mu\left[\mu_i^{k-1}\right] = \mu_i^k \in \mathcal{X}$ and $\mu^*\left[\mu\left[\mu_i^{k-1}\right]\right] = \mu_i^k \in \mathcal{Y}$. For further notational convenience, and when there is no confusion, denote $x_i = i$ and $y_j = j$, using the order in I to refer the actual argument x_i or x_j. Thus $\phi[i, \mu_i, v(\mu_i)]$ is equivalent to $\phi[x_i, \mu(x_i), v(\mu_i)]$.

By construction, and because μ is a finite permutation matrix, at some point there exist K such that $\mu_i^K = x_i$. For any $k < K$, μ_i^k cannot indicate the same player more than once. All those players are part of a chain. A chain of length K is a submatrix $\mu(X \times Y, K)$ of μ, where K is the smallest integer such that $\mu_i^K = i$. For some permutation matrices μ, there may be multiple chains that partition μ.[11] In order to cover the entire permutation matrix μ, the convention will be that μ_i^K of a chain of length K "jumps" to any player outside those already covered so that μ_i^k denotes $\mu_{i'}^k$ rather than i. This goes on until the entire matrix μ is partitioned in chains.

At this point, it is worth noting that in this formulation of the model, we do not identify agents by type. This is without loss of generality. Suppose on the contrary that there are $k, m \leq n$ different types $\widetilde{x}, \widetilde{y}$ with match surplus $\widetilde{\phi}(\widetilde{x}, \widetilde{y}, v)$ and distributions $G(\widetilde{x}), H(\widetilde{y})$. Then under the candidate equilibrium allocation μ^*, agents match whose types are $G(\widetilde{x}) = H(\widetilde{y})$. We can then transform this matching problem into one with a uniform type distribution where $\phi(x, y, v) = \widetilde{\phi}(x, G^{-1}(H(y)), v)$.

Existence of equilibrium of this matching problem follows from Kaneko (1982). The model here is a special case of his cooperative game. The proof in Kaneko is a variation on Shapley and Scarf (1974).

Theorem. (Kaneko 1982, Theorem 1) *A stable matching equilibrium exists for any matching problem*

[10] In this sense, we take the candidate equilibrium allocation as given. For example, from the observed outcome in a model that is estimated (e.g. on-line dating, Hortaçsu e.a. (2006)).

[11] Consider the following 5×5 example of μ where the matrix $\mu^* - \mu$ is given by:

$$\mu^* - \mu = \left(\begin{array}{ccc|cc} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right).$$

There are two chains, one of length 3 and one of length 2. Starting at $i = 1, \mu_1 = 2, \mu_1^2 = 3$ and $\mu_1^3 = 1$. Starting at $i = 4, \mu_4 = 5, \mu_4^2 = 4$.

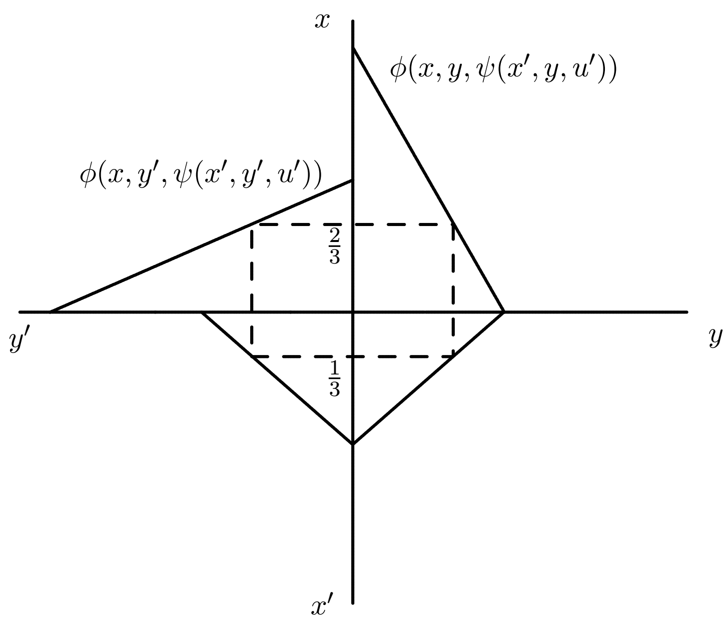
$(\mathcal{X}, \mathcal{Y}, \Phi)$.

For the purpose of this paper, we are interested in the allocations, i.e., who matches with whom. With finite types, and even if there is a unique allocation, there may be multiple payoffs (possibly a continuum) that support the same allocation. For the remainder of the paper, when we mention uniqueness, we refer to the allocation, not to the allocation and payoff vectors.

Next, we illustrate by virtue of a simple example that the set of stable allocation is not typically a singleton. Moreover, both extremes of positive and negative assortative matching can be an equilibrium for the same matching problem.

Example. Consider a 2×2 matching problem with $\mathcal{X} = \{x, x'\}, \mathcal{Y} = \{y, y'\}$ and $\phi(x, y, v) = 2 - 2v, \phi(x, y', v) = 1 - \frac{1}{2}v$ and $\phi(x', y, v) = \phi(x', y', v) = 1 - v$. The Pareto frontiers are represented in Figure 1 in a graphical representation following Legros and Newman (2002):

Figure 1



One way to interpret this figure is to fix the level of utility obtained by x' at u'. Then compare the utility obtained by x in each of the alternative matches with y and $y' : \phi(x, y, v) \lesseqgtr \phi(x, y', v')$. Since the utility of x' is fixed, we can derive the utility each of the players y, y' can obtain from matching with x' from ψ. We therefore calculate

$$
\begin{aligned}
u &= \phi(x, y, \psi(y, x', u')) \\
&= 2 - 2(1 - u') = 2u'
\end{aligned}
$$

and

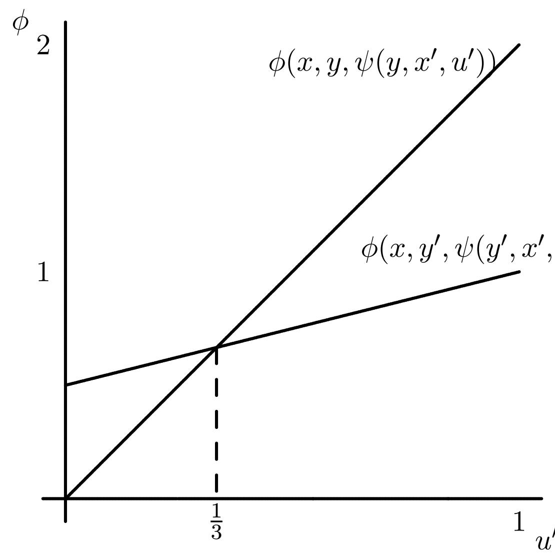
$$
\begin{aligned}
u &= \phi(x, y', \psi(y', x', u')) \\
&= 1 - \frac{1}{2}\left(1 - u'\right) = \frac{1}{2} + \frac{1}{2}u'
\end{aligned}
$$

Then it is immediate that for a level of utility to player x' equal to $u' > 1/3$, the payoff under both match $\phi(x, y, \psi(y, x', u')) > \phi(x, y', \psi(y', x', u'))$. Equivalently, the payoff to x of a match with y dominates a match with y' if $u > 2/3$ since $u = 2u'$. There are two equilibrium allocations

$$\left\{\mu\left(x, x'\right) = \left(y, y'\right), u' \geq 1/3, u \geq 2/3\right\}$$
$$\left\{\mu\left(x, x'\right) = \left(y', y\right), u' \leq 1/3, u \leq 2/3\right\}.$$

The payoffs v and v' follow immediately from the definition of ψ. Plotting $\phi(x, y, \psi(y, x', u'))$ and $\phi(x, y', \psi(y', x', u'))$, we observe increasing linear functions that intersect at $u' = 1/3$. See Figure 2.

Figure 2



Given payoffs u, u' large enough, there is no blocking pair. Player x and y' (or x' and y) cannot form a match that makes both better off. It becomes clear from this example that multiplicity does not hinge on the non-linearity of the frontiers, but rather on the differential slopes of the frontiers.

4 Local Supermodularity

We now derive the main results. First, we define the condition **U** for the general matching problem and derive two useful properties in Lemmas 1 and 2. We show the main theorem which links condition **U** to uniqueness. Then we characterize condition **U** for the case of the symmetric assignment game. We also extend the characterization for a continuum of agents. Finally, we show that uniqueness properties are identical for those economies where agents have common preferences over output produced in a match.

4.1 The General Result

Condition U. *For any μ, $u_i > \phi(\mu, u_i)$ for all feasible u_i, where*

$$\phi(\mu, u_i) = \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[\mu_i, \mu_i^2, \psi\left(\mu_i^2, \mu_i^2, \phi\left[...\psi\left(\mu_i^n, \mu_i^n, u_i\right)\right]\right)\right]\right)\right].$$

First, we show that this condition is invariant whichever player i or j one starts with. That is, when comparing the payoffs of agent i, taking into account all players' outside options induced by the match μ, the same condition holds when comparing any other player j. In other words, it does not matter with which player one starts to derive the inequality.

Lemma 1. *For a given μ, condition $\mathbf{U}$ for i, and condition $\mathbf{U}$ for j, are equivalent.*

Proof. To see this, start from condition $\mathbf{U}$ for a given j :

$$u_j > \phi\left[j, \mu_j, \psi\left(\mu_j, \mu_j, \phi\left[\mu_j, \mu_j^2, \psi\left(\mu_j^2, \mu_j^2, \phi\left[...\psi\left(\mu_j^n, \mu_j^n, u_j\right)\right]\right)\right]\right)\right].$$

Since μ is a permutation matrix over the entire domain of $\mathcal{X}$ and $\mathcal{Y}$, there exists a $k < n$ such that $\mu_i^k = j$.

Observe that $\phi(\cdot, \cdot, v)$ is decreasing in v by assumption. Since ψ is the inverse in v of ϕ, $\psi(\cdot, \cdot, u)$ is decreasing in u. Therefore $\phi[\cdot, \cdot, \psi(\cdot, \cdot, u)]$ is increasing in u. This is also true for any sequence of pairs of transformations $\phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \cdot\right]\right)\right]$. Apply then the increasing transformation $\phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \cdot\right]\right)\right]$ to get

$$\begin{aligned}&\phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, u_j\right]\right)\right]\\ &\quad > \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \phi\left[j, \mu_j, \psi\left(\mu_j, \mu_j, \phi\left[\mu_j, \mu_j^2, \psi\left(\mu_j^2, \mu_j^2, \phi\left[...\psi\left(\mu_j^n, \mu_j^n, u_j\right)\right]\right)\right]\right)\right]\right]\right)\right].\end{aligned}$$

Now define $u_i = \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, u_j\right]\right)\right]$ and substitute on the left-hand side

$$u_i > \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \phi\left[j, \mu_j, \psi\left(\mu_j, \mu_j, \phi\left[\mu_j, \mu_j^2, \psi\left(\mu_j^2, \mu_j^2, \phi\left[...\psi\left(\mu_j^n, \mu_j^n, u_j\right)\right]\right)\right]\right)\right]\right]\right)\right].$$

Observe that the right-hand side has $k + n$ iterations of the permutation μ: k by μ_i and n by μ_j. Moreover, k was chosen such that $\mu_i^k = j$. Therefore, $\mu_i^{k+1} = \mu_j, \mu_i^{k+2} = \mu_j^2, ..., \mu_i^n = \mu_j^{n-k}, ..., \mu_i^{n+k} = \mu_j^n$. Therefore,

$$\begin{aligned}&u_i >\\ &\phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \phi\left[\mu_i^k, \mu_i^{k+1}, \psi\left(\mu_i^{k+1}, \mu_i^{k+1}, \phi\left[\mu_i^{k+1}, \mu_i^{k+2}, \psi\left(\mu_i^{k+2}, \mu_i^{k+2}, \phi\left[...\psi\left(\mu_i^{n+k}, \mu_i^{n+k}, u_j\right)\right]\right)\right]\right)\right]\right]\right)\right]\end{aligned}$$

and using the fact that $\mu_i^n = i, \mu_i^{n+1} = \mu_i, \mu_i^{n+2} = \mu_i^2, ..., \mu_i^{n+k} = \mu_j^k$ we get

$$\begin{aligned}&u_i >\\ &\phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \phi\left[\mu_i^k, \mu_i^{k+1}, \psi\left(\mu_i^{k+1}, \mu_i^{k+1}, ...\psi\left(\mu_i^n, \mu_i^n, \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[...\psi\left(\mu_i^k, \mu_i^k, u_j\right)\right]\right)\right]\right)\right)\right]\right]\right)\right]\end{aligned}$$

Substituting again (now in the right-hand side) for u_i defined above, we rewrite as

$$u_i > \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, ...\psi\left[\mu_i^k, \mu_i^k, \phi\left[\mu_i^k, \mu_i^{k+1}, \psi\left(\mu_i^{k+1}, \mu_i^{k+1}, ...\psi\left(\mu_i^n, \mu_i^n, u_i\right)\right)\right]\right]\right)\right].$$

QED.

The next Lemma is instrumental in establishing the uniqueness result. At this point, it is worth recalling the notation we are using and that is common in the matching literature. For an alternative allocation μ, say where x_1 is matched with y_2 and x_3 is matched with y_1 in a 3×3 market, then μ_i for $i = 1$ refers to y_2 and x_3 respectively. Therefore $\phi\left[i, \mu_i, v(\mu_i)\right]$ and $\psi\left[\mu_i, i, u(\mu_i)\right]$ for $i = 1$ are $\phi\left[1, 2, v_2\right]$ and $\psi\left[3, 1, u_3\right]$ respectively.

Lemma 2. *Consider payoff vectors u, v and an alternative allocation $\mu \neq \mu^*$ then the following statements are equivalent:*

$$
\begin{aligned}
u_i &> \phi\left[i, \mu_i, v(\mu_i)\right], \text{ for all } i \\
&\Updownarrow \\
v_j &> \psi\left[\mu_j, j, u(\mu_j)\right], \text{ for all } j
\end{aligned}
$$

and for any i

$$
\begin{aligned}
u_i &= \phi\left[i, i, v_i\right] \\
&\Updownarrow \\
v_i &= \psi\left[i, i, u_i\right]
\end{aligned}
$$

Proof. The second part follows immediately from the definition of ψ as being the inverse of $\phi(\cdot, \cdot, v)$ in the third argument. The first part uses the same property. Since $u_i > \phi\left[i, \mu_i, v(\mu_i)\right]$, then after applying the inverse in the third argument, $\psi\left[i, \mu_i, u_i\right] < v(\mu_i)$ with the sign changed since ϕ, and therefore its inverse ψ, is monotonically decreasing. Therefore, for $y_j = \mu_i$ we have $v_j > \psi\left[\mu_j, j, u(\mu_j)\right]$ since $i = \mu_j$. Now for any $i, u_i > \phi\left[i, \mu_i, v(\mu_i)\right]$ this is equivalent to $v_j > \psi\left[\mu_j, j, u(\mu_j)\right]$ for some j. Since $u_i > \phi\left[i, \mu_i, v(\mu_i)\right]$ holds n times for all i, there are associated n inequalities $v_j > \psi\left[\mu_j, j, u(\mu_j)\right]$. Because μ is a permutation matrix, each of these n inequalities is different, and therefore the condition holds for all j. QED.

Theorem 1. *A matching problem $(\mathcal{X}, \mathcal{Y}, \Phi)$ satisfies condition* **U** *for all $\mu \neq \mu^*$ if and only if it has a unique equilibrium μ^*.*

Proof. $(\Leftarrow)$: Consider a candidate alternative equilibrium allocation μ. The allocation μ^* is the unique equilibrium allocation, so a set of feasible payoff vectors u, v satisfies

$$
\begin{aligned}
u_i &> \phi\left[i, \mu_i, v(\mu_i)\right] \\
v_i &> \psi\left[\mu_i, i, u(\mu(y_i))\right]
\end{aligned}
$$

for all i. Together with the definitions

$$\begin{aligned} u_i &= \phi[i, i, v_i] \\ v_i &= \psi[i, i, u_i]. \end{aligned}$$

for all i. From Lemma 2, we only need to consider $u_i > \phi[i, \mu_i, v(\mu_i)]$ and $v_i = \psi[i, i, u_i]$ for all i. Now starting from the equilibrium condition

$$u_i > \phi[i, \mu_i, v(\mu_i)]$$

and substituting $v(\mu_i)$ for the definition $v(\mu_i) = \psi(\mu_i, \mu_i, u(\mu_i))$ we get

$$u_i > \phi[i, \mu_i, \psi(\mu_i, \mu_i, u(\mu_i))].$$

Using the equilibrium condition $u(\mu_i) > \phi\left[\mu_i, \mu_i^2, v(\mu_i^2)\right]$ it follows that

$$u_i > \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[\mu_i, \mu_i^2, v(\mu_i^2)\right]\right)\right]$$

since ϕ and ψ are monotonically decreasing. We can now continue to substitute each $u(\mu_i^k)$ from the inequality $u(\mu_i^k) > \phi\left[\mu_i^k, \mu_i^{k+1}, v(\mu_i^{k+1})\right]$, and each $v(\mu_i^k)$ from the definition $\psi\left(\mu_i^k, \mu_i^k, u(\mu_i^k)\right)$. Since $\mu_i^n = i$, after n substitutions, $v(\mu_i^n) = v_i = \psi(\mu_i^n, \mu_i^n, u_i)$ and we obtain

$$u_i > \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[\mu_i, \mu_i^2, \psi\left(\mu_i^2, \mu_i^2, \phi[...\psi(\mu_i^n, \mu_i^n, u_i)]\right)\right]\right)\right].$$

This is condition **U**. Moreover, this applies to any candidate alternative equilibrium allocation $\mu \neq \mu^*$.

($\Rightarrow$) : We establish this part of the proof by contradiction. Suppose there exist a candidate alternative equilibrium allocation μ and payoff vectors u', v' such that both (μ^*, u, v) and (μ, u', v') are a stable allocation. Then for all i

$$\begin{aligned} u_i &> \phi[i, \mu_i, v(\mu_i)] \\ v_i &= \psi[i, i, u_i] \end{aligned}$$

and, for all i

$$\begin{aligned} u_i' &> \phi\left[i, i, v_i'\right] \\ v_i' &= \psi\left[\mu_i, i, u_i'\right]. \end{aligned}$$

We can immediately apply to the conditions for the equilibrium (μ^*, u, v) the iterative substitution from the first part of this proof, which gives

$$u_i > \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[\mu_i, \mu_i^2, \psi\left(\mu_i^2, \mu_i^2, \phi[...\psi(\mu_i^n, \mu_i^n, u_i)]\right)\right]\right)\right].$$

For the equilibrium (μ, u', v'), we first invoke Lemma 2 and use instead the conditions

$$\begin{aligned} v'_i &> \psi\left[i, i, u'_i\right] \\ u'(\mu_i) &= \phi\left[\mu_i, i, v'_i\right] \end{aligned}$$

for all i. Let j be such that $y_j = \mu(x_i)$ and therefore the corresponding payoff for x_i satisfies

$$u'_i = \phi\left[i, \mu_i, v'(\mu_i)\right].$$

After substitution for $v'(\mu_i) > \psi\left[\mu_i, \mu_i, u'(\mu_i)\right]$

$$u'_i < \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, u'(\mu_i)\right)\right],$$

since ϕ is monotonically decreasing in its third argument. Now we will substitute $u'(\mu_i) = \phi\left[\mu_i, \cdot, v'(\cdot)\right]$. Again, there is a k such that $y_k = \mu(\mu_i) = \mu_i^2$ and thus $u'(\mu_i) = \phi\left[\mu_i, \mu_i^2, v'(\mu_i^2)\right]$. Therefore,

$$u'_i < \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[\mu_i, \mu_i^2, v'(\mu_i^2)\right]\right)\right],$$

and after substitution repeatedly for $v'(\mu_i^m) > \psi\left[\mu_i^m, \mu_i^m, u'(\mu_i^m)\right]$ and $u'(\mu_i^m) = \phi\left[\mu_i^m, \mu_i^{m+1}, v'(\mu_i^{m+1})\right]$ we obtain

$$u_i < \phi\left[i, \mu_i, \psi\left(\mu_i, \mu_i, \phi\left[\mu_i, \mu_i^2, \psi\left(\mu_i^2, \mu_i^2, \phi\left[...\psi\left(\mu_i^n, \mu_i^n, u_i\right)\right]\right)\right]\right)\right].$$

The candidate equilibrium (μ, u', v') implies the opposite of condition **U**, therefore a contradiction. QED.

4.2 Local Supermodularity for the Assignment Game

We now consider the assignment game, i.e. where $\phi(x, y, v) = f(x, y) - v$. Because we can represent the payoffs now in matrix form, we introduce the following notation. First, the profile of match values Φ can now be represented by the matrix $\mathbf{F}$. An n agent matching problem is therefore $(\mathcal{X}, \mathcal{Y}, \mathbf{F})$. The objective now is to obtain a short-hand notation in terms of the cross-partial difference. Denote

$$\Delta(x_i, y_j) = f\left(x_i, y_j\right) + f\left(x_{i+1}, y_{j+1}\right) - f\left(x_{i+1}, y_j\right) - f\left(x_i, y_{j+1}\right).$$

As before, when there is no confusion we use the short-hand notation $\Delta(i, j)$. The n agent matching problem with an $n \times n$ matrix $\mathbf{F}$ with element $f(x_i, y_j)$ can also be represented as an $(n-1) \times (n-1)$ matrix $\boldsymbol{\Delta}$

$$\boldsymbol{\Delta} = \begin{pmatrix} \Delta(1,1) & ... & \Delta(1, n-1) \\ \Delta(2,1) & & \\ ... & & ... \\ \Delta(n-1, 1) & ... & \Delta(n-1, n-1) \end{pmatrix}.$$

Observe that in $\boldsymbol{\Delta}$ is a transformation of $\mathbf{F}$:

$$\boldsymbol{\Delta} = \mathbf{T}\mathbf{F}\mathbf{T}'$$

where $\mathbf{T}$ is an $(n-1)\times n$ matrix:

$$\mathbf{T} = \begin{pmatrix} 1 & -1 & ... & & 0 \\ 0 & 1 & -1 & & 0 \\ ... & & & & ... \\ 0 & ... & 0 & 1 & -1 \end{pmatrix}$$

Supermodularity or increasing differences is most often defined as follows: f is supermodular if $f(x_i, y_j) + f(x_k, y_l) > f(x_i, y_l) + f(x_k, y_j)$ for any pair (x_i, y_j) and (x_k, y_l) with $x_i > x_k, y_j > y_l$,. Observe that if this is satisfied for any adjacent pair (x_i, y_j) and (x_{i+1}, y_{j+1}), i.e. $\Delta(x_i, y_j) > 0$, then by induction, it follows that it is also satisfied for any non-adjacent pair. This definition is well-known, especially for continuous functions, where positive supermodularity is often defined by positive cross-partial derivatives.

Definition. *A matching problem $(\mathcal{X}, \mathcal{Y}, \mathbf{F})$ satisfies strict supermodularity if for all $i, j \in \{1, ..., n-1\}, \Delta(i,j) > 0$.*

Next, we derive condition for uniqueness in the case of the symmetric assignment game. Condition **U** for a given μ is equivalent to

$$f(i,i) - f(i,\mu_i) + f\mu_i,\mu_i) - f(\mu_i, \mu_i^2) + ... + f(\mu_i^{n-1}, \mu_i^{n-1}) - f(\mu_i^{n-1}, \mu_i^n) > 0.$$

For the case of a symmetric functions $f(x,y) = f(y,x)$, the local supermodularity condition can be expressed in two distinct ways, referred to as **LS1** and **LS2**. The second formulation is in terms of the cross-partial differences $\Delta(i,j)$.

Definition. *($LS1$) A symmetric function $f(x,y)$ of the assignment game satisfies local supermodularity provided*

$$f(i,i) + f(j,j) - f(i,j) - f(j,i) > 0,$$

for all $i, j \in \{1, ..., n\}$.

Definition. *($LS2$) A symmetric function $f(x,y)$ of the assignment game satisfies local supermodularity provided the sum of the elements of the connected principal minors of $\mathbf{\Delta}$ are positive, i.e.,*

$$\sum_{k=i}^{j}\sum_{l=i}^{j}\Delta(k,l) > 0,$$

for all $i, j \in \{1, ..., n-1\}$ ($j > i$).

First, we show that these two conditions are equivalent.

Lemma 3. *$LS1$ and $LS2$ are equivalent.*

Proof. Consider any i, j (let $j \geq i$), then **LS1**

$$f(i,i) + f(j,j) - f(i,j) - f(j,i) > 0$$

for all $i, j \in \{1, ..., n\}$. Now starting at $f(i,i)$, subtract and add all the entries in the columns and rows in $\mathbf{F}$ between i and j, once in the first and last row and column, twice in all other rows and columns. Then **LS1** is equivalent to

$$\begin{array}{cccccccccccc} & f(i,i) & - & f(i,i+1) & + & f(i,i+1) & -... & + & f(i,j-1) & - & f(i,j) & \\ - & f(i+1,i) & + & f(i+1,i+1) & - & f(i+1,i+1) & +... & - & f(i+1,j-1) & + & f(i+1,j) & \\ + & f(i+1,i) & - & f(i+1,i+1) & + & f(i+1,i+1) & -... & + & f(i+1,j-1) & - & f(i+1,j) & \\ - & f(i+2,i) & + & f(i+2,i+1) & - & f(i+2,i+1) & +... & - & f(i+2,j-1) & + & f(i+2,j) & . \\ & ... & & & & & & & & & ... & \\ + & f(j-1,i) & - & f(j-1,i+1) & & & ... & + & f(j-1,j-1) & - & f(j-1,j) & \\ - & f(j,i) & + & f(j,i+1) & & & ... & - & f(j,j-1) & + & f(j,j) & > 0 \end{array}$$

for all $i, j \in \{1, ..., n\}$.Substitute everywhere with the definition of $\Delta(k,l) = f(k,l) + f(k+1, l+1) - f(k, l+1) - f(k+1, l)$ and we get

$$\begin{array}{cccccccc} & \Delta(i,i) & + & \Delta(i,i+1) & +... & + & \Delta(i,j-1) & \\ + & \Delta(i+1,i) & + & \Delta(i+1,i+1) & +... & + & \Delta(i+1,j-1) & \\ & ... & & & & & ... & \\ + & \Delta f(j-1,i) & & & +... & + & \Delta(j-1,j-1) & > 0 \end{array}$$

for all $i, j \in \{1, ..., n-1\}$. This condition is equivalent to

$$\sum_{k=i}^{j} \sum_{l=i}^{j} \Delta(k,l) > 0$$

for all $i, j \in \{1, ..., n-1\}$, i.e., **LS2**. QED.

We now show that local supermodularity as defined above is a necessary and sufficient condition for uniqueness.

Theorem 2. *For the symmetric assignment game, the allocation μ^* with positive assortative matching is the unique equilibrium if and only if $f(x,y)$ satisfies* **LS1** *(***LS2***).*

Proof. We show **LS1** $\Leftrightarrow$ **U**, i.e. that **LS1** is equivalent to condition **U**. Invoking Theorem 1 thereby establishes the result.

($\Rightarrow$) Consider any $\mu \neq \mu^*$. Then we need to establish that condition **U** for that particular μ follows from **LS1**. For a given μ, condition **U** satisfies

$$\sum_j \left(f(\mu^j, \mu^j) - f(\mu^j, \mu^{j+1}) \right) > 0.$$

Using **LS1** for all i, j, we now derive condition **U**. First, consider any $f(\mu^j, \mu^{j+1})$. From **LS1** it follows that $f(\mu^j, \mu^j) + f(\mu^{j+1}, \mu^{j+1}) - f(\mu^j, \mu^{j+1}) - f(\mu^{j+1}, \mu^j) > 0$. Summing over all j we get

$$\sum_j \left[f(\mu^j, \mu^j) + f(\mu^{j+1}, \mu^{j+1}) - f(\mu^j, \mu^{j+1}) - f(\mu^{j+1}, \mu^j) \right] > 0$$

Since μ is a permutation matrix, $\sum_j f(\mu^j, \mu^j) = \sum_j f(\mu^{j+1}, \mu^{j+1}) = tr(\mathbf{F})$ and equal to trace of the matrix $\mathbf{F}$. Given symmetry, $f(\mu^j, \mu^{j+1}) = f(\mu^{j+1}, \mu^j)$

$$2 \sum_j \left[f(\mu^j, \mu^j) - f(\mu^j, \mu^{j+1}) \right] > 0.$$

Up to a factor 2, this is exactly condition **U**.

($\Leftarrow$) Condition **U** is satisfied for all $\mu \neq \mu^*$. That includes those μ that are identical to μ^* except for the pairs (i, j) and (j, i). Then condition **U** for μ is given by $f(i, i) + f(j, j) - f(i, j) - f(j, i) > 0$ since all other terms cancel. This is **LS1**. QED.

A simple example illustrates that the result crucially hinges on the symmetry assumption. For a 3×3 matching market, consider the alternative allocation

$$\mu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Condition (**U**) here is

$$f(1,1) + f(2,2) + f(3,3) - f(1,2) - f(2,3) - f(3,1) > 0$$

and follows from summing

$$\begin{aligned} f(1,1) + f(2,2) - 2f(1,2) &> 0 \\ f(1,1) + f(3,3) - 2f(1,3) &> 0 \\ f(2,2) + f(3,3) - 2f(2,3) &> 0 \end{aligned}$$

since $f(1,3) = f(3,1)$. In contrast, in the absence of symmetry (say the only source of asymmetry is $f(1,3) \neq f(3,1)$), we get from summing the **LS1** conditions

$$2f(1,1) + 2f(2,2) + 2f(3,3) - 2f(1,2) - 2f(2,3) - f(1,3) - f(3,1) > 0.$$

This does not necessarily imply condition (**U**), for example when $f(1,3)$ is sufficiently large. This derives from the fact that $f(1,3)$ is not in the domain of μ, whereas under symmetry it is. This is the case for the following payoff matrix

$$\mathbf{F} = \begin{pmatrix} 2 & 1 & x \\ 1 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}$$

whenever $x \in (5,7)$, say $x = 6$. In that case, **LS1** is satisfied. The crucial deviation involving $(1,3)$ satisfies $x + 0 < 5 + 2$ for $x < 7$), yet the unique stable allocation is $(3,1,2) = \mu(1,2,3)$ and generates output of 10 instead of 9 (larger than 9 for any $x > 5$) under μ^*.

Observe that **LS1** implies supermodularity, but not vice versa. To see this, note that **LS1** is only required along the diagonal, not everywhere. For example, consider any $i > i'$ and $j > j'$ then supermodularity requires

$$f(i,i') + f(j,j') - f(i,j') - f(i',j) > 0,$$

which does not follow from **LS1**.

A continuum of types. We define equivalent of the conditions derived above for continuous functions. Assume that agents are distributed uniformly over a compact set $\mathcal{X} \times \mathcal{Y}$, where x and y are one-dimensional. I should point out here that local supermodularity does not require continuity or differentiability of f. In fact, even if $f(x,y)$ is continuous and differentiable, the order of the equilibrium allocation will generally not coincide, leading the function to be discountinous in the orders $\mathcal{X}$ and $\mathcal{Y}$, and therefore to discontinuous matching sets (for an example, see Anderson and Smith (2004)). For multi-dimensional types, this is likely to be the case as long as each dimension is not perfectly correlated. Nonetheless, continuous equivalents may be usefull for applications whenever f is differentiable (or piece-wise differentiable).

Definition. *For a continuous Assignment Game, for all* $(x,y) \in \mathcal{X} \times \mathcal{Y}$*:*

$$\begin{aligned}
\mathbf{SM} \quad &: \quad f_{xy}(x,y) > 0 \\
\mathbf{LS1} \quad &: \quad f(x,x) + f(y,y) - f(x,y) - f(y,x) > 0 \\
\mathbf{LS2} \quad &: \quad \int_{i=x}^{y} \int_{i=x}^{y} f_{12}(i,j)\,didj > 0
\end{aligned}$$

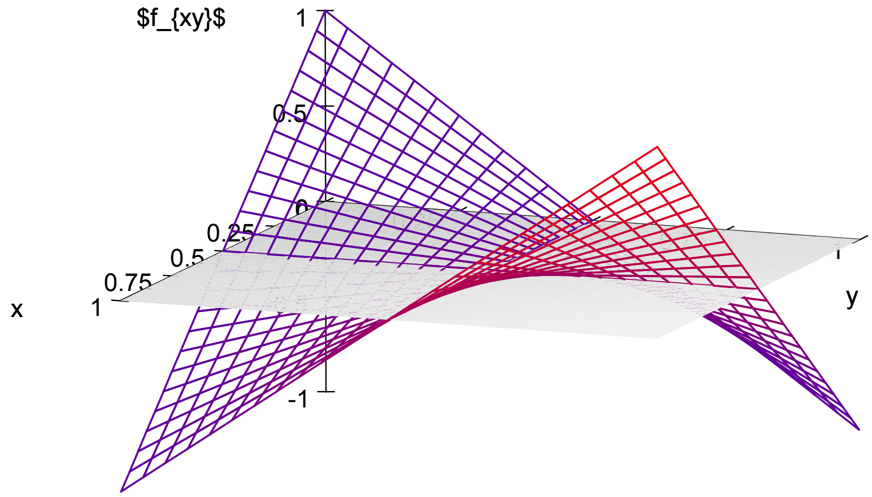
Consider a continuous type example. Let $\mathcal{X} \times \mathcal{Y} = [0,1]^2$, and

$$f(x,y) = xy(x-\alpha)(y-\alpha),$$

with the parameter $\alpha \in [0,1]$. Then

$$f_{xy} = (2x-\alpha)(2y-\alpha).$$

This function is not supermodular (except for the case where $\alpha = 0$). For example, the cross-partial derivative is negative $f_{xy} < 0$ for all $(x,y) \in [0, \frac{\alpha}{2}) \times (\frac{\alpha}{2}, 1]$ or $(x,y) \in (\frac{\alpha}{2}, 1] \times [0, \frac{\alpha}{2})$. The cross-partial over $[0,1]^2$ is plotted below



Nonetheless, there is positive assortative matching. The equilibrium allocation μ^* satisfies $y = \mu^*(x)$. The equilibrium aggregate output is

$$\begin{aligned}\int\int f(x,x)dxdx &= \int_0^1\int_0^1 x^2(x-\alpha)^2 dxdx \\ &= \frac{1}{30} - \alpha\frac{1}{10} + \alpha^2\frac{1}{12}.\end{aligned}$$

For example, for $\alpha = 1$ aggregate output is 1/60, for $\alpha = 1/2$ it is 1/240 and for $\alpha = 0$ it is 1/30.

The local supermodularity condition requires that for all connected "principal minors", the integral $\int_{i=x}^{y}\int_{i=x}^{y} f_{12}(i,j)didj > 0$ be positive for all x, y, which is equivalent to requiring supermodularity along the diagonal only

$$f(x,x) + f(y,y) - f(x,y) - f(y,x) > 0$$

for any pair i, j. For the example, this condition is equivalent to

$$[x\,(y-\alpha) - y\,(y-\alpha)]^2 > 0$$

for all $x, y \in [0,1]$ which is satisfied for any $\alpha \neq 1$. The general result is in the Proposition below:

Proposition 1. *All symmetric, multiplicatively separable (nowhere constant) differentiable functions $f(x,y) = g(x)g(y)$ (where $g(\cdot)$ is univariate) are locally supermodular and induce unique positive assortative matching.*

Proof. The cross-partial is $f_{xy} = g'(x)g'(y)$. LS2 requires $\int_{i=x}^{y}\int_{i=x}^{y} f_{12}(i,j)didj > 0$ be positive for all x, y, which is equivalent to

$$f(x,x) + f(y,y) - f(x,y) - f(y,x) > 0$$

or

$$\begin{aligned} g^2(x) + g^2(y) - 2g(x)g(y) &> 0 \\ [g(x) - g(y)]^2 &> 0. \end{aligned}$$

This is satisfied everywhere as long as $g(\cdot)$ is nowhere constant. QED.

Next, we derive a property for a more general class of functions. Because these functions are not symmetric, they does not necessarily induce unique positive assortative matching and the property is therefore not local supermodularity.

Proposition 2. *All multiplicatively separable (non-constant) differentiable functions $f(x,y) = g(x)h(y)$ (where $g(\cdot)$ is univariate) satisfy*

$$f(i,i) + f(j,j) - f(i,j) - f(j,i) > 0$$

for all i, j provided $g'(x)h'(x) > 0$ for all *x.*

Proof. The cross-partial is $f_{xy} = g'(x)h'(y)$. **LS1** requires $\int_{i=x}^{y}\int_{i=x}^{y} f_{12}(i,j)didj > 0$ be positive for all x, y, which is equivalent to

$$f(x,x) + f(y,y) - f(x,y) - f(y,x) > 0$$

or

$$\begin{aligned} g(x)h(y) + g(y)h(y) - g(x)h(y) - g(y)h(x) &> 0 \\ [g(x) - g(y)]\,[h(x) - h(y)] &> 0. \end{aligned}$$

This is satisfied as long as $g'(x)h'(x) > 0$ for all x and $f(\cdot)$ is not a constant. QED.

4.3 Common Preferences

Suppose all agents on each side have common monotonically increasing, concave, differentiable preferences $U(\cdot), V(\cdot)$ over consumption. Consumption is produced according to a general production technology $f(x,y)$ (not necessarily supermodular) with a division of consumption p to type y and and $f(x,y) - p$ to type x. Then the following result holds.

Proposition 3. *Suppose all agents on each side have common monotonically increasing, concave, differentiable preferences over consumption $U(\cdot), V(\cdot)$ and suppose the match value matrix of output* *is given by* $\mathbf{F}$ *with characteristic element* $f(x, y)$*. Then for any symmetric* $f(x, y)$*, the allocation* μ^* *satisfying positive assortative matching is the unique equilibrium if and only if* $f(x, y)$ *satisfies local supermodularity.*

Proof. The payoffs are given by:

$$
\begin{aligned}
u &= \phi(x, y, v) = U(f(x, y) - p) = U(f(x, y) - V^{-1}(v)) \\
v &= \psi(x, y, u) = V(p) = V((f(x, y) - U^{-1}(u))).
\end{aligned}
$$

Now consider any outside option to an agent x. If matching with y generates higher utility than matching with y', then:

$$
\begin{aligned}
\phi(x, y, v_y) &> \phi(x, y', v_{y'}) \\
U(f(x, y) - V^{-1}(v_y)) &> U(f(x, y') - V^{-1}(v_{y'})).
\end{aligned}
$$

Since $U(\cdot)$ is a common monotonic transformation, it follows that

$$
f(x, y) - V^{-1}(v_y) > f(x, y') - V^{-1}(v_{y'}).
$$

Likewise, because $V(\cdot)$ is a common monotonic transformation, this transformed matching problem is the assignment game, and the results from concerning local supermodularity extend immediately. QED.

Likewise, a match value function that is additively separable in (x, y) and v, i.e. $\phi_{13}(x, y, v) = 0$, can be decomposed in monotonic transformations of the assignment game, the following holds:

Corollary 1. *Let* $\phi(x, y, v)$ *be additively separable in* (x, y) *and* v*, i.e.* $\phi_{13}(x, y, v) = 0$ and $\phi_{23}(x, y, v) = 0$*. Then for any* ϕ *symmetric in* (x, y)*, the allocation* μ^* *satisfying positive assortative matching is the unique equilibrium if and only if* $\phi(x, y)$ *satisfies local supermodularity in* (x, y)*.*

5 Discussion

5.1 Same Preferences, Different Distribution

The objective in this paper is to characterize a given economy and to establish whether or not there is unique assortative matching. This is driven by applications of matching models, for example to online dating. There, the researcher observes an equilibrium allocation (i.e., who matches with whom) and therefore takes as given the candidate equilibrium allocation and the induced ranking of agents. As we specified in the outset of the model, our approach here is to write down the model in terms of the agents (some of which can be identical in observables) in the model, rather than in terms of types. Nonetheless, it is obvious that there is a one-to-one mapping between both approaches, as was illustrated above. In the latter case (that of a distribution of types), one question one could ask is whether a change in the distribution of types can render the outcome not to satisfy assortative matching, when under the initial distribution there was positive assortative matching. The answer is affirmative. We illustrate with an example.

Consider the initial example in section 2, where the value X of being matched to a partner of the same ethnicity is equal to 1. The types are the same as before, the only difference is that there are now 4 agents on each side of the market. Suppose that $\mathcal{X} = \{(3, A), (2, B), (2, B), (1, C)\}$. $\mathcal{Y} = \{(3, C), (2, B), (2, B), (1, A)\}$. Then the matrix of match values is

$$\mathbf{F}(x, y) = \begin{pmatrix} \mathbf{9} & 6 & 6 & 4 \\ 6 & \mathbf{5} & 5 & 2 \\ 6 & 5 & \mathbf{5} & 2 \\ 4 & 2 & 2 & \mathbf{1} \end{pmatrix}$$

and modulo multiplicity of identical types, the allocation is essentially unique. Suppose on the contrary now on the $\mathcal{X}$ side of the market there are two agents of type $(3, A)$ and just one of type $(2, B)$: $\mathcal{X} = \{(3, A), (3, A), (2, B), (1, C)\}$; and on the $\mathcal{Y}$ side there are two agents of type $(1, A)$ and just one of type $(2, B) : \mathcal{Y} = \{(3, C), (2, B), (1, A), (1, A)\}$. Then match values are given by

$$\mathbf{F}(x, y) = \begin{pmatrix} \mathbf{9} & 6 & 4 & 4 \\ 9 & 6 & \mathbf{4} & 4 \\ 6 & \mathbf{5} & 2 & 2 \\ 4 & 2 & 1 & \mathbf{1} \end{pmatrix}.$$

While the types remain the same, under the new distribution, positive assortative matching is no longer the equilibrium allocation. The new economy violates local supermodularity: on the candidate equilibrium allocation μ^*, it is not the case that $f[(3, A), (2, B)] + f[(2, B), (1, A)] > f[(3, A), (1, A)] + f[(2, B), (2, B)]$ since $6 + 2 < 5 + 4$.

5.2 The Distributive Lattice Property

A well-known property of two-sided matching models is that the equilibrium set is a distributive lattice. This is true for matching without transfers (Gale-Shapley (1963)) as well as for the assignment game. Out of all stable allocations, agents on one side of the market agree on the ranking of the stable matchings, and there is an $\mathcal{X}$-preferred and a $\mathcal{Y}$-preferred stable matching. In the case of the assignment game, even if there is only one allocation, there may be multiple price-vectors that support that allocation, and the agents on one side of the market all agree on the ranking of equilibrium payoffs induced by those prices.

When there are multiple equilibrium allocations, the upper envelope of $\phi(\mu, u_i)$ over all μ as defined in condition **U**, provides bounds on the payoff to player i over all equilibrium allocations. More specifically, the upper envelope is $\max_\mu \{\phi(\mu, u_i)\}$. We next show that $\phi(\mu, u_i)$ is monotonically increasing in u_i.

Lemma 4. For any $u_i > u_i'$,

$$\phi(\mu, u_i) - \phi(\mu, u_i') \geq 0.$$

With ϕ_3 and ψ_3 negative, and an even number of multiplications, this expression is positive.

Proof. Take the total derivative of ϕ with respect to u_i:

$$\frac{d\phi(\mu, u_i)}{du_i} = \underbrace{\phi_3\psi_3\phi_3\psi_3...\phi_3\psi_3}_{2n \text{ times}} > 0$$

QED.

For example, for the assignment game, $\frac{d\Phi(\mu,u_i)}{du_i} = 1$. Since for each μ, $\phi(\mu, u_i)$ is increasing, the upper envelope is as well. This is true for all players on one side of the market. And since, ϕ (and ψ) are decreasing in the other player's payoff, an increase in the equilibrium payoff on one side of the market implies a decrease on the other side of the market. As a result, in payoffs, the set of stable matchings is a lattice.

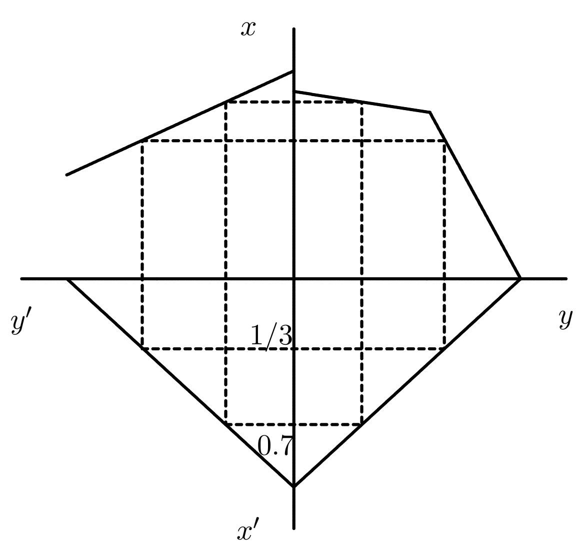
Surprisingly, that is not the case when we consider allocations. Now in the generalized model, it is possible that there are multiple stable allocations, and that agents do not agree on the ranking. Therefore, in allocations the set of stable matchings is not a lattice. We provide a simple example to illustrate this. The example is a modification of the earlier one in section 3.

Let the pairwise Pareto frontiers be identical as before, but except for the x, y pair (before, $\phi(x, y, v) = 2 - 2v$ for all v):

$$\phi(x, y, v) = \begin{cases} 2 - 2v & \text{if } v \geq 0.6 \\ 0.9 - \frac{1}{6}v & \text{if } v \leq 0.6 \end{cases}$$

as illustrated in Figure 3.

Figure 3

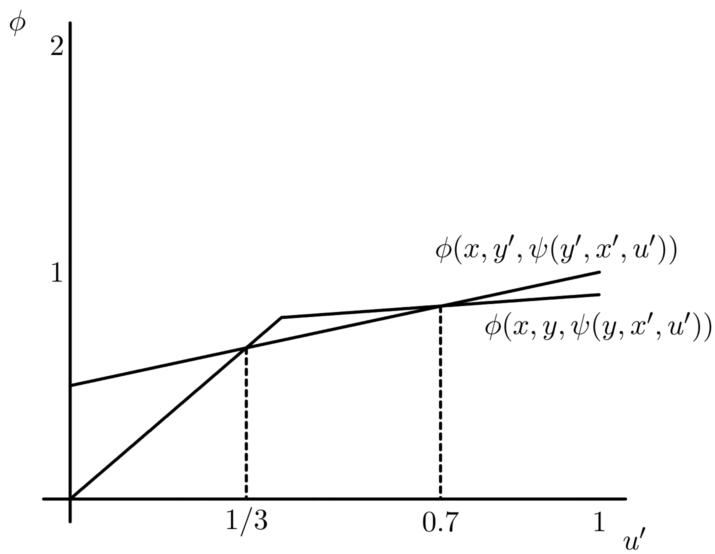


Then for $v \leq 0.6$ (and correspondingly, for $\phi \geq 0.8$), $\phi(x, y', \psi(y', x', u)) = 0.733 + \frac{1}{6}u$. When deriving condition $\mathbf{U}$, there is now one additional intersection between $\phi(x, y', \psi(y', x', u))$ and $\phi(x, y', \psi(y', x', u))$ at $u = 0.7$. There payoff space is not partitioned into 3 regions, each corresponding to a different equilibrium:

$$\left\{\mu\left(x, x'\right) = \left(y', y\right), u' \in [0, 1/3], u \in [0, 2/3]\right\}$$
$$\left\{\mu\left(x, x'\right) = \left(y, y'\right), u' \in [1/3, 0.7], u \in [2/3, 0.85]\right\}$$
$$\left\{\mu\left(x, x'\right) = \left(y', y\right), u' \in [0.7, 1], u \in [0.85, 0.9]\right\}.$$

The payoff function ϕ with corresponding outside options are plotted in Figure 4.

Figure 4



Observe that even though in payoffs, the set of equilibria is ranked by the agents on one side of the market, but not the allocations. Depending on what the corresponding payoffs are, $\mathcal{X}$ agents will prefer $\mu(x, x') = (y', y)$ over $\mu(x, x') = (y, y')$ or vice versa.

5.3 Beckerian Assortative Matching

Becker's (1973) view on Assortative Matching is that for multidimensional characteristics, one observes an equilibrium allocation, and then one derives from the allocation whether each of the individual characteristics is positively assorted. In other words, since the order on $\mathcal{X}$ and $\mathcal{Y}$ is arbitrary, the order must be determined by the equilibrium outcome. Then whether there is positive/negative assortative matching is determined for each characteristic individually. This implicitly presupposes that both characteristics are present on both sides of the market. Let x and y be vectors $(x^{\alpha}, x^{\beta}, ...)$ and $(y^{\alpha}, y^{\beta}, ...)$. Then the Beckarian definition of PAM/NAM is as follows:

Definition PAM. *Consider ordered sets $\mathcal{X}$ and $\mathcal{Y}$, such that $f(x_i^\alpha; x^{-\alpha}, y) > f(x_j^\alpha; x^{-\alpha}, y)$ and $f(y_i^\alpha; x, y^{-\alpha}) > f(y_j^\alpha; x, y^{-\alpha})$ for all $x_i^a > x_j^a$ and $y_i^a > y_j^a$. The there is PAM in trait α if there exists an equilibrium allocation such that $y_i^a = \mu(x_i^a)$. Analogously for NAM where $y_i^a = \mu(x_{n+1-i}^a)$.*

If x and y are scalars, then PAM/NAM is completely determinate and pinned down by the first derivative of f with respect to x and y. If x and y are vectors, then it may well be the case that there is PAM according to one variable and NAM according to another. Consider for example the marriage example with preferences for wealth/education and an i.i.d. characteristic (nearness, ethnicity,...). Then there is PAM in the x^α (education or wealth), yet NAM in x^β (ethnicity, nearness,...). In the example

$$\mathbf{F} = \begin{pmatrix} 9 & 6 & 3+X \\ 6 & 4+X & 2 \\ 3+X & 2 & 1 \end{pmatrix}.$$

If X is small ($X < 2$), then there is PAM with respect to x^α, y^α and NAM with respect to x^β, y^β. In contrast, if X is large ($X > 2$), then the unique allocation is. For example, with $X = 3$

$$\mathbf{F} = \begin{pmatrix} 9 & 6 & \mathbf{6} \\ 6 & \mathbf{7} & 2 \\ \mathbf{6} & 2 & 1 \end{pmatrix}.$$

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