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Competing Teams

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In many economic applications of matching, the teams that form compete later in market structures with strategic interactions or with knowledge spillovers. Such post-match competition introduces externalities at the matching stage: a team's payoff depends not only on their members' attributes but also on those of other matched teams. This article develops a large market model of matching with externalities, in which first teams form, and then they compete. We analyse the sorting patterns that ensue under competitive equilibrium as well as their efficiency properties. Our main results show that insights substantially differ from those of the standard model without externalities: there can be multiple competitive equilibria with different sorting patterns; both optimal and competitive equilibrium matching can involve randomization; and competitive equilibrium can be inefficient with a matching that can drastically deviate from the optimal one. We also shed light on the economic relevance of our matching model with externalities. We analyse two economic applications that illustrate how our model can rationalize the trend in within- and between-firm inequality, and also the evolution of markups of sectors where firms have market power.

Key words: Matching with externalities, Sorting, Strategic interaction, Knowledge spillovers, Wage inequality, Market power.

JEL Codes:

1. INTRODUCTION

The success of a firm, a team, or a partnership usually depends on its competitors' decisions. For instance, in a patent race between pharmaceutical companies, where the winner takes all the benefit, a competing team affects the other teams' performance negatively. If the competitor discovers the blockbuster drug, the rest gets nothing. Competing teams may also affect the outcome positively whenever the performance of the competitor generates knowledge spillovers that boost the own firm's performance. The success in the discovery of the structure of DNA for which James Watson and Francis Crick at Cambridge gained credit would not have been possible without the spillovers from the team led by Rosalind Franklin and Maurice Wilkins at Kings College.[1]

1. Watson and Crick posited the model of DNA as a double helix, fruit of their brilliant intuition and the meeting of different but complementary minds. Yet, Watson was motivated by a talk by Wilkins on the molecular structure of DNA

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It has long been recognized that externalities between firms have important implications for the provision of effort and therefore for the efficient allocation of resources.[2] In this article, we build further on the insights from this literature, but focus instead on the effects of these externalities on *team composition*, an equally crucial determinant of performance. If a pharmaceutical firm gets the best scientists, it is more likely to make new discoveries and hence obtain a patent. The fact that firms spend so much time and resources carefully choosing their skilled workers—often poaching them away from competitors—is direct evidence that team composition is an important strategic tool in the competition with other firms. Consulting firms, banks, and law firms try to hire the best young talent; university departments constantly attempt to attract the most productive academics; and research divisions in technology companies lure the best engineers.

This article sheds light on robust qualitative features of the effects of externalities on matching. We develop a tractable model with a large number of heterogeneous agents, just like the standard matching model (Becker, 1973), but now in the presence of externalities. In a competitive labour market, a continuum of firms hire heterogeneously skilled workers to form teams. Once the teams are formed, those firms then compete in an output market with externalities. Since the performance of a team or firm in this downstream competition depends on the composition of the other teams, this interdependence feeds back into the labour market matching problem. Indeed, the hiring decision now takes into account not only the complementarities between the workers hired, but also the feedback effects coming from the composition of other teams that form. We allow for different ways to structure competition among firms after hiring their teams. Indeed, we consider economy-wide externalities, where each firm exerts an external effect on all other firms—as in models with knowledge spillovers. And we also consider pairwise externalities where firms compete with one other firm only, and where the pairs of teams that compete against each other are set either at random—as in settings where the competitor is initially unknown—or deterministically in an *ex ante* way—as in oligopolistic markets where each firm knows the identity of its competitors. This variety of settings encompasses many potential economic applications of our framework.

We show that externalities have profound implications and derive insights that have no counterpart in the standard model. First, we show that despite the presence of complementarities both optimal and equilibrium matching may be stochastic—for example, a fraction of the population matches in a positively assortative way (PAM) and the rest negatively assortatively (NAM). Second, we show that there can be multiple equilibria with different sorting patterns. Third, we provide interpretable conditions under which the optimal matching differs, sometimes in a drastic way, from the equilibrium one. Intuitively, complementarities are no longer the sole determinant of sorting, as they interact with the externalities to pin down who matches with whom.

Progress in characterizing the equilibrium allocation has been hindered by problems of existence of competitive equilibrium as well as by the combinatorial complexity of the matching problem that ensues.[3] In our tractable setup, we sidestep some of these problems and provide

in 1951, and together with Crick, before coming up with their model, they had access to Franklin's X-ray photographs that documented the helical structure.

2. Most notably, the literature on tournaments, contests, and patent races has extensively focused on important aspects such as long-term, repeated interaction (Che and Yoo, 2001) and the optimal provision of effort (Che and Gale, 2003).

3. See the seminal paper by Koopmans and Beckmann (1957). Their "quadratic assignment problem", which need not have a competitive equilibrium, has generated a huge literature in Operations Research and Combinatorial Optimization. Despite its apparent simplicity, it is considered to be one of the most difficult NP-Hard problems, in the sense that unless one proves that P = NP, one cannot even obtain f-approximation algorithms for any constant f (see the recent survey Loiola *et al.* (2007)). This speaks to the overall difficulty of the topic of matching with externalities.

conditions on the match output function under which we can construct a competitive equilibrium and show that there could be multiple ones including cases entailing stochastic matching. All this in a relatively elementary fashion using standard tools.

These insights are robust—they obtain under both binary as well as a continuum of characteristics, and under relatively weak assumptions—and cannot arise without externalities. More importantly, they have implications for the optimal labour force composition in firms. One striking feature is that when a competitive equilibrium is inefficient, the optimal matching and thus team composition can look quite different from the equilibrium matching. For example, the outlook of the market is very different if the planner's matching is NAM (diversity within teams, homogeneity between teams) while the competitive equilibrium matching is PAM (homogeneity within teams, diversity between teams). These different outcomes can arise even for small changes in the technology, for example, when the differential externality under PAM and NAM becomes slightly stronger.[4]

To assess the economic relevance of our model, we tackle two economic applications that are of interest in Macro/Labor Economics and Industrial Organization. We claim that our insights help us understand important economic phenomena, and we derive several analytical properties that are in line with recent empirical evidence that has received attention in the literature. In our main application to the macroeconomics of knowledge spillovers, we highlight a mechanism that can account for the evolution of wage inequality. Recent evidence establishes that nearly all of the increase in inequality is driven by between-firm inequality and not by within-firm inequality.[5] At the same time, a different literature shows that knowledge spillovers are an important determinant of the firm size distribution.[6] We link the wage inequality literature with the firm inequality literature by adding a matching stage where workers sort into firms. In equilibrium, knowledge spillovers drive between-firm inequality and worker complementarities drive within-firm inequality.

We find that, in the unique competitive equilibrium with stochastic matching, increased complementarity between workers leads to more positive sorting within firms. This implies that the composition of workers within firms is more alike while the composition between firms looks more distinct. We can then provide conditions under which between-firm wage inequality can increase significantly while within-firm wage inequality barely changes. Moreover, due to the externalities, equilibrium is inefficient. The planner favours an allocation with less positive sorting and hence a more mixed composition of workers within firms, which results in lower wage inequality. Note that this smooth evolution of wage inequality cannot arise in a standard matching model since it requires a mixture between PAM and NAM allocations. Indeed, this is determined by the presence of externalities.

The second application explores the relationship between market power and the composition of skills in the firm. Recent evidence shows a sharp rise in market power in U.S. firms. We apply the model to an oligopolistic output market setting where competing firms hire their workforce on the economy-wide labour market. We show that the degree of complementarity between workers affects the equilibrium distribution of markups in the output market. More complementarity leads to more dispersion of markups as well as higher markups. This establishes that changes

4. The discontinuity of the equilibrium allocation in the properties of the technology is of course well-known from the assignment game without externalities (Becker, 1973): as the cross-partial derivative of the match surplus switches from positive to negative, the allocation discontinuously jumps from PAM to NAM. The novelty here is that it is driven by the presence of externalities.

5. See Card *et al.* (2013) for Germany, Song *et al.* (2015) and Barth *et al.* (2014) for the US, Benguria (2015) for Brazil, and Vlachos *et al.* (2015) for Sweden.

6. See Lucas and Moll (2014), Perla and Tonetti (2014), König *et al.* (2016), and Eeckhout and Jovanovic (2002), amongst others.

in technology affect the composition of skills across firms and, equally importantly, that labour market composition affects the extent of the inefficiency due to market power.

As mentioned, this article contributes to the analysis of matching with externalities, an important topic that has received scant attention in the matching literature, despite the pervasiveness of externalities in economic applications. Its importance was recognized in the seminal matching paper by Koopmans and Beckmann (1957), who analyse a variation of their matching problem between locations and plants in the presence of transportation costs between locations, which generate externalities in the optimal assignment. They show that in their model a competitive equilibrium does not exist, and left the problem open for future research. Sasaki and Toda (1996) provide a suitable concept of stability in matching with externalities, and analyse its implications for the marriage model and assignment games. A recent paper by Pycia and Yenmez (2017) generalizes the analysis of stable matchings to many-to-many and many-to-one matching problems, and show several properties of core allocations, including some comparative statics.[7] What distinguishes our article from the rest of the literature is our focus on large markets and equilibrium and optimal sorting patterns. We study both the optimal matching problem from a planner's perspective and a decentralized version using a standard notion of market equilibrium with externalities. Our parsimonious model affords a fairly complete solution to the problem in most instances and an explicit comparison between the equilibrium allocation and the planner's solution. Moreover, we shed light on the intuition underlying the inefficiencies that we derive, and flesh out in detail some economic applications.

The rest of the article proceeds as follows. In the next section, we start with a simple example. Section 3 describes the model. Section 4 contains the main results regarding sorting patterns and the inefficiency that the externalities can generate. In Section 5, we develop two economic applications, one that focuses on the role of sorting in the presence of economy-wide knowledge spillovers, and the other on market power. Section 6 concludes. The Appendix contains all the proofs as well as some additional extensions omitted from the text.

2. AN EXAMPLE

To illustrate some of the main results of the article, consider the following simple two-stage matching problem.[8] There is a unit measure of agents, half of them with a high productive attribute H and half with a low one L. In the first stage, agents match pairwise, and thus form teams. They have utility linear in money and are free to make transfers among each other. In the second stage, the formed pairs randomly match with each other and "compete". The payoff structure from this (reduced-form) competition is as follows: if two teams with the same composition compete, then each obtains a payoff of 1, while if they have different composition each obtains 0. Finally, if an agent is unmatched, then his payoff is normalized to zero. We will examine the competitive equilibria of the first stage and the sorting patterns that can emerge.

We first show that there is a PAM equilibrium (only HH and LL teams form) with supporting wages $1/4$ for both H and L. To see this, note that if an H conjectures PAM, then in the second stage his team will match with equal probability with a team HH and LL. So if he matches with (hires) another H in the first stage he obtains $(1/2)\times 1+(1/2)\times 0-1/4=1/4$, while if he matches with an L, then she obtain $0-1/4=-1/4$. Hence, each agent H strictly prefers to form a team with another H. Similarly, each L strictly prefers to match with another L. Thus, PAM

7. Other papers in this literature, which study matching with externalities focusing on markets with a small number of agents, are Bando (2012), Hafalir (2008), Fisher and Hafalir (2016), and Chen (2016).

8. We are grateful to a referee for suggesting the basic example with random matching of this section.

along with wages equal to $1/4$ is a competitive equilibrium. The aggregate output is $(1/2)\times(1/2)$ (one-half measure of teams, each with expected output $1/2$).

There is also a NAM equilibrium (only *HL* teams form) with supporting wages $1/2$ for both H and L. To see this, note that both types of agents prefer to hire an agent of the opposite characteristic if they conjecture NAM. This is because they match with probability one with a mixed team *HL*, so they obtain 1 if they also form a mixed team, and 0 otherwise. The aggregate output of this equilibrium is $(1/2)\times 1$.

Finally, there is an equilibrium where agents randomize their choice of partners and wages are given by $1/3$ for each type of agent. Suppose that agents conjecture that matching is PAM with probability $\alpha \in (0,1)$ and NAM with $1-\alpha$ (equivalently, a fraction α of each type matches in a PAM way and $1-\alpha$ in a NAM way). Consider an agent H: if he matches with another H, then the team's expected output is $\alpha/2$ (with probability α matching is PAM and the formed *HH* team is matched with another *HH* team with probability $1/2$, while in any other event the team's output is 0); if he matches with an L, the resulting team's expected output is $(1-\alpha)\times 1$. A similar analysis holds for L. From the incentive constraints of H and L, a necessary condition for an equilibrium where matching is stochastic and given by α is that each type of agent is indifferent between hiring an H or an L. For each type the incentive constraint is $(\alpha/2)-(1/3)=(1-\alpha)-(1/3)$, and so $\alpha=2/3$. Since each type obtains a positive payoff, we have constructed a competitive equilibrium with stochastic matching. Aggregate expected output is $(1/2)\times(1/3)$ (one half measure of teams, each with expected output $1/3$).

The planner in this setting—assuming she can intervene in the first stage but not in the second—can choose any way to match the agents pairwise, which can be summarized as the choice of the fraction $\alpha^P \in [0,1]$ of agents that she matches as PAM, and the rest as NAM. Thus, her problem is $\max_{\alpha^P\in[0,1]}(1/2)\times((\alpha^P/2)^2+(\alpha^P/2)^2+(1-\alpha^P)^2)$. This is simply the measure of teams $1/2$ multiplied by the sum of expected payoffs of the teams (*e.g.* there are $\alpha^P/2$ teams *HH* and each obtains, under α^P, expected output $\alpha^P/2$, and similarly with the other terms). It is easy to check that the maximum is achieved at $\alpha^P=0$, so NAM is efficient.

If instead we assume that a team generates an expected output of 1 if matched with a team of a different composition, and 0 otherwise, then following the same steps as above one can show that there is a unique equilibrium matching that is efficient and is stochastic with $\alpha=2/3$.

So far we have assumed that competing teams are randomly assigned pairwise. Another alternative would be to add an exogenous initial stage in which half of the agents of each type are assigned pairwise, with each pair being future competitors. Then each agent in the competing teams matches with a partner, thus forming the two teams that will face off downstream.

To verify that similar results as in the random assignment case obtain, assume that before the interaction starts, half of the population is assigned pairwise in a PAM way. That is, half of the H's are matched together and similarly for half of the L's. (Assuming NAM instead yields similar results.) Each of these pairs are competitors. Consider the matching stage, where each of these competitors hires a partner and then they compete. If two competing teams have the same composition, then each obtains 1, and 0 otherwise.

Proceeding as before, we can show that there is a competitive equilibrium with PAM and also one with NAM, each supported by wages $1/2$ for both H and L. To see this, consider an agent with H who conjectures PAM, and so he assumes that his team will compete against an *HH* team (recall that he was initially assigned to a competing H). By hiring another H he obtains $1-(1/2)=1/2$, while hiring an L yields $0-(1/2)=-1/2$. The same applies to an agent with L. Hence, there is an equilibrium with PAM and aggregate output equal to $1/2$, and a similar logic yields one with NAM and aggregate output equal to $1/2$. Finally, there is a stochastic equilibrium matching with $\alpha=1/2$ and wages given by $1/4$, with aggregate output of $1/4$. To see this, note that an agent with H obtains $\alpha-(1/4)$ when hiring another H (since with probability α the initial member H of the competing team hires another H), and $1-\alpha-(1/4)$ when hiring an agent with L, and similarly for the incentive constraint an agent with attribute L. Hence, indifference yields $\alpha=1-\alpha$, and thus $\alpha=1/2$ is the equilibrium stochastic matching. Regarding the planner, she solves $\max_{\alpha^P}(1/2)\times((\alpha^P)^2+(1-\alpha^P)^2)$. To see this, note that under matching α^P there are $\alpha^P/4$ teams HH, $\alpha^P/4$ teams LL, and $(1-\alpha^P)/2$ teams HL. Now, a team HH, which already has an H in its competing team, competes with HH with probability α^P and with HL with probability $1-\alpha^P$: thus, its expected payoff is α^P, and similarly for a team LL. Finally, a team HL or LH competes with an identical team with probability $1-\alpha^P$ and with a different one with α^P, so its expected payoff is $1-\alpha^P$. Thus, the planner's objective function is $(\alpha^P/4)\times\alpha^P+(\alpha^P/4)\times\alpha^P+((1-\alpha^P)/2)\times(1-\alpha^P)=(1/2)\times((\alpha^P)^2+(1-\alpha^P)^2)$, and the optimal matchings are $\alpha^P=1$ and $\alpha^P=0$. Hence, the competitive equilibria with PAM and NAM are both efficient while the stochastic matching competitive equilibrium is inefficient.

These examples reveal that competition in the second stage turns the matching problem in the first stage into one with externalities: the composition of teams in the market affects the payoff of any given team, as each team competes against another one in the second stage. As a result, the first stage can have *multiple* equilibria with drastically different sorting patterns, including one in which matching is *stochastic*.[9] Moreover, equilibrium can be *inefficient*. None of these results obtain in the standard case (as in Becker (1973)) without externalities.

3. THE MODEL

3.1. *Overview*

We consider an economy with a large number of heterogeneous agents who match pairwise. For instance, this could be a labour market where skilled workers form teams or partnerships. Equivalently, one could envision a large number of identical firms that hire pairs of heterogeneous workers and make zero profits. Absent externalities, if agents can perfectly transfer utility, then this would be a standard matching problem (*e.g.* as in Becker (1973)).

Implicit in our model, however, is another stage after matching in which the formed teams compete. Continuing with the labour market example, firms, after they hire their workers, compete in an output market.

Competition among teams can take a different form depending on the economic application under consideration. Indeed, each team could compete with exactly one other team, whose identity could be known before the second stage, or it could be drawn at random from the pool of teams. Alternatively, competition might take place among all of them. Our model will encompass all these alternatives in a reduced form by assuming that the payoff function of each team depends not only on its composition but also on the composition of other teams.

The crucial feature that our model captures is that competition in the second stage feeds back into the formation of teams in the first stage. This turns the team formation problem into a matching problem with externalities, which creates a wedge between equilibrium and optimal matchings.

9. In this example, both PAM and NAM can emerge in equilibrium because the externality changes the team's payoff function from supermodular to submodular. This effect cannot arise in the standard model, where a pair's payoff is determined only by the members' characteristics and the production complementarities.

3.2. *The general framework*

There is a unit-measure continuum of agents. Each agent is indexed by a characteristic $x \in [0,1]$, whose distribution in the population is given by a cdf $F:\mathbb{R}\to[0,1]$. The cdf F has either a finite support (discrete case), in which case it is an increasing step function, or it has support $[0,1]$ (continuous case), in which case we assume that it is strictly increasing and continuous on $[0,1]$. Following the standard assumption in matching models that focus on sorting (*e.g.* see Chade *et al.* (2017)), agents match pairwise and thus form teams of size two. A (deterministic) matching is thus a (measurable) one-to-one function $\mu:[0,1]\to[0,1]$ that is measure-preserving (matches measurable sets of $[0,1]$ of equal F-measure).[10] The most important instances for our purposes are the matching μ that is increasing (PAM), denoted by μ_+, and the decreasing one (NAM), denoted by μ_-. In the continuous case μ_+ and μ_- are given by $\mu_+(x)=x$ and $\mu_-(x)=F^{-1}(1-F(x))$ for all x.

Let $\mathbb{M}$ be the set of matchings μ. Match payoff or output is given by a function $\mathcal{V}:[0,1]^2\times\mathbb{M}\to\mathbb{R}_+$ such that, if an agent with characteristic x matches with one with characteristic x' and matching is given by μ, then the match payoff of team (x,x') is $\mathcal{V}(x,x'|\mu)$. The function $\mathcal{V}(\cdot,\cdot|\mu)$ is twice continuously differentiable for each $\mu\in\mathbb{M}$. Agents value match output and their preferences are quasilinear in money, so utility is perfectly transferable among agents.[11] For simplicity, we assume that unmatched agents produce zero and we normalize their payoff to zero as well. The dependence of $\mathcal{V}$ on μ captures the effects of a second stage where the formed teams compete. The precise functional form of $\mathcal{V}$ will vary across applications and will depend on the precise nature of competition in the second stage, which we will describe in more detail in the next subsection.

We focus on the competitive equilibria of this matching problem with externalities. Our definition of competitive equilibrium is fairly standard (see *e.g.* Mas-Colell *et al.* (1995), or Chapter 6 in Arrow and Hahn (1971)). When choosing the composition of teams, agents take as given both market wages as well as the matching. This implies that each firm behaves as if its own choice does not affect the candidate equilibrium allocation, a conjecture that is consistent with our large economy environment. More precisely, a competitive equilibrium of the matching problem consists of a wage function $w:[0,1]\to\mathbb{R}$ and a matching function μ such that, for all x, $\mu(x)\in\text{argmax}_{x'}\mathcal{V}(x,x'|\mu)-w(x')$ (*i.e.* each agent with attribute x finds it optimal to match with a partner $\mu(x)$ given w), agents obtain positive payoffs, and the market clears.

The planner's objective is to find the matching that maximizes the aggregate match output, given that utility is transferable. Since the planner prefers to match everyone, her problem is to find a $\mu\in\mathbb{M}$ that maximizes $\int_0^1\mathcal{V}(x,\mu(x)|\mu)dF(x)$. Denote any maximizer by μ^P. When $\mathcal{V}$ does not depend on μ, so that $\mathcal{V}(x,x'|\mu)\equiv\hat{\mathcal{V}}(x,x')$ for all (x,x'), this problem has a well-known solution in the following cases: if $\hat{\mathcal{V}}$ supermodular in (x,x') then the optimal matching is PAM, and if it is submodular then the optimal matching is NAM.

If we allow matching to be stochastic, then a matching is now a measure π on (the Borel σ-field of) $[0,1]^2$ such that its marginals coincide with F, that is, $\pi(E\times[0,1])=\pi([0,1]\times E)=\int_E dF$ for each Borel set $E\subset[0,1]$. Denote by $\mathcal{M}$ the set of such measures, and with some abuse of notation, let $\mathcal{V}(x,x'|\cdot):\mathcal{M}\to\mathbb{R}_+$ for each $(x,x')\in[0,1]^2$. A competitive equilibrium consists of a w and a π such that each agent chooses a partner optimally, obtains a positive payoff, and market clears. Each agent with characteristic x must be indifferent among all the x' in the support of partners with whom x can match under π, and must prefer to match than to remain unmatched.

10. We endow $[0,1]$ with its Borel σ-field and measurable in this article should be understood as Borel measurable.

11. We leave for future research the more complex analysis when utility is imperfectly transferable, which would call for an extension of the general model in Legros and Newman (2007) to the case with externalities.

That is, $\mathcal{V}(x,x'|\pi)-w(x')\geq 0$ and is constant for all such x'. In turn, the planner's problem is to choose $\pi\in\mathcal{M}$ that maximizes $\int_{[0,1]^2}\mathcal{V}(x,x'|\pi)d\pi(x,x')$, and any maximizer is denoted by π^P.

3.3. *Competing teams' assignment*

To avoid confusion, we use the term "assignment" to denote how teams are matched in the second stage, and reserve the term "matching" to denote how team members match in the first stage. We will focus on cases where competition in the second stage takes places in one of the following forms:

(1) *Pairwise competing teams with local spillovers:* The interaction in the second stage takes place between pairs of teams. We will explore the following instances of this case:

(1.*i*) *Ex post random assignment of teams:* After teams are formed, they are randomly assigned pairwise. Since ex ante all teams are potential competitors, the composition of all teams are payoff relevant, but ex post each team will compete with only one team. Let $V:[0,1]^4\to\mathbb{R}_+$ be given by $V(x,x'|s,s')$, which is the output of a team with composition (x,x') if it competes against a team with (s,s'). We will assume that the (measurable) function V is symmetric in its first and second argument, and also in its third and fourth argument.[12] For any μ, let $G(\mu)=\{(s,s')|s'=\mu(s),s\in[0,1]\}$ be its graph. When the third and fourth coordinates of V are restricted to be in the set $G(\mu)$ for a given μ, then $V(x',x|s,\mu(s))$ is the output of (x,x') competing with a team with composition $(s,\mu(s))$. Then for all $(x,x')\in[0,1]^2$, under ex post random assignment we have

$$\mathcal{V}(x,x'|\mu)\equiv\int_0^1 V(x,x'|s,\mu(s))dF(s). \tag{3.1}$$

The assumptions on V imply that $\mathcal{V}(\cdot,\cdot|\mu)$ is symmetric in (x,x'). An intuitive interpretation of (3.1) is that each team competes against teams of a given composition a fraction of time, represented by their presence in the overall population. An example is sports competition, where each team plays every other team, and hence the sum of the outcomes of all competitors is equal to the expected value of competing with a random team multiplied by a constant (the measure of teams). For another example, consider firms that, after hiring their teams of skilled workers, they bid for contracts without knowing *ex ante* the identity of their competitor.

(1.*ii*) *Ex ante deterministic assignment of teams:* Teams compete pairwise in the second stage and each team knows in advance its opponent. We assume that there is a function V as in (1.*i*), that is symmetric in its first and second coordinates, and also in the third and fourth. Now, if (x,x') knows that, given a matching μ, it will compete against a team with composition $(s,\mu(s))$, then the only relevant part of $G(\mu)$ for each team is the point (pair) that represents the competitor's composition, and thus $V(x,x'|s,\mu(s))$ will be the output of (x,x') in this case. In addition, we will assume that there is an exogenous initial assignment $\eta\in\mathbb{M}$ that takes place before the first stage, in which half of the population with composition given by F is assigned pairwise. For example, if F has a density f, then, for each x, $f(x)$ is divided by two, and all agents in one of the halves match pairwise.[13] Each pair consists of competitors that will interact in the second stage, and

12. Symmetry is standard in one-sided matching problems without externalities and rules out task-specific productivity, *e.g.*, where the same worker is more productive if assigned to task 1 (say, manager) than to task 2 (say, mechanic). See Kremer and Maskin (1996) for the analysis of matching with asymmetric match output and task assignment.

13. We could also model *ex ante* random assignment of teams, where all the agents are randomly matched with a competitor. The problem now is that two agents with the same characteristics can be matched with competitors with different attributes, which precludes the equal treatment of these agents.

each member of this pair matches in the first stage with a partner from the remaining one-half measure of agents. The resulting teams then compete in the second stage. In the example above, $s=\eta(x)$ and thus agents with x and s are competitors; under the matching μ, x conjectures it will compete with $(s,\mu(s))$ and chooses a partner x'. Then for all $(x,x')\in[0,1]^2$

$$\mathcal{V}(x,x'|\mu)\equiv V(x,x'|\eta(x),\mu(\eta(x))), \tag{3.2}$$

where with some abuse of notation we have omitted η from $\mathcal{V}$. Note that in this case we cannot assert that $\mathcal{V}(\cdot,\cdot|\mu)$ is symmetric in (x,x') since in the right side $V(x,x'|\eta(x),\mu(\eta(x)))$ need not be equal to $V(x',x|\eta(x'),\mu(\eta(x')))$. That is, an asymmetry ensues between those agents who are initially assigned and those each of them hires.

A natural interpretation of this type of exogenous deterministic assignment is that there is a large number of local markets, each with two firms that hire workers and then compete downstream. This captures, for instance, Industrial Organization applications in which firms hire workers in competitive markets and then compete in oligopolistic product markets (*e.g.* Coca Cola and Pepsi, or Visa and MasterCard). Whenever we deal with this case, we will assume that the assignment η is PAM, so $\eta(x)=x$, since the analysis for NAM is similar.

(2) *Competing teams with aggregate spillovers:* All the teams compete against each other in the second stage, and competition entails spillover effects that enter the payoff of each team as a common aggregate externality. Let $\xi:\mathbb{M}\to\mathbb{R}$ and let $S:[0,1]^2\times\mathbb{R}\to\mathbb{R}_+$. Then for all $(x,x')\in[0,1]^2$

$$\mathcal{V}(x,x'|\mu)\equiv S(x,x',\xi(\mu)). \tag{3.3}$$

We will assume in this case that the function $S(\cdot,\cdot,\xi(\mu))$ is symmetric in (x,x'), and thus so is $\mathcal{V}(\cdot,\cdot|\mu)$. The functional form of the aggregate externality component ξ depends on the application at hand. An intuitive one is where the composition of the labour force of all firms determines the production of knowledge within firms, which generates spillover effects (positive externalities) on all the firms.

In all these cases, we have described $\mathcal{V}$ under the assumption that matching is deterministic and given by μ, but it is straightforward to modify it if instead matching is stochastic, and we will do so below on an as-needed basis. Also, note that the assignment of competing teams is exogenously given and cannot be altered either by the firms or the planner. This restriction is what generates externalities that cannot be completely internalized, leading to inefficiencies in the equilibrium composition of teams. As in any model with externalities, the problem trivializes if there are no "missing markets".[14] Intuitively, if teams could *choose* their competitor and set up the appropriate transfers among them, then the externality problem could be eliminated as the allocation of competing teams in the second stage would be efficient. It is hardly plausible that in a large market setup firms will internalize the externalities in this way, especially in the case of aggregate spillovers, where firms compete among all of them and not pairwise. Moreover, there could also be a technological (such as differentiated products), legal (such as patent legislation), or geographical constraints that restrict firms to compete only in a specific sector or location, as it would be too costly for them to, for example, switch from consumer marketing and retail to cement production. For completeness, we discuss this issue in Appendix A.5, and illustrate how *endogenous assignment* of competing teams plus transfers between teams can restore equilibrium efficiency.

14. Indeed, in line with standard Coasian arguments, if teams can set up contracts among them, then they will find a way to achieve efficiency. In all of our applications this does not seem likely to happen.

4. MAIN RESULTS

4.1. *Binary characteristics*

We start with a benchmark case in which agents' characteristics are binary: $x \in \{\underline{x}, \overline{x}\}$, where $0 \leq \underline{x} < \overline{x} \leq 1$ and where exactly half of the agents are of type $\overline{x}$ and half are of type $\underline{x}$.[15] This case is rich enough to present, in an elementary and intuitive fashion, the main insights that emerge in our matching setting with externalities.

There are three possible team configurations: a team with two agents with $\overline{x}$, or a team with two $\underline{x}$ members, or a mixed team with one member of each characteristic. In cases (1.*i*) and (2) above, mixed teams $(\underline{x}, \overline{x})$ and $(\overline{x}, \underline{x})$ are treated symmetrically. In the case of deterministic assignment of competing teams (case 1.*ii*) above), however, we will need to distinguish between $(\underline{x}, \overline{x})$ and $(\overline{x}, \underline{x})$. For example, when the η assignment is PAM, a team $(\underline{x}, \overline{x})$ is interpreted as follows: η initially assigned member $\underline{x}$ to a competitor $\underline{x}$, and then he hired $\overline{x}$. Similarly for $(\overline{x}, \underline{x})$, where $\overline{x}$ is assigned by η to an agent with $\overline{x}$, and then hired a partner with $\underline{x}$.

We will proceed in a general way and allow for a stochastic matching π, which in this setting is characterized by a number $0 \leq \alpha \leq 1$. This number represents the fraction of the population that matches à la PAM, with $1-\alpha$ matching in a NAM way. Clearly, the corner $\alpha = 1$ represents μ_+, and $\alpha = 0$ represents μ_-. In this way, α spans all the possible matchings in this economy. Also for notational economy, in this section we will set $\mathcal{V}(x,x'|\alpha) \equiv \mathcal{V}(x,x'|\pi)$, $\mathcal{V}(x,x'|1) \equiv \mathcal{V}(x,x'|\mu_+)$, and $\mathcal{V}(x,x'|0) \equiv \mathcal{V}(x,x'|\mu_-)$.

Competitive equilibria. The binary case permits a complete description of the set of competitive equilibria and their sorting properties. To this end, we introduce the function $\Gamma : [0,1] \to \mathbb{R}$ given by

$$\Gamma(\alpha) = \mathcal{V}(\overline{x}, \overline{x}|\alpha) + \mathcal{V}(\underline{x}, \underline{x}|\alpha) - \mathcal{V}(\underline{x}, \overline{x}|\alpha) - \mathcal{V}(\overline{x}, \underline{x}|\alpha), \tag{4.4}$$

which represents the gain/loss from rematching two teams as PAM instead of as NAM if the matching is α. Although $\mathcal{V}(x,x'|\cdot)$ can be a nonlinear function of α, it turns out that it is linear in several cases of interest. Indeed, in the case of pairwise competing teams with random assignment of teams (case 1.*i*) in the previous section), or with *ex ante* deterministic assignment (case 1.*ii*), or when there are aggregate spillovers (case 2) that are multiplicative and linear in α, this function can be written as follows:

$$\mathcal{V}(x,x'|\alpha) = \alpha \mathcal{V}(x,x'|1) + (1-\alpha)\mathcal{V}(x,x'|0). \tag{4.5}$$

That is, $\mathcal{V}(x,x'|\alpha)$ is the expected match output for a team with composition (x,x') when it is assigned to a competing team in a PAM way with probability α (or aggregate spillovers are as in PAM, which occurs with probability α) and obtains $\mathcal{V}(x,x'|1)$, or in a NAM way with probability $1-\alpha$ and obtains $\mathcal{V}(x,x'|0)$.[16]

15. The equal number of agents with high and low characteristics is made for convenience. Otherwise in the case of negative sorting, one needs to keep track of the measure of agents with the characteristic present in more than half of the population who match among themselves once cross matches are exhausted. Since this extension does not lead to new insights, and since it will be relaxed in the case with a continuum of characteristics below, we focus here on the uniform case.

16. With *ex post* random assignment of competing teams, under α the $1/2$ measure of teams consists of $\alpha/4$ teams with two members with $\overline{x}$, $\alpha/4$ teams with two members with $\underline{x}$, and $(1-\alpha)/2$ mixed teams. Hence, the expected payoff of a team with composition (x,x') is, using equation (3.1), $\mathcal{V}(x,x'|\alpha) = (\alpha/2)V(x,x'|\underline{x},\underline{x}) + (\alpha/2)V(x,x'|\overline{x},\overline{x}) + (1-\alpha)V(x,x'|\underline{x},\overline{x}) = \alpha((V(x,x'|\underline{x},\underline{x}) + V(x,x'|\overline{x},\overline{x}))/2) + (1-\alpha)V(x,x'|\underline{x},\overline{x}) = \alpha\mathcal{V}(x,x'|1) + (1-\alpha)\mathcal{V}(x,x'|0)$. With *ex ante* deterministic assignment η PAM it is immediate, since $\mathcal{V}(\overline{x},\overline{x}|\alpha) = \alpha V(\overline{x},\overline{x}|\overline{x},\overline{x}) + (1-\alpha)V(\overline{x},\overline{x}|\overline{x},\underline{x}) = \alpha\mathcal{V}(\overline{x},\overline{x}|1) +$

Using (4.5) we can express equation (4.4) as

$$\Gamma(\alpha)=\alpha\Gamma(1)+(1-\alpha)\Gamma(0). \tag{4.6}$$

In cases where Γ is not linear in α, we will assume that it is a continuous function.

A wage function w in this setup reduces to a pair of wages $\underline{w}\equiv w(\underline{x})$ and $\overline{w}\equiv w(\overline{x})$. A competitive equilibrium with PAM and wages $(\underline{w},\overline{w})$ must satisfy the following incentive constraints:

$$\mathcal{V}(\overline{x},\overline{x}|1)-\overline{w}\geq\mathcal{V}(\overline{x},\underline{x}|1)-\underline{w} \tag{4.7}$$

$$\mathcal{V}\left(\underline{x},\underline{x}|1\right)-\underline{w}\geq\mathcal{V}(\underline{x},\overline{x}|1)-\overline{w}. \tag{4.8}$$

Adding both constraints reveals that a necessary condition for a PAM equilibrium is $\Gamma(1)\geq 0$, or, equivalently, that $\mathcal{V}(\cdot|1)$ is supermodular in (x,x').[17]

Similarly, a competitive equilibrium with NAM and wages $(\underline{w},\overline{w})$ satisfies

$$\mathcal{V}(\overline{x},\underline{x}|0)-\underline{w}\geq\mathcal{V}(\overline{x},\overline{x}|0)-\overline{w} \tag{4.9}$$

$$\mathcal{V}(\underline{x},\overline{x}|0)-\overline{w}\geq\mathcal{V}\left(\underline{x},\underline{x}|0\right)-\underline{w}, \tag{4.10}$$

and the corresponding necessary condition is $\Gamma(0)\leq 0$ or $\mathcal{V}(\cdot|0)$ submodular in (x,x').

Finally, in a competitive equilibrium with $\alpha\in(0,1)$ and wages $(\underline{w},\overline{w})$, agents must be indifferent between hiring a low or a high type. That is, the following equations must hold:

$$\mathcal{V}(\overline{x},\overline{x}|\alpha)-\overline{w}=\mathcal{V}(\overline{x},\underline{x}|\alpha)-\underline{w} \tag{4.11}$$

$$\mathcal{V}\left(\underline{x},\underline{x}|\alpha\right)-\underline{w}=\mathcal{V}(\underline{x},\overline{x}|\alpha)-\overline{w}, \tag{4.12}$$

and the corresponding necessary condition is that $\Gamma(\alpha)=0$, that is, $\mathcal{V}(\cdot,\cdot|\alpha)$ is modular in (x,x').

We now show that if Γ is continuous in α, then the necessary conditions derived for a competitive equilibrium with PAM, NAM, and stochastic matching are *sufficient* for equilibrium existence.

Proposition 1 *If Γ is continuous in α, then a competitive equilibrium exists. It exhibits PAM if $\Gamma(1)\geq 0$, NAM if $\Gamma(0)\leq 0$, and it is interior with $0<\alpha<1$ if $\Gamma(\alpha)=0$.*

Figure 1 depicts the competitive equilibria for the case in which Γ is linear in α. Except for the nongeneric case in which $\Gamma(\alpha)=0$ for all $\alpha\in[0,1]$, there is either a unique competitive equilibrium or three of them. There are non-existence examples in the matching literature in the presence of externalities (*e.g.* see the quadratic example in Koopmans and Beckmann (1957)). In our binary case a competitive equilibrium *always* exists, and there can be *multiple* ones including one with *stochastic matching* $\alpha\in(0,1)$. Multiplicity (with different sorting patterns) and interiority cannot arise without externalities (as in Becker (1973)). In particular, multiplicity emerges when the

$(1-\alpha)\mathcal{V}(\overline{x},\overline{x}|0)$, and similarly for $(\underline{x},\underline{x})$, $(\underline{x},\overline{x})$, and $(\overline{x},\underline{x})$. Finally, with aggregate spillovers that are multiplicative and linear in α, so that, say, $S(x,x',\xi(\alpha))=\zeta+k(x,x')\ell(\alpha)$ with $\zeta>0$, $k>0$, and $\ell(\alpha)=a+b\alpha=\alpha\ell(1)+(1-\alpha)\ell(0)$, we have $\mathcal{V}(x,x'|\alpha)=\alpha k(x,x')\ell(1)+(1-\alpha)k(x,x')\ell(0)=\alpha\mathcal{V}(x,x'|1)+(1-\alpha)\mathcal{V}(x,x'|0)$. We will use this tractable multiplicative case in several examples to illustrate some of the results.

17. Recall that a function f defined on a lattice of $\mathbb{R}^2$ is supermodular if given any two points (x,y) and (x',y'), $f(x\vee x',y\vee y')+f(x\wedge x',y\wedge y')\geq f(x,y)+f(x',y')$; it is submodular if the inequality is reversed; and it is modular if it holds with equality.

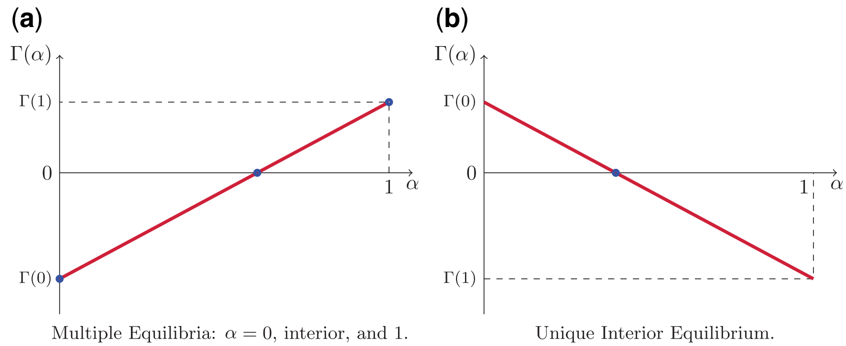

FIGURE 1

Equilibrium set. If Γ is positive for all α, then only a PAM equilibrium exists, and if it is negative for all α, then there is only an equilibrium with NAM. There is an interior equilibrium if Γ changes sign. (a) Multiple equilibria: $\alpha=0$, interior, and 1. (b) Unique interior equilibrium.

externalities change the match output from being a submodular to a supermodular function in team composition as α goes from 0 to 1. In turn, equilibrium is unique if the complementarity properties of the match output does not change with α, or if it is neither submodular at $\alpha=0$ nor supermodular at $\alpha=1$.

Example 1. Let $\mathcal{V}$ be given by $\mathcal{V}(x,x'|\alpha)=\zeta+k(x,x')\ell(\alpha)$, where k is strictly positive and symmetric in (x,x'), the aggregate spillover effect is multiplicative, with ℓ continuous and strictly increasing in α, and ζ is positive and large enough to ensure that $\mathcal{V}\geq 0$ for all teams and for all values of α. Then $\Gamma(\alpha)=(k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x}))\ell(\alpha)$. Assume that k is strictly supermodular in (x,x'), and thus $k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x})>0$. Then if $\ell(1)>0>\ell(0)$, there are three equilibria: a PAM equilibrium $\alpha=1$, a NAM equilibrium $\alpha=0$, and an interior one. If $\ell(1)>0$ and $\ell(0)>0$, there is only a PAM equilibrium, and similarly for the other cases. And if ℓ is instead continuous and strictly decreasing in α with $\ell(0)>0>\ell(1)$, there is a unique equilibrium that is interior. Note that if ℓ is a nonlinear function of α, then Γ is also a nonlinear function of α.

The planner's problem. The planner takes the structure of competition after matching as given, and her objective is to choose the matching $\alpha\in[0,1]$ that maximizes the total expected output of the economy. Formally

$$\max_{\alpha\in[0,1]}\frac{1}{2}\left(\frac{\alpha}{2}\mathcal{V}(\overline{x},\overline{x}|\alpha)+\frac{\alpha}{2}\mathcal{V}(\underline{x},\underline{x}|\alpha)+\frac{(1-\alpha)}{2}\mathcal{V}(\underline{x},\overline{x}|\alpha)+\frac{(1-\alpha)}{2}\mathcal{V}(\overline{x},\underline{x}|\alpha)\right).$$

To explain the objective function, note that there is a measure $1/2$ of teams, of which a fraction $\alpha/2$ contains two agents with $\overline{x}$, and each of these teams obtains $\mathcal{V}(\overline{x},\overline{x}|\alpha)$; a fraction $\alpha/2$ are teams with two agents with $\underline{x}$ and each of these teams obtains $\mathcal{V}(\underline{x},\underline{x}|\alpha)$; a fraction $(1-\alpha)/2$ are of composition $(\underline{x},\overline{x})$ and each of these teams obtains $\mathcal{V}(\underline{x},\overline{x}|\alpha)$; and a fraction $(1-\alpha)/2$ are of composition $(\overline{x},\underline{x})$ and each of these teams obtains $\mathcal{V}(\overline{x},\underline{x}|\alpha)$.

We will focus on the case in which Γ is linear in α (which holds for a broad class of problems), as it contains most of the insights of the planner's problem, and then discuss the nonlinear case. Using (4.5) and the definition of Γ, the problem can be written as

$$\max_{\alpha\in[0,1]}\frac{1}{2}\left(\frac{\alpha^2}{2}A+\frac{\alpha}{2}B+C\right), \tag{4.13}$$

where $A \equiv \Gamma(1)-\Gamma(0)$, $B \equiv \Gamma(0)+(\mathcal{V}(\underline{x},\overline{x}|1)+\mathcal{V}(\overline{x},\underline{x}|1)-\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\overline{x},\underline{x}|0))$, and $C \equiv (\mathcal{V}(\underline{x},\overline{x}|0)+\mathcal{V}(\overline{x},\underline{x}|0))/2$. The following proposition characterizes the solution α^p to the planner's problem in terms of A, B, and C.

Proposition 2 *Assume that Γ is linear in α and either $A \neq 0$ or $B \neq 0$.*[18] *The optimal matching is as follows:*

(i) If $A \geq 0$, then the planner chooses $\alpha^p = 1$ if $A+B \geq 0$ and $\alpha^p = 0$ if $A+B < 0$;
(ii) If $A < 0$ and $B \leq 0$, then the planner chooses $\alpha^p = 0$;
(iii) If $A < 0$, $B > 0$, and $B+2A \geq 0$, then the planner chooses $\alpha^p = 1$;
(iv) If $A < 0$, $B > 0$, and $B+2A < 0$, then the planner chooses $\alpha^p = -B/2A \in (0,1)$.

The intuition is straightforward. The planner's objective function is a quadratic function of α, and thus it is either convex or concave: if convex, then the solution is at a corner (part (*i*)), while if concave, it is at a corner if the objective function is monotone (parts (*ii*) and (*iii*)), and it is interior otherwise.

When $\mathcal{V}$ is nonlinear in α, as it can be in the case of aggregate spillovers, there is no sweeping characterization of the planner's problem as in the linear case. But it is possible to provide simple sufficient conditions on $\mathcal{V}$ such that the optimal matching is interior, which is the surprising result in Proposition 2. To this end, rewrite the planner's objective as follows (recall that in this case $\mathcal{V}(\cdot,\cdot|\alpha)$ is symmetric in (x,x') and thus $\mathcal{V}(\underline{x},\overline{x}|\alpha) = \mathcal{V}(\overline{x},\underline{x}|\alpha)$):

$$\frac{1}{4}\Big(\alpha\mathcal{V}(\overline{x},\overline{x}|\alpha)+\alpha\mathcal{V}(\underline{x},\underline{x}|\alpha)+2(1-\alpha)\mathcal{V}(\underline{x},\overline{x}|\alpha)\Big) = \frac{1}{4}\Big(\alpha\Gamma(\alpha)+2\mathcal{V}(\underline{x},\overline{x}|\alpha)\Big). \tag{4.14}$$

Since $\mathcal{V}$ is positive, so is the planner's objective function. The optimal matching is interior if the objective function is strictly increasing at $\alpha = 0$ and strictly decreasing at $\alpha = 1$. Formally, assume that $\mathcal{V}(x,x'|\cdot)$ is differentiable in α for all (x,x'). Then the solution to the planner's problem is interior if

$$\Gamma(0)+2\mathcal{V}_\alpha(\underline{x},\overline{x}|0) > 0, \tag{4.15}$$

which holds if each term is positive and one of them strictly positive; and

$$\Gamma(1)+\Big(\mathcal{V}_\alpha(\overline{x},\overline{x}|1)+\mathcal{V}_\alpha(\underline{x},\underline{x}|1)\Big) < 0, \tag{4.16}$$

which holds if each term is negative and one of them strictly negative.

Intuitively, the planner's solution trivializes when there are no externalities, for then $\mathcal{V}(x,x'|\cdot)$ is independent of α. As a result, the planner's objective function becomes linear in α, and the optimal matching is either PAM ($\alpha^P = 1$) or NAM ($\alpha^P = 0$), depending on whether $\mathcal{V}$ is supermodular or submodular in (x,x'). Unlike the standard case, when externalities are present an *interior* matching can be optimal. Moreover, in the standard case a marginal change in complementarities that changes $\mathcal{V}$ from being supermodular to submodular changes the matching from PAM to NAM, that is, from one corner to the other one. With externalities such a change does not have the same impact, for now the sorting pattern depends in a more complex way on the properties of $\mathcal{V}$, as the planner weighs not only the complementarities but also the externality effect.

Example 2. Assume that $\mathcal{V}$ is given by $\mathcal{V}(x,x'|\alpha) = \zeta + k(x,x')\ell(\alpha)$, and suppose that $k(\overline{x},\overline{x}) + k(\underline{x},\underline{x}) - 2k(\underline{x},\overline{x}) < 0$. Consider first the linear case $\ell(\alpha) = a + b\alpha$, with $b > 0$, $a < 0$, and $a+b > 0$. Then $A = (k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x}))b$, $B = (k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x}))a+2k(\underline{x},\overline{x})b$, and thus $B+$

18. This is just to avoid the nongeneric case in which all matchings are optimal.

$2A=(k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x}))(a+b)+(k(\overline{x},\overline{x})+k(\underline{x},\underline{x}))b$. Then it is easy to verify that $A<0$, $B>0$, and that $B+2A<0$ if and only if $2k(\underline{x},\overline{x})>((a+2b)/(a+b))(k(\overline{x},\overline{x})+k(\underline{x},\underline{x}))$, in which case the optimal matching is interior. Assume now that ℓ is nonlinear with $\ell'>0$, $\ell(0)<0$, and $\ell(1)>0$. Then $\Gamma(0)=(k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x}))\ell(0)>0$ and $\mathcal{V}_\alpha(\underline{x},\overline{x}|0)=k(\underline{x},\overline{x})\ell'(0)>0$, and thus (4.15) holds. Also, $\Gamma(1)=(k(\overline{x},\overline{x})+k(\underline{x},\underline{x})-2k(\underline{x},\overline{x}))\ell(1)<0$ while $\mathcal{V}_\alpha(\overline{x},\overline{x}|1)+\mathcal{V}_\alpha(\underline{x},\underline{x}|1)=(k(\overline{x},\overline{x})+k(\underline{x},\underline{x}))\ell'(1)>0$, and (4.16) holds if and only if $2k(\underline{x},\overline{x})>(1+(\ell'(1)/\ell(1)))(k(\overline{x},\overline{x})+k(\underline{x},\underline{x}))$, which generalizes the condition for the linear case, and depends on primitives.

Comparing equilibrium and efficient matching. To understand the efficiency properties of equilibria, it is instructive to focus on the planner's marginal incentives to increase α, that is, the first derivative of her objective function, which we will denote by Γ^p. For simplicity, we first analyse the case of Γ linear in α and then discuss the general case. In the linear case, the derivative Γ^p is given by

$$\begin{aligned}\Gamma^p(\alpha)&=\frac{1}{2}\left(\alpha A+\frac{B}{2}\right)\\&=\frac{1}{2}\left(\alpha\Gamma(1)+(1-\alpha)\Gamma(0)-\Gamma(0)+\frac{\mathcal{V}(\underline{x},\overline{x}|1)+\mathcal{V}(\overline{x},\underline{x}|1)-\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\overline{x},\underline{x}|0)}{2}+\frac{\Gamma(0)}{2}\right)\\&=\frac{1}{2}\left(\Gamma(\alpha)-\frac{D}{2}\right),\end{aligned}\tag{4.17}$$

where the first equality in (4.17) follows from differentiation of the planner's objective with respect to α, the second from replacing A and B and adding and subtracting $\Gamma(0)$, and the third from replacing $\alpha\Gamma(1)+(1-\alpha)\Gamma(0)$ by $\Gamma(\alpha)$ and from defining D as

$$D\equiv\mathcal{V}(\overline{x},\overline{x}|0)+\mathcal{V}\left(\underline{x},\underline{x}|0\right)-\mathcal{V}(\underline{x},\overline{x}|1)-\mathcal{V}(\overline{x},\underline{x}|1).\tag{4.18}$$

At the corners, we have $\Gamma^p(1)=(\Gamma(1)-(D/2))/2$ and $\Gamma^p(0)=(\Gamma(0)-(D/2))/2$.[19]

The constant $D/2$ summarizes the difference between the *private* and *social* incentives to increase α, and contains useful information about the efficiency properties of equilibria. The constant D measures the difference between the value of matching two teams in a PAM way when the equilibrium is NAM minus the value of matching them in a NAM way when the equilibrium is PAM. Indeed, we immediately obtain that an *interior* equilibrium is *inefficient* except in the nongeneric case in which $D=0$ (as in the second example in Section 2, where there was a unique equilibrium that was efficient). This is because for an interior equilibrium $\alpha\in(0,1)$ we must have $\Gamma(\alpha)=0$, but then the planner's marginal incentive to increase α is given by $\Gamma^p(\alpha)=-D/2$, which is generically not equal to zero and thus she prefers either a bigger or smaller α^p depending on the sign of D. Similarly, assume that $\Gamma(0)\le 0$, so there is an equilibrium with NAM. Then if D is negative and large the planner will choose $\alpha^p\neq 0$. Writing in full the planner's marginal incentives at $\alpha=0$ we obtain, after simplification

$$\Gamma^p(0)=\frac{1}{4}\left(\Gamma(0)+\left[\mathcal{V}(\underline{x},\overline{x}|1)-\mathcal{V}(\overline{x},\underline{x}|1)-V(\underline{x},\overline{x}|0)-\mathcal{V}(\overline{x},\underline{x}|0)\right]\right),$$

where the first term $\Gamma(0)$ summarizes the complementarities in $\mathcal{V}$ under $\alpha=0$, and the second term in square brackets reflects the externality effect from increasing α away from zero. So a NAM

19. We are grateful to an anonymous referee for suggesting the use of D to compare the solutions of the planner and the market.

equilibrium is inefficient if the externality effect is *strong* enough, and it is efficient otherwise. A similar analysis holds PAM.

Beyond the linear Γ case, we can see the wedge between the private and social incentives by using the planner's objective function (4.14), which yields (under the assumption that $\mathcal{V}(x,x'|\cdot)$ is differentiable for all (x,x'))

$$\Gamma^p(\alpha)=\frac{1}{4}(\Gamma(\alpha)+\alpha\Gamma'(\alpha)+2\mathcal{V}_\alpha(\underline{x},\overline{x}|\alpha)).$$

In an interior equilibrium $\Gamma(\alpha)=0$, and the planner's marginal incentives to increase α is given by $\Gamma^p(\alpha)=(1/4)(\alpha\Gamma'(\alpha)+2\mathcal{V}_\alpha(\underline{x},\overline{x}|\alpha))$, which is generically nonzero and can be positive or negative depending on the primitives. For example, in the multiplicative case above this wedge is $\alpha\Gamma'(\alpha)+2\mathcal{V}_\alpha(\underline{x},\overline{x}|\alpha)=(\alpha(k(\overline{x},\overline{x})+k(\underline{x},\underline{x}))+(1-\alpha)2k(\underline{x},\overline{x}))\ell'(\alpha)$, whose sign depends on the sign of ℓ': if positive (negative), the planner's optimal choice is a larger (smaller) α^P than the value of α in the interior competitive equilibrium.

Summarizing, unlike the standard case, equilibrium with binary characteristics can only be efficient if it is a corner and the externality effect is either small or reinforces the sign of complementarities in match output. In all other cases, equilibrium is *inefficient*. We shall see that similar results hold beyond the binary case.

4.2. *Continuum of characteristics*

The binary case affords a fairly complete derivation of the set of competitive equilibria, optima, and the main properties that make matching under externalities different from its counterpart without externalities. These properties are multiplicity of competitive equilibria, stochastic matching—either in competitive equilibrium or as the planner's optimal matching—and inefficiency. The general finite case clearly adds combinatorial complexity to the analysis, but an educated guess is that the main insights derived in the binary case will still obtain. Indeed, in Appendix A.3, we analyse the case with three characteristics (low, medium, and high), derive the necessary and sufficient conditions for PAM, NAM, and stochastic matching, as well as the main properties of the planner's problem. Moreover, we fully solve an example that is robust to the number of characteristics and that clearly illustrates the main insights (multiplicity, stochastic matching, and inefficiency).

Instead of continuing with the finite case, in this section we extend most of these insights to the case when the agents' characteristic x lie in $[0,1]$, continuously distributed according to a cdf F. This is a case that is commonly used in economic applications of matching without externalities, and this section provides a tractable extension with externalities, which we will use in the economic applications in the next section.

Following the pattern of the binary case, we will provide sufficient conditions for competitive equilibria with PAM and NAM, as well as for multiplicity of equilibria. A nice feature of the continuum case is that wages have a closed-form solution that is uniquely pinned down, and its properties (monotonicity, curvature, and interpretation) can be easily described. Moreover, we will show that the planner's solution can be "interior" (either stochastic or deterministic but away from PAM and NAM), and that competitive equilibrium can be inefficient.

Competitive equilibria with PAM and NAM. Consider first the cases of either *ex post* random pairwise assignment of competing teams or when all teams compete under aggregate spillovers. In these cases, the match output function $\mathcal{V}(\cdot,\cdot|\mu)$ is symmetric in (x,x') for any given μ, and we will also assume throughout the analysis that it is twice continuously differentiable in its first and second arguments.

Let us first construct a competitive equilibrium with PAM, that is, $\mu_+(x)=x$ for all x. The problem an agent with characteristic x faces when the market wage function is w is

$$\max_{x'} \mathcal{V}(x,x'|\mu_+)-w(x').$$

The first-order condition of this problem is simply $\mathcal{V}_2(x,x'|\mu_+)=w'(x')$, where (henceforth) the notation $\mathcal{V}_n$ denotes the derivative of $\mathcal{V}$ with respect to its n-th argument, and similarly for second derivatives (denoted $\mathcal{V}_{nm}$ for the second derivative). To transform this into an equilibrium condition we posit that it must hold along the assignment μ_+, that is, each agent matches with an agent with the same characteristic, and thus $\mathcal{V}_2(x,x|\mu_+)=w'(x)$. Integrating yields the wage function w given by $w(x)=w(0)+\int_0^x \mathcal{V}_2(s,s|\mu_+)ds$. Since $\mathcal{V}(\cdot,\cdot|\mu_+)$ is symmetric in (x,x'), it follows that under PAM partners divide match output equally, and so $w(x)=0.5\mathcal{V}(x,x|\mu_+)$ for all x.[20]

We claim that (w,μ_+), that is, the derived wage function and the PAM matching function, constitute a competitive equilibrium if and only if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular in (x,x'). Necessity follows as in the binary case: simply take the incentive constraints for x and x' (of not mimicking each other) and add them up. Regarding sufficiency, it will follow if each agent with characteristic x finds it (globally) optimal to choose a partner with the same characteristic when he conjectures that the prevailing matching in the market is PAM and he faces wages given by w. Without loss of generality, consider two potential partners of an agent with x, one with characteristic x and the other with x', with $x'<x$; then $\mathcal{V}(x,x|\mu_+)-w(x)\geq\mathcal{V}(x,x'|\mu_+)-w(x')$ if and only if

$$\mathcal{V}(x,x|\mu_+)-\mathcal{V}(x,x'|\mu_+)\geq\int_{x'}^{x}\mathcal{V}_2(s,s|\mu_+)ds,$$

where the inequality follows from the definition of the wage function. But the left side is equal to $\int_{x'}^{x}\mathcal{V}_2(x,s|\mu_+)ds$, and thus we need to show that

$$\int_{x'}^{x}(\mathcal{V}_2(x,s|\mu_+)-\mathcal{V}_2(s,s|\mu_+))\,ds\geq 0,$$

which follows from the supermodularity of $\mathcal{V}(\cdot,\cdot|\mu_+)$ in (x,x'). The argument for $x'\geq x$ is symmetric. Thus, each agent finds it optimal to choose a partner of the same characteristic. Hence, a competitive equilibrium (w,μ_+) with PAM exists when $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular in (x,x').[21] The equilibrium wage function is strictly increasing in x, and it is convex if $\mathcal{V}_{22}(x,x|\mu_+)\geq 0$ for all x.[22]

A similar argument shows that a NAM equilibrium exists if and only if $\mathcal{V}(\cdot,\cdot|\mu_-)$ is submodular in (x,x') when agents conjecture that there is NAM in the market. Necessity is as usual, and to prove sufficiency, let $\mu_-(x)=F^{-1}(1-F(x))$ be the matching function and let the wage function w be $w(x)=w(0)+\int_0^x\mathcal{V}_2(\mu_-^{-1}(s),s|\mu_-)ds$. Let $\mu_-(x)=x'$ and consider $x''<x'$; then choosing x' is a global optimum for and agent with x if and only if

$$\mathcal{V}(x,x'|\mu_-)-\mathcal{V}(x,x''|\mu_-)\geq\int_{x''}^{x'}\mathcal{V}_2(\mu_-^{-1}(s),s|\mu_-)ds.$$

20. To see that the two expressions for w coincide, it suffices to point out that $w'(x)=0.5(\mathcal{V}_1(x,x|\mu_+)+\mathcal{V}_2(x,x|\mu_+))=0.5(2\mathcal{V}_2(x,x|\mu_+))=\mathcal{V}_2(x,x|\mu_+)$ using the symmetry of $\mathcal{V}(\cdot,\cdot|\mu_+)$ in (x,x').

21. An alternative proof in the PAM case is as follows: since the wage equals half the match output, if x chooses a partner of the same characteristic then his payoff is $0.5\mathcal{V}(x,x|\mu_+)$. If he chooses $x'>x$ then his payoff is $\mathcal{V}(x,x'|\mu_+)-0.5\mathcal{V}(x',x'|\mu_+)$, and this is less than $0.5\mathcal{V}(x,x|\mu_+)$ if and only if $\mathcal{V}(x',x'|\mu_+)+\mathcal{V}(x,x|\mu_+)\geq 2\mathcal{V}(x,x'|\mu_+)$, that is, if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular.

22. This follows from differentiating twice $w(x)=0.5\mathcal{V}(x,x|\mu_+)$ and using $\mathcal{V}_{12}(x,x|\mu_+)\geq 0$ for all x.

But the left side is equal to $\int_{x''}^{x'} \mathcal{V}_2(x,s|\mu_-)ds$, and thus we must show that

$$\int_{x''}^{x'} \Big(\mathcal{V}_2(x,s|\mu_-) - \mathcal{V}_2(\mu_-^{-1}(s),s|\mu_-)\Big) ds \geq 0.$$

Since μ_- is decreasing, so is μ_-^{-1}. Then $x=\mu_-^{-1}(x')$ implies that for any $x'' \leq s \leq x'$ we have $\mu_-^{-1}(s) \geq \mu_-^{-1}(x')=x$, and thus the result follows from the submodularity of $\mathcal{V}$, since the integrand in the expression above is positive as $\mathcal{V}_2$ is decreasing in its first argument. The argument for $x'' \geq x'$ is analogous. Hence, choosing a partner in accordance with μ_- is optimal for each agent, and as a result (w,μ_-) constitutes a competitive equilibrium. As in the PAM case, one can verify that w is convex if $\mathcal{V}_{22}(\mu_-^{-1}(x),x|\mu_-) \geq 0$ for all x.[23]

As in the binary case, we can have multiple equilibria with *different* assignments. In particular, a PAM and a NAM equilibrium can coexist if $\mathcal{V}(\cdot,\cdot|\mu)$ switches from supermodular to submodular in (x,x') when μ changes from μ_+ to μ_-. The following example illustrates the presence of *multiple* competitive equilibria.

Example 3. Let $\mathcal{V}$ be given by $\mathcal{V}(x,x'|\mu)=\zeta+k(x,x')\xi(\mu)$ for all (x,x') and μ, so there is an aggregate spillover effect that is multiplicative, with $\xi(\mu_+)>0>\xi(\mu_-)$, and $\zeta>0$ and large enough to ensure that match payoff is positive for all teams and for $\mu \in \{\mu_+,\mu_-\}$. Also, assume that $k_{12} \geq 0$. Then there is a competitive equilibrium with PAM and wages given by $w(x)= w(0)+\xi(\mu_+)\int_0^x k_2(s,s)ds$ for all x. Similarly, there is an equilibrium with NAM with $w(x)= w(0)+\xi(\mu_-)\int_0^x k_2(\mu_-^{-1}(s),s)ds$ for all x. So as in the binary case, we can have multiple equilibria. For a closed-form solution, let $k(x,x')=xx'$, F uniform on $[0,1]$, $\xi(\mu_+)=1$, and $\xi(\mu_-)=-1$. Then in a competitive equilibrium with PAM the wage function w is given by $w(x)=0.5(\zeta+x^2)$, which is strictly increasing and strictly convex in x. Under NAM, $\mu_-(x)=1-x$ and thus $w(x)= w(0)-0.5x(2-x)$. To find $w(0)$, note that we must have $w(x)+w(1-x)=\zeta-x(1-x)$ for all x, and thus $w(1)+w(0)=\zeta$, while from the wage function $w(1)=w(0)-0.5$. Hence, $w(0)=0.5\zeta+0.25$. The wage function is strictly decreasing and strictly convex. Both types of equilibria coexist in this setting since the externality switches $\mathcal{V}$ from supermodular to submodular in (x,x') depending on the matching that agents conjecture will prevail in the market.

Consider now the case of *ex ante* deterministic assignment of teams. That is, before teams are formed, half of the agents with characteristic x are matched with a future competitor with characteristic $\eta(x)=x$. We will assume that the V is twice continuously differentiable in its arguments, which more than suffices for our purposes.

Let us construct a competitive equilibrium with PAM, that is, with $\mu_+(x)=x$ for all x. The problem of an agent with x facing a wage function w is

$$\max_{x'} V(x,x'|x,x)-w(x').$$

The interpretation of this setup is as stated in Section 3.3: an agent with characteristic x is assigned, before the team formation stage, to a competitor with the same characteristic, and he conjectures PAM in this market, that is, that his competitor will team up with another agent with characteristic x.

From the first-order condition evaluated at μ_+ we obtain $V_2(x,x|x,x)=w'(x)$, and thus $w(x)= w(0)+\int_0^x V_2(s,s|s,s)ds$. We now show that this wage function along with the PAM assignment

23. Here $w''(x)=\mathcal{V}_{12}(\mu_-^{-1}(x),x|\mu_-)(\mu_-^{-1})'(x)+\mathcal{V}_{22}(\mu_-^{-1}(x),x|\mu_-)$, with $V_{12}(\mu_-^{-1}(x),x|\mu_-) \leq 0$ and $(\mu_-^{-1})'(x)<0$ for all x.

μ_+ constitute a competitive equilibrium if V is supermodular in its first two arguments, that is $V_{12} \geq 0$, and if $V_{23} + V_{24} \geq 0$, for which it suffices that V is supermodular in its second and third arguments, and also in its second and fourth arguments.[24] To see this, consider $x' < x$; then $V(x,x|x,x) - w(x) \geq V(x,x'|x,x) - w(x')$ if and only if

$$V(x,x|x,x) - V(x,x'|x,x) \geq \int_{x'}^{x} V_2(s,s|s,s)ds.$$

Since the left side is equal to $\int_{x'}^{x} V_2(x,s|x,x)ds$, this inequality is equivalent to

$$\int_{x'}^{x} (V_2(x,s|x,x) - V_2(s,s|s,s))\, ds \geq 0.$$

But

$$\int_{x'}^{x} (V_2(x,s|x,x) - V_2(s,s|s,s))\, ds \geq \int_{x'}^{x} (V_2(x,s|x,x) - V_2(s,s|x,x))\, ds \geq 0,$$

where the first inequality follows from $V_{23} + V_{24} \geq 0$, and the second from $V_{12} \geq 0$. The argument for $x \geq x'$ is analogous. Hence, a competitive equilibrium with PAM exists under the stated assumptions on V.

A similar argument yields a competitive equilibrium with NAM μ_- given by $\mu_-(x) = F^{-1}(1 - F(x))$. To see this, note that the problem of an agent with characteristic x is

$$\max_{x'} V(x,x'|x,\mu_-(x)) - w(x').$$

Proceeding as before, the first-order condition is $V_2(x,x'|x,\mu_-(x)) = w'(x')$ and thus the equilibrium condition is $V_2(\mu_-^{-1}(x'),x'|\mu_-^{-1}(x'),x') = w'(x')$. Integrating yields w given by $w(x) = w(0) + \int_0^x V_2(\mu_-^{-1}(s),s|\mu_-^{-1}(s),s)ds$. We claim that (w,μ_-) constitute a competitive equilibrium if $V_{12} + V_{23} \leq 0$, and $V_{24} \geq 0$. To prove it, it suffices to show that under these conditions it is a global optimum for an agent with characteristic x to choose $x' = \mu_-(x)$. Consider $x'' < x'$ (the argument for $x'' \geq x'$ is analogous); then x' is an optimal choice for x if and only if

$$V(x,x'|x,\mu_-(x)) - V(x,x''|x,\mu_-(x)) \geq \int_{x''}^{x'} V_2(\mu_-^{-1}(s),s|\mu_-^{-1}(s),s)ds.$$

Since the left side is equal to $\int_{x''}^{x'} V_2(x,s|x,\mu_-(x))ds = \int_{x''}^{x'} V_2(\mu_-^{-1}(x'),s|\mu_-^{-1}(x'),x')ds$, this inequality is equivalent to

$$\int_{x''}^{x'} \Big(V_2(\mu_-^{-1}(x'),s|\mu_-^{-1}(x'),x') - V_2(\mu_-^{-1}(s),s|\mu_-^{-1}(s),s) \Big) ds \geq 0.$$

It thus suffices to show that under the stated assumptions about V we have that $V_2(\mu_-^{-1}(\cdot),s|\mu_-^{-1}(\cdot),\cdot)$ is an increasing function or, equivalently, that $(V_{21} + V_{23})(\mu_-^{-1})' + V_{24} \geq 0$. This is satisfied if $V_{21} + V_{23} \leq 0$ and $V_{24} \geq 0$.[25] Thus, a competitive equilibrium with NAM exists under the stated assumptions.

We summarize all these results in the following proposition:

24. This holds if V is multiplicatively separable $k(x,x')z(y,y')$ with k supermodular, and k and z positive and increasing.

25. If V is multiplicatively separable $k(x,x')z(y,y')$, it holds if k and z are positive and increasing, and k_{12} is sufficiently negative.

Proposition 3 *(i) If competing teams are either ex post randomly assigned or if they all compete under aggregate spillovers, then there is a competitive equilibrium (w,μ_+) that exhibits PAM (a competitive equilibrium (w,μ_-) that exhibits NAM) if and only if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular ($\mathcal{V}(\cdot,\cdot|\mu_-)$ is submodular).*

(ii) If competing teams are ex ante deterministically assigned in a PAM way by an assignment η, then there is a competitive equilibrium (w,μ_+) that exhibits PAM (a competitive equilibrium (w,μ_-) that exhibits NAM) if $V_{12}\geq 0$ and $V_{23}+V_{24}\geq 0$ (if $V_{12}+V_{23}\leq 0$, and $V_{24}\geq 0$).

In short, we have shown that the insights regarding existence of competitive equilibria with PAM and NAM, and that they may coexist, extend to the case with a continuum of types. In addition, with a continuum of characteristics we can derive properties of the wage function that supports PAM or NAM under each of the competing team assignment environments. A useful feature of our analysis is that it permits the derivation of sorting and wage properties under externalities in a tractable way that resembles the analysis without externalities. This is helpful in economic applications of our model, as we will illustrate in the next section.

We close with a brief discussion of stochastic matching with a continuum of types. A full analysis of this case is much more challenging than in the binary case and beyond the scope of this article. We can, however, say a few things about it. The most important one is that, as in the binary case, for a measure π to be part of a competitive equilibrium with stochastic matching we must have that, conditional on π, the function $\mathcal{V}(\cdot,\cdot|\pi)$ is modular in (x,x'). In differential terms, this is equivalent to finding a measure π such that $\mathcal{V}_{12}(x,x'|\pi)=0$ for all (x,x'). To see this, consider any two agents with characteristics x and x' facing a wage function w. Among all the indifference conditions for these types, the following must hold:

$$\mathcal{V}(x,x|\pi)-w(x)=\mathcal{V}(x,x'|\pi)-w(x')$$
$$\mathcal{V}(x',x'|\pi)-w(x')=\mathcal{V}(x',x|\pi)-w(x).$$

Adding them up we obtain that $\mathcal{V}(\cdot,\cdot|\pi)$ must be modular in (x,x'). Furthermore, assume that $\mathcal{V}_{12}(x,x'|\pi)=0$ for all (x,x'). Let $w(x)=w(0)+\int_0^x\mathcal{V}_2(x,s|\pi)ds$. Since $\mathcal{V}$ is modular in its first two arguments, the integrand is independent of x. Then an agent with characteristic x solves $\max_{x'}\mathcal{V}(x,x')-w(x')$ and the first-order condition is $\mathcal{V}_2(x,x'|\pi)=\mathcal{V}_2(x',x'|\pi)$, which holds for all x' by the modularity premise. Hence, each agent is indifferent about whom to hire, and is therefore willing to randomize according to π.

For a simple example in the spirit of the binary case, assume π is such that a fraction α of agents of each characteristic match in a PAM way and $1-\alpha$ in a NAM way. Assume that F is uniform on $[0,1]$, and that $\mathcal{V}(x,x'|\pi)=\mathcal{V}(x,x'|\alpha)=\zeta+k(x,x')\ell(\alpha)$, with $k_{12}>0$, $\ell(0)<0$, $\ell(1)>0$, and $\ell'>0$, and ζ large enough to make payoffs positive. The aggregate spillover depends only on the fraction of agents matched à la PAM, and there is a value of α such that $\mathcal{V}_{12}(x,x'|\pi)=k_{12}(x,x')\ell(\alpha)=0$. Although not conclusive, this suggests that competitive equilibria with stochastic matching can also exist with a continuum of characteristics.

The planner's problem. A full characterization of the planner's problem like the one given for the binary case is not available for the continuum case, as it requires a nontrivial extension of optimal transport theory to handle problems where the measure being chosen also appears in the integrand, as in $\max_{\pi\in\mathcal{M}}\int_{[0,1]^2}\mathcal{V}(x,x'|\pi)d\pi(x,x')$. It is possible, however, to shed light on the planner's solution without solving the full-blown problem, but instead building on the analysis of the binary case.

Consider the following restricted planner's problem, where the feasible set is any combination $\alpha\in[0,1]$ of PAM and NAM. More precisely, let f be the density associated with F and suppose that, for each x, $\alpha f(x)$ is matched in a PAM way and $(1-\alpha)f(x)$ in a NAM way. This restricted problem is

$$\max_{\alpha\in[0,1]}\frac{1}{2}\left(\alpha\int_0^1\mathcal{V}(x,x|\alpha)dF(x)+(1-\alpha)\int_0^1\mathcal{V}(x,\mu_-(x)|\alpha)dF(x)\right).$$

We will focus on the case in which $\mathcal{V}$ is linear in α, that is, $\mathcal{V}(x,x'|\alpha)=\alpha\mathcal{V}(x,x'|\mu_+)+(1-\alpha)\mathcal{V}(x,x'|\mu_-)$, and $\mathcal{V}(\cdot,\cdot|\alpha)$ is symmetric in (x,x'). Simple algebra allows us to rewrite the planner's problem as follows:

$$\max_{\alpha\in[0,1]}\frac{1}{2}\left(\alpha^2A'+\alpha B'+C'\right),$$

where

$$\begin{aligned}
A'=&\left(\int_0^1\mathcal{V}(x,x|\mu_+)dF(x)-\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_+)dF(x)\right)\\
&-\left(\int_0^1\mathcal{V}(x,x|\mu_-)dF(x)-\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x)\right)\\
B'=&\left(\int_0^1\mathcal{V}(x,x|\mu_-)dF(x)-\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x)\right)\\
&+\left(\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_+)dF(x)-\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x)\right)\\
C'=&\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x).
\end{aligned}$$

Note that these expressions are straightforward analogues of those in the planner's problem in the binary case. It is clear that if the planner were restricted to choose between combinations of PAM and NAM, then an adaptation of Proposition 2 would give a sharp characterization of the optimal matching in this restricted problem. Without this restriction, we can still use the restricted problem to show the following result:

Proposition 4 *If $A'<0$, $B'>0$, and $B'+2A'<0$, then the efficient matching is neither PAM nor NAM.*

The proof is immediate. By Proposition 2, we know that under the premises the solution to the planner's restricted problem is an interior matching. This shows that there is a stochastic matching that dominates PAM and NAM. As a result, the optimal solution to the unrestricted planner's problem cannot be PAM or NAM. This can happen even if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular or submodular in (x,x').

Example 4. Consider the following continuous version of Example 2. Let x be uniformly distributed on $[0,1]$, and let $\mathcal{V}$ be given by $\mathcal{V}(x,x'|\alpha)=\zeta+k(x,x')\hat{\hat{\xi}}(\pi)$, with $k_{12}<0$, so there is an aggregate spillover that is multiplicative. Assume also that when π matches a fraction α in a PAM way and the rest in a NAM way, $\hat{\hat{\xi}}(\pi)$ reduces to $\ell(\alpha)=a+b\alpha$, with $b>0$, $a<0$, and $a+b>0$. That is, $\mathcal{V}(x,x'|\mu_+)=\zeta+k(x,x')\ell(1)$, $\mathcal{V}(x,x'|\mu_-)=\zeta+k(x,x')\ell(0)$, and $\mathcal{V}(x,x'|\alpha)=\alpha\mathcal{V}(x,x'|\mu_+)+(1-\alpha)\mathcal{V}(x,x'|\mu_-)$. Then $A'=(\int_0^1 k(x,x)dx-\int_0^1 k(x,1-x)dx)b$, which is negative since k_{12} implies that the first term is strictly negative. Also, $B'=(\int_0^1 k(x,x)dx-\int_0^1 k(x,1-x)dx)a+b\int_0^1 k(x,1-x)dx>0$ since $a<0$. Hence, $B'+2A'=(a+2b)\int_0^1 k(x,x)dx-(a+b)\int_0^1 k(x,1-x)dx$, which is strictly negative if and only if $\int_0^1 k(x,1-x)dx>((a+2b)/(a+b))\int_0^1 k(x,x)dx$. This condition asserts that k needs to be "sufficiently" submodular. If so, then the optimal matching is neither PAM or NAM, since both are dominated by a stochastic matching that pairs a fraction in a PAM way and the rest in a NAM way.

Inefficiency of competitive equilibrium. A competitive equilibrium with PAM or NAM can be inefficient. To see this, assume that the conditions in Proposition 3 (*i*) or (*ii*) hold, so that there is a competitive equilibrium with PAM (the analysis for a competitive equilibrium with NAM is similar). Consider now the planner's choice between *just* PAM or NAM: under PAM welfare is $0.5\int_0^1\mathcal{V}(x,x|\mu_+)dF(x)$ while under NAM is $0.5\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x)$. To show that the PAM equilibrium is inefficient it suffices to show that welfare is strictly higher under NAM (obviously, this does not imply that the planner will choose NAM), so that $\int_0^1\mathcal{V}(x,x|\mu_+)dF(x)<\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x)$. Adding and subtracting $\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_+)dF(x)$ to both sides of the inequality, we obtain that NAM dominates PAM if and only if

$$\begin{aligned}\int_0^1\mathcal{V}(x,x|\mu_+)dF(x)&-\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_+)dF(x)\\&<\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_-)dF(x)-\int_0^1\mathcal{V}(x,\mu_-(x)|\mu_+)dF(x).\end{aligned}\tag{4.19}$$

The left side can be interpreted as the efficiency gains from PAM instead of from NAM when everybody conjectures that the prevailing matching is PAM. The right side represents the effect of the externality in the welfare under NAM versus PAM. If the externality effect is strong enough, then PAM is dominated by NAM and any competitive equilibrium exhibiting PAM is inefficient. Note that this can happen even if $\mathcal{V}(\cdot,\cdot|\mu_+)$ is supermodular in (x,x'), which cannot arise without externalities.

Example 5. Let x be uniformly distributed on $[0,1]$, and let $\mathcal{V}$ be given by $\mathcal{V}(x,x'|\mu)=k(x,x')e(\mu)$ for all (x,x') and μ, where the functions k and e are strictly positive. Also, assume that $k_{12}>0$, so there is an equilibrium with PAM. Welfare under NAM is strictly greater than under PAM if and only if $\int_0^1 k(x,x)dx<(e(\mu_-)/e(\mu_+))\int_0^1 k(x,1-x)dx$. Since $\int_0^1 k(x,x)dx>\int_0^1 k(x,1-x)dx$ by strict supermodularity of k, it follows that welfare under NAM is greater than under PAM if and only if the externality under NAM is sufficiently greater (in relative terms) than that under PAM.

5. ECONOMIC RELEVANCE

This section sheds light on the economic relevance of the theory. The objective is 2-fold. First, to illustrate how post-match competition introduces externalities at the matching stage in some natural economic applications, which in turn leads to insights that do not obtain without externalities. Second, to show how the insights of our framework can contribute in a novel way to provide answers to some issues that have received ample attention in the Macro/Labor and Industrial Organization literatures. Section 5.1 contains our main application, which develops a setting of competing teams with aggregate spillovers. It generates results that are consistent with the empirical evidence on the evolution of inequality within and between firms. In Section 5.2, we analyse team formation in a competitive input market when firms compete downstream in an oligopolistic output market, where competing firms are deterministically assigned. Finally, in Section 5.3, we briefly discuss further applications.

5.1. *Knowledge spillovers*

In this application, knowledge spillovers in the downstream market affect how each firm chooses the skill composition of its team and the wages it pays to workers of different characteristics, which in turn affect the firm's investment in capital (knowledge). The equilibrium interaction between the competitive input market and the imperfectly competitive output market can provide crucial insights on an important current issue, the evolution of between- and within-firm inequality in the last decades. In particular, evidence for different countries shows that the increase in wage inequality in recent decades can be explained almost exclusively by the increase in *between-firm* inequality while there is hardly any change in *within-firm* inequality.[26] We show that a matching model with externalities generated by knowledge spillovers downstream can rationalize these facts.

To provide some context for the model we develop below, the literature on endogenous growth models initiated by Romer (1986) and Lucas (1988) introduced the idea that total factor productivity (TFP) is not exogenous, but depends on the decisions of other agents in the economy. For example, the productivity of any given worker is higher if the economy's workforce is more productive. This can be modelled by letting TFP be a function of the aggregate investment, whether it be in education or in technology. This has been taken further by Jovanovic and Rob (1989), Eeckhout and Jovanovic (2002), Lucas and Moll (2014), Perla and Tonetti (2014), Benhabib *et al.* (2017), and König *et al.* (2016), who observe that those spillovers may affect agents differentially, which naturally leads to inequality and to a distribution of firms.

We build on this literature and add a matching stage in which teams form, the theme of this article. This allows us to capture firm heterogeneity in skill and wages. We then ask how technological change, which takes the form of an increase in production complementarities, affects skill and wage inequality within and between firms.

In the first stage, teams of two workers form in a competitive labour market. In the second stage, teams make investment decisions where the return on investment is a function of the distribution of investment in the entire economy. The externality in the second stage is general, with non-internalized effects across all agents.

We first provide simple parametric conditions under which competitive equilibrium is unique and exhibits a fraction of the agents matched in a PAM way and the rest in a NAM way.

We then show how technological change affects the composition of skills, both in the equilibrium allocation as well as in the planner's solution. An increase in complementarities leads to more positive sorting in equilibrium, that is, to a larger fraction being matched in a PAM fashion. This implies that the variance of skills within firms goes down, while the one between firms goes up. We show that the latter increases by a much larger magnitude than the decrease in the former, which accords well with the evidence.[27] Similarly, the evidence on wages shows that the variance of wages within firms has remained relatively constant while the variance of

26. See Card *et al.* (2013) for Germany, Song *et al.* (2015) and Barth *et al.* (2014) for the US, Benguria (2015) for Brazil, and Vlachos *et al.* (2015) for Sweden.

27. Vlachos *et al.* (2015) have detailed information on aptitude tests for Sweden and show that skill inequality has increased between firms but not within.

wages between firms has increased, a stylized fact our model can rationalize as well under certain parametric conditions.[28]

Finally, we show that the competitive equilibrium is inefficient, and that the planner prefers less positive sorting that the market. In particular, the market exhibits an inefficiently high level of between-firm inequality.

Formally, we consider a setup with binary types in which firms copy the technology of more productive firms. The advantage of the binary setting is that we can explicitly solve for an interior equilibrium.

In the matching stage, firms hire two workers of type $x \in \{\underline{x}, \overline{x}\}$, $0 \leq \underline{x} < \overline{x} \leq 1$, in a competitive labour market. And in the second stage, firms make an investment decision k, the payoff of which depends on their team composition, which is equal to the sum of the characteristics of its members, $x_1 + x_2$. For notational economy, we will denote this sum by $\overline{X} \equiv 2\overline{x}$, $\hat{X} \equiv \overline{x} + \underline{x}$, and $\underline{X} \equiv 2\underline{x}$.

Firms receive spillovers from other firms and these spillovers vary with both the rank the firm has based on its capital and with its team composition X. We model this rank dependence by assuming that the spillover S results from copying the technology of higher ranked firms. The higher is the own capital of a firm, the fewer higher ranked firms there are and the less there is to copy. The cost of investment is quadratic and is inversely proportional to X^{γ}, $\gamma \geq 1$, where higher γ yields stronger complementarity between worker characteristics. The composition of workers affects the optimal investment: a firm with composition X that invests k obtains a direct return $A\lambda k$ as well as a spillover $AS(k,X)k$, where with a slight abuse of notation we denote by $A > 0$ the economy-wide TFP, while the parameter $\lambda > 0$ ensures a direct positive gross benefit from investment even if the firm does not enjoy any spillover effect from other firms.

Using the notation from the binary case in Section 4.1, consider an arbitrary matching $\alpha \in [0,1]$, where as usual α is the fraction of teams matched according to PAM, and denote by $\mathcal{V}(X|\alpha)$ the payoff of a team with composition X given a prevailing matching α. We assume that $\mathcal{V}(X|\alpha)$ is given by

$$\mathcal{V}(X|\alpha) = \max_{k \geq 0} \left(A(\lambda + S(k,X))k - \frac{k^2}{2X^{\gamma}} \right). \tag{5.20}$$

For values $\underline{\kappa} < \hat{\kappa} < \overline{\kappa}$, the spillover function S is defined as follows:

$$S(k,\overline{X}) = 0 \quad \forall k, \tag{5.21}$$

$$S(k,\hat{X}) = \begin{cases} 1 - \frac{\alpha}{2} - (1-\alpha) & \text{if } k \in [0,\overline{\kappa}) \\ 0 & \text{if } k \geq \overline{\kappa}, \end{cases} \tag{5.22}$$

$$S(k,\underline{X}) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } k \in [0,\hat{\kappa}) \\ 1 - \frac{\alpha}{2} - (1-\alpha) & \text{if } k \in [\hat{\kappa},\overline{\kappa}) \\ 0 & \text{if } k \geq \overline{\kappa}. \end{cases} \tag{5.23}$$

The intuition behind the function S is that a firm with a given composition can, by choosing k, learn about the production technology of firms with better composition. Firms with high k have less to copy than firms with low k. Note that the magnitude of the positive spillover a firm enjoys depends on the measure of teams with higher k. The higher the k chosen the smaller is that measure.

28. These results are not straightforward, since the distribution of team composition also changes when there is an increase in complementarities, which complicates the comparative statics analysis. Hence the need for conditions under which the results obtain.

We will look for a second-stage equilibrium where each firm with composition $\underline{X}$ chooses $k=\underline{\kappa}$, $\hat{X}$ chooses $k=\hat{\kappa}$, and $\overline{X}$ chooses $k=\overline{\kappa}$, where $\underline{\kappa}=A\underline{X}^{\gamma}(\lambda+1-(\alpha/2))$, $\hat{\kappa}=A\hat{X}^{\gamma}(\lambda+(\alpha/2))$, and $\overline{\kappa}=A\overline{X}^{\gamma}\lambda$ (these expressions come from the maximization problem (5.20), ignoring the kinks in S). As a result, the candidate $\mathcal{V}(\cdot|\alpha)$ is given by $\mathcal{V}(\underline{X}|\alpha)=A^2\underline{X}^{\gamma}(\lambda+1-\alpha/2)^2/2$, $\mathcal{V}(\hat{X}|\alpha)=A^2\hat{X}^{\gamma}(\lambda+\alpha/2)^2/2$, and $\mathcal{V}(\overline{X}|\alpha)=A^2\overline{X}^{\gamma}\lambda^2/2$. We now provide sufficient conditions for these choices to be a second-stage equilibrium of the market.

Lemma 1 *Let $\alpha\in[0,1]$, $\underline{\kappa}=A\underline{X}^{\gamma}(\lambda+1-(\alpha/2))$, $\hat{\kappa}=A\hat{X}^{\gamma}(\lambda+(\alpha/2))$, and $\overline{\kappa}=A\overline{X}^{\gamma}\lambda$. If $\lambda\geq 1$ and $\underline{x}/\overline{x}<1/3$, then $0<\underline{\kappa}<\hat{\kappa}<\overline{\kappa}$ and $(\underline{\kappa},\hat{\kappa},\overline{\kappa})$ is a second-stage equilibrium.*

We can now analyse the competitive equilibria in the labour market in the first stage. In this case, the function $\Gamma(\alpha)=\mathcal{V}(\overline{X}|\alpha)+\mathcal{V}(\underline{X}|\alpha)-2\mathcal{V}(\hat{X}|\alpha)$, which is helpful for computing equilibria, is given by

$$\begin{aligned}\Gamma(\alpha)&=\frac{A^2\overline{X}^{\gamma}\lambda^2}{2}+\frac{A^2\underline{X}^{\gamma}\left(\lambda+1-\frac{\alpha}{2}\right)^2}{2}-2\frac{A^2\hat{X}^{\gamma}\left(\lambda+\frac{\alpha}{2}\right)^2}{2}\\&=\frac{A^2\overline{X}^{\gamma}\lambda^2}{2}\left(1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1-\frac{\alpha}{2}}{\lambda}\right)^2-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{\alpha}{2\lambda}\right)^2\right),\end{aligned}\tag{5.24}$$

where the second equality follows by multiplying and dividing the second and third term by the first one and then simplifying. It is clear from (5.24) that Γ strictly decreases in α (the second term in the expression in parenthesis strictly decreases in α while the third strictly increases but has a minus sign in front of it). Hence, it follows from the analysis in Section 4.1 that, given the second-stage equilibrium in Lemma 1, there exists a *unique* equilibrium at the matching stage, which is either PAM, or NAM, or interior. We now provide some simple parametric conditions under which the unique equilibrium is interior and the proportion of teams matched in a PAM way is strictly increasing in the complementarity parameter γ.

Proposition 5 *Assume the second-stage equilibrium described in Lemma 1. If $\lambda\geq 1$, $1\leq\gamma<1+2(\log(1+(1/2\lambda))/\log 2)$, and $\underline{x}/\overline{x}$ is sufficiently small, then there is a unique competitive equilibrium in the first stage, which is interior (i.e. $\alpha\in(0,1)$). Moreover, the equilibrium α is strictly increasing in γ.*

As an illustration we consider a closed-form example. Assume $\underline{x}=0$ and $\lambda=1$. Then $\Gamma(\alpha)=0$ if and only if $1-2^{1-\gamma}(1+(\alpha/2))^2=0$, which yields

$$\alpha=2\left(2^{\frac{1}{2}(\gamma-1)}-1\right),\tag{5.25}$$

and this is strictly positive if $\gamma>1$, and strictly less than one if $\gamma<1+2(\log(3/2)/(\log 2))\cong 2.17$. Differentiating with respect to γ reveals immediately that α is strictly increasing in γ.

We now compute the between- and within-firm variance of *wages*. For clarity, denote by $\alpha^{\star}$ the equilibrium value of α, and assume that parameters are as in Proposition 5, so that $\alpha^{\star}$ is strictly increasing in γ. We know from Section 4.1 that wages are $\underline{w}=0.5\mathcal{V}(\underline{X}|\alpha^{\star})=A^2\underline{X}^{\gamma}(\lambda+1-(\alpha^{\star}/2))^2/4$ and $\overline{w}=0.5\mathcal{V}(\overline{X}|\alpha^{\star})=A^2\overline{X}^{\gamma}\lambda^2/4$.

We show in Appendix A.8 that the within-firm variance of wages is given by the following expression:

$$\mathrm{Var}[w|\alpha^\star]=\frac{A^4\lambda^4}{128}(1-\alpha^\star)\left(\overline{X}^\gamma-\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2. \tag{5.26}$$

Note that the within-firm variance can increase or decrease in γ, since although the second term in parenthesis, which is proportional to $(\overline{w}-\underline{w})^2$, increases in γ, the change is tempered by the decrease in $1-\alpha^\star$ when γ increases. In particular, if $\underline{x}=0$, $\overline{x}=1$, and $\lambda=1$, the within-firm variance of wages is equal to $(1-\alpha^\star)2^{2\gamma}=(3-2^{\frac{1}{2}(\gamma+1)})2^{2\gamma}$ times $A^4/128$, where we have used (5.25). Then the variance is strictly concave and non-monotone in γ, first increasing for values of γ near 1, and then decreasing after reaching a peak at $\gamma\cong 1.52$. Thus, for values of γ near the peak, the within-firm variance barely changes. The more general insight is that in this set up, *the within-firm variance need not change much with an increase in* γ.

Let us turn now to the between-firms variance of wages. To compute it, we need to take into account that the fraction of PAM/NAM teams changes with γ. We show in Appendix A.8 that the variance between firms is:

$$\mathrm{Var}[w_i+w_j|\alpha^\star]=\frac{A^4\lambda^4}{128}\left(\frac{\alpha^\star}{2}\left(3\overline{X}^\gamma-\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2+\frac{\alpha^\star}{2}\left(\overline{X}^\gamma-3\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right.$$
$$\left.+(1-\alpha^\star)\left(\overline{X}^\gamma+\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right). \tag{5.27}$$

To see that it is easy to generate cases where this variance is strictly increasing, assume $\underline{x}=0$, so that $\underline{X}=0$, and $\overline{x}>1/2$, so that $\overline{X}>1$. Then simple algebra yields

$$\mathrm{Var}[w_i+w_j|\alpha^\star]=\frac{A^4\lambda^4}{256}\overline{X}^{2\gamma}(8\alpha^\star+2),$$

which is clearly strictly increasing and strictly convex in γ. By continuity, the same holds for $\underline{x}$ small.[29] The more general insight is that in this set up, *the between-firm variance changes significantly with an increase in* γ.

Similar insights obtain if we consider the effect of changes in γ on the within-firm and between-firm inequality in *skill* instead of wages. Indeed, the variance of skills across teams increases with technological change, that is, with an increase in γ. In turn, the variance of skills within firms decreases but by a significantly smaller amount.[30] This comparative statics result accords well with the evidence that most of the increase in skill inequality can be explained by the increase in between-firm inequality, with little impact on within-firm inequality.

The planner's solution in Appendix A.9 also features an interior solution for α^p, with $\alpha^p<\alpha^\star$, so the competitive equilibrium is inefficient. Moreover, the optimal α^p is strictly increasing in the

29. Replacing $\alpha^\star$ by its closed form expression and calculating the solution numerically reveals that the increase in between-firm variance when γ increases is a general property.

30. A similar derivation as in Appendix A.8 reveals that the between-firm variance of skills increases by $0.5(\overline{x}-\underline{x})^2(\partial\alpha^\star/\partial\gamma)$, while the within-firm variance decreases by just $0.125(\overline{x}-\underline{x})^2(\partial\alpha^\star/\partial\gamma)$.

complementarity γ: technological change induces the planner to choose strictly more positive sorting.

To close, we stress that these insights cannot obtain in a frictionless matching model without externalities (Becker, 1973). To see this, note first that without externalities sorting is not affected at all by a marginal increase in complementarities, except in the knife-edge case where technology switches from supermodular to submodular, and hence the allocation switches from PAM to NAM. Our model can have equilibria with stochastic matching, where technological change has a smooth impact on the allocation towards more positive or negative sorting. Second, although under PAM and no externalities between-firm wage inequality increases with complementarities—simply due to higher firms having more surplus to distribute—there is zero within-firm variance of wages. In the competitive equilibrium with stochastic matching analysed above, within-firm variance of wages is strictly positive. Moreover, it is *a priori* unclear how both between- and within-firm variance of wages behave, since one needs to deal with the additional effect that team composition changes with a change in complementarities. Third, with PAM and no externalities there is no between- or within-firm skill inequality, something that our model delivers. Finally, without externalities inequality is efficient, while in our case it is not.

5.2. *Market power*

Our second application is to oligopoly. Intuitively, in an oligopoly the profits of a firm with a given workforce composition depend on the workforce composition of all the other firms—since workers' skills affect the firms' marginal cost of production. This downstream oligopolistic competition leads to a matching problem with externalities in the team formation stage. We will illustrate how the addition of this matching stage in which firms hire their workers in a competitive market can affect firms' market power downstream, measured by the markup of price over marginal cost. Recent evidence establishes that this measure has risen steeply in the last few decades. Most of the rise in average markups is driven by an increase in the upper percentiles, *i.e.,* due to higher dispersion and more skewness in the markup distribution (see De Loecker and Eeckhout (2017)). We will see that, in the equilibrium with PAM that we derive, an increase in production complementarities between workers lead to higher and more spread out markups, especially near the top, which is in line with the evidence.

Since firms in most oligopolistic markets know who their competitors are, we assume that teams are deterministically assigned in stage two as described in Section 3.3. Hence, spillovers and market power arise in narrowly defined sectors, yet firms hire on the economy-wide labour market. Indeed, this is precisely the case in markets where firms have market power and compete in a specific product market, say Coca Cola and Pepsi in soft drinks, and Visa and MasterCard in credit cards, with all these firms competing in the upstream labour market when hiring marketing and sales professionals.

We analyse a market structure with a large number of sectors each with two firms that in the second stage compete à la Cournot in a product market; in the first stage, they hire skilled labour in a competitive labour market with heterogeneous workers with characteristic $x \in [0,1]$ distributed with continuous cdf F. More precisely, we assume that half of the agents of each type are initially assigned according to PAM with a future competitor (see Section 3.3). Hence, there is a continuum of "locations" containing pairs of "firms" (each with one agent) that compete downstream à la Cournot. After this initial stage, each firm hires a "partner" in a competitive market. At the end of this stage, in each location there will be a pair of firms, each with two agents.

In each location, the demand for the product is linear, given by $p=a-b(q_i+q_j)$, with $a>0$ and $b>0$, where q_i and q_j are the outputs of the two firms.[31] The cost of production for firm $k=i,j$ when the output level q_k and the firm composition is (x_k,x'_k) is given by $C(x_k,x'_k,q_k)=c(x_k,x'_k)q_k$, where (x_k,x'_k) is the workforce composition of firm $k=i,j$. As a result, each firm $k=i,j$ maximizes $pq_k-c(x_k,x'_k)q_k$ with respect to q_k. The cost-per-unit function c is given by $c(x_k,x'_k)=\nu-\beta x_k x'_k$, with $\nu>\beta\bar{x}^2$, $\beta>0$. That is, firms with better team composition (higher x_k and x'_k) have lower marginal cost. Moreover, c is strictly submodular, that is, $c_{12}=-\beta$, with "degree" of submodularity indexed by β. To ensure interior solutions we will assume that $a>2c(0,0)$.

As is well-known, we obtain that in any given sector the unique Nash equilibrium quantities are $q_i=(a-2c(x_i,x'_i)+c(x_j,x'_j))/(3b)$ and $q_j=(a-2c(x_j,x'_j)+c(x_i,x'_i))/(3b)$, with equilibrium price $p=(a+c(x_i,x'_i)+c(x_j,x'_j))/3$. The profits of the two firms are given by

$$V(x_i,x'_i|x_j,x'_j)=\frac{(a-2c(x_i,x'_i)+c(x_j,x'_j))^2}{9b}=\frac{(a-2(\nu-\beta x_i x'_i)+\nu-\beta x_j x'_j)^2}{9b} \tag{5.28}$$

$$V(x_j,x'_j|x_i,x'_i)=\frac{(a-2c(x_j,x'_j)+c(x_i,x'_i))^2}{9b}=\frac{(a-2(\nu-\beta x_j x'_j)+\nu-\beta x_i x'_i)^2}{9b}. \tag{5.29}$$

At the matching stage, in a competitive equilibrium with PAM we have that $\mu_+(x)=x$, that is, each team (x,x) is paired with an identical team (since η is PAM too), and wages are given by $w(x)=w(0)+\int_0^x V_2(s,s|s,s)ds$. It follows from (5.28) to (5.29) that $V_2(x,x|x,x)=-4c_2(x,x)(a-c(x,x))/(9b)>0$. Hence, equilibrium wages are equal to

$$\begin{aligned} w(x)&=w(0)-\frac{4}{9b}\int_0^x c_2(s,s)(a-c(s,s))ds \\ &=w(0)+\frac{4\beta}{9b}\int_0^x s(a-\nu+\beta s^2)ds \\ &=w(0)+\frac{4\beta}{9b}\left((a-\nu)\frac{x^2}{2}+\beta\frac{x^4}{4}\right), \end{aligned}$$

where the second equality uses the functional form of c, and the third follows by integration.

The following result shows that a PAM equilibrium exists and describes some equilibrium properties.

Proposition 6 *If a is large enough, then there exists a competitive equilibrium with PAM. Wages increase in a and decrease in b, and firms with better composition of their labour force set higher markups.*

The bound on a is derived in Appendix A.10, and it ensures that the downstream market demand is large enough to encourage bidding for the best workers in the first stage.[32]

Regarding the properties of the competitive equilibrium in the matching stage, it is immediate that the wage function is strictly increasing and strictly convex in x, strictly increasing in a—a

31. We assume that a and b is the same across sectors but we could allow for sector-specific demand parameters.

32. We also show in Appendix A.10 that if a is large enough, then the planner prefers PAM to any convex combination of PAM and NAM, providing a "partial" efficiency result.

larger downstream market increases the incentives to hire the right workers and this drives wages up—and strictly decreasing in b —loosely, a less sensitive demand reduces the profitability of each sector, and thus of each firm in each sector, and this lowers wages since the marginal revenue of hiring someone of a better type decreases.

Turning to markups in the downstream market, under PAM sectors where firms have better labour force composition charge higher markups. To see this, let ϵ be the price-elasticity of demand in a given location in the Nash equilibrium of each sector. It is easy to show that at the equilibrium price and total quantity produced it is given by $\epsilon(x) = -0.5(a+2c(x,x))/(a-c(x,x))$. The Lerner index ϱ of each firm is then

$$\varrho(x) \equiv \frac{p-c(x,x)}{p} = -\frac{1}{2\epsilon(x)} = \frac{a-\nu+\beta x^2}{a+2\nu-2\beta x^2}.$$

Intuitively, $\varrho_x > 0$, so firms in sectors with better workforce composition—which have lower marginal cost of production—set strictly higher markups.

But we can say much more. Differentiating once again with respect to x one verifies that $\varrho_{xx} > 0$, and so markups are strictly convex in workforce composition, increasing faster for firms with better workers. More interestingly, $\varrho_\beta > 0$ and $\varrho_{\beta x} > 0$, so markups are *strictly increasing* in workers' complementarities, and this increase is more pronounced for firms with better skill composition. Since ϱ is strictly increasing and strictly convex in x, it follows that markups become more *spread out* as β increases, and more so for *better firms*. Indeed, the variance of markups, $\mathrm{Var}(\varrho)$, strictly increases in β. To see this, by definition

$$\mathrm{Var}(\varrho) = \int_0^1 \varrho(x)^2 dF(x) - \left(\int_0^1 \varrho(x) dF(x) \right)^2,$$

and it follows by differentiation with respect to β that $\partial \mathrm{Var}(\varrho)/\partial\beta > 0$ if and only if

$$\int_0^1 \varrho(x)\varrho_\beta(x) dF(x) - \int_0^1 \varrho(x) dF(x) \int_0^1 \varrho_\beta(x) dF(x) > 0,$$

which holds by Chebyshev's order inequality since both ϱ and ϱ_β are strictly increasing in x.[33]

In short, technological change in the form of an increase in production complementarities can lead to changes in market power, as measured by markups, that accord well with the evidence cited above. This provides a novel mechanism that can qualitatively explain the increase in market power, especially in the upper tail of the distribution of firms, and the increase in the variance of markups. Moreover, it has the potential to be testable.

To close this section, we note that these features would also emerge if instead of a duopoly there was a monopoly in each location, which reduces the problem to a standard matching model without externalities. Thus, the main role of this application is to uncover the mechanism described and to illustrate that these insights extend to a more realistic model with oligopolistic competition, which calls for our framework with externalities.[34]

33. Chebyshev's order inequality asserts that if $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous and increasing, then $\mathbb{E}[f(x)g(x)] \geq \mathbb{E}[f(x)]\mathbb{E}[g(x)]$, with strict inequality if both functions are strictly increasing.

34. There are some differences though. For example, if one compares the behaviour of markups in the monopoly case and in our case, one can show that the effects of changes in complementarities are more pronounced near the top in our case than under monopoly. This could be useful for applied work since a big part of the changes have taken place in the upper tail of the distribution.

5.3. *Other applications*

There are several additional economic applications. We briefly discuss two that we have explored.

The first one is in Appendix A.11, where we analyse a model with spillovers from a patent race with a continuum of characteristics. This is a variation of the knowledge spillover model with copying in Section 5.1. The continuum setup allows for a calculus-based solution of an equilibrium with PAM. We show that it solves a differential equation derived from the interaction of the two stages in the model. The analysis serves as an illustration of how to construct a competitive equilibrium in this more complex setup with aggregate spillovers.

Another application we have explored — we omit the details in this version —, is the design of competitions of sports teams, and the impact of policy interventions. A sports league where first teams form, and then they compete facing each team a few times, fits well our setting with random assignment of competing teams. These markets have a zero-sum contest aspect where teams exert a negative externality on each other, which creates inefficiencies at the team formation stage. Market forces lead to PAM, while the optimal matching might entail NAM, which implies having a diverse set of players in each team. One can then study policy interventions that mitigate these inefficiencies, such as tax/subsidy schemes, salary caps, and the rookie draft that is a common practice in the U.S. In particular, the last one can be quite effective if the team with lowest characteristics gets to choose first.

6. CONCLUSION

In many market settings, the presumption that firms and teams operate in isolated output markets is tenuous. Often, there is strategic interaction between competing teams, for example due to knowledge spillovers, market power, or patents. This generates externalities and has implications for the labour market. While it is well known that the inefficiencies in the output market affect the optimal provision of effort, in this article, we argue that they also affect the composition of skilled workers in teams.

Our analysis reveals that the features of competitive equilibria in matching with externalities differ in several dimensions from the standard matching model that is a workhorse in economic applications. In particular, we show that there can be multiple equilibria with varying sorting patterns; both optimal and equilibrium matching can involve randomization; equilibrium can be inefficient with a matching that can drastically deviate from the optimal one; and match complementarities interact with externalities to determine sorting.

We derive these results under different assumptions regarding heterogeneity of agents (finite number of characteristics or a continuum), and also under different forms in which teams can compete downstream (aggregate spillovers, random assignment, and deterministic assignment of teams). Hence, our model encompasses a large variety of economic settings with matching.

In addition to these insights, we argue that our framework is economically relevant. We show that a version of the model with general knowledge spillovers can account for the recently observed empirical fact that that the rise in wage inequality is mainly driven by between-firm inequality rather than within-firm inequality. Our model combines the effect of general knowledge spillovers on the firm size distribution with within-firm complementarities, which generates increased between-firm sorting, and hence the predicted effect on between- and within-firm wage inequality, and similarly for skill inequality. We believe this economic insight is novel and important. We also show how sorting can affect markups in an oligopolistic output market, thus providing a rationale for the empirically observed evolution of the distribution of markups.

Although there are many open questions for future research, we only mention three that seem important. Obviously, it would be interesting to have a full characterization of the planner's problem and competitive equilibrium with stochastic matching when there is a continuum of characteristics. Also, analysing the model under imperfectly transferable utility would enlarge the number of economic applications (*e.g.* contracting problems in market settings) that can be analysed with externalities. Finally, we have abstracted from search frictions that are important in some labour markets, and that would be of interest incorporating in the formation of teams.

We can also see at least two directions for empirical work. First, with sufficiently detailed data on team composition (research teams, sports teams, class rooms, etc.) and individual performance, one could identify the nature of externalities in conjunction with the nature of complementarities. Clearly, if a market setting is estimated with a model without externalities, the obtained estimates for complementarities will be biased. Second, the model can be estimated using wage data. Wages reflect the allocation and if the allocation is inefficient, this will be evident in the wage distribution. Even if the allocation is efficient (say PAM in equilibrium as well as PAM by the planner), wages nonetheless will incorporate the inefficiency and will not be set at private marginal product. Data on markups in output markets for example will therefore give an indication of the extent of the externality, and as a consequence of the extent to which wages are set inefficiently.

A. APPENDIX

A.1. *Proof of Proposition 1*

We will first prove the result for the case of *ex post* random assignment of competing teams and aggregate spillovers (cases (1.*i*) and (2) in Section 3.3), since in these cases $\mathcal{V}(\cdot,\cdot|\alpha)$ is symmetric in (x,x').

Assume $\Gamma(1)\geq 0$. We will construct an equilibrium with PAM. Inequalities (4.7)–(4.8) reveal that we need to find $\overline{w}$ and $\underline{w}$ that satisfy them. Set $\overline{w}=0.5\mathcal{V}(\overline{x},\overline{x}|1)$ and $\underline{w}=0.5\mathcal{V}\left(\underline{x},\underline{x}|1\right)$. These wages satisfy (4.7)–(4.8) and yield a positive payoff to both $\underline{x}$ and $\overline{x}$. Hence, we have constructed a competitive equilibrium with PAM.

If $\Gamma(0)\leq 0$, then to construct an equilibrium with NAM, we will set wages that satisfy inequalities (4.9)–(4.10), yield positive payoffs to both $\underline{x}$ and $\overline{x}$, and are such that $\overline{w}+\underline{w}=\mathcal{V}(\underline{x},\overline{x}|0)$. If $\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\underline{x},\underline{x}|0)\geq 0$, then it is easy to verify that $\underline{w}=0.5\mathcal{V}\left(\underline{x},\underline{x}|0\right)$ and $\overline{w}=\mathcal{V}(\underline{x},\overline{x}|0)-\underline{w}$ satisfy (4.9)–(4.10) and provide a positive payoff to both $\underline{x}$ and $\overline{x}$. If $\mathcal{V}(\underline{x},\overline{x}|0)-\mathcal{V}(\underline{x},\underline{x}|0)<0$, then $\overline{w}=0.5\mathcal{V}(\overline{x},\overline{x}|0)$ and $\underline{w}=\mathcal{V}(\underline{x},\overline{x}|0)-\overline{w}$ do the job. Thus, in each case these wages along with $\alpha=0$ constitute a competitive equilibrium with NAM.

Assume that $\Gamma(\alpha)=0$ for some $0<\alpha<1$. Then (4.11)–(4.12) imply $\overline{w}-\underline{w}=\mathcal{V}(\overline{x},\overline{x}|\alpha)-\mathcal{V}(\overline{x},\underline{x}|\alpha)=\mathcal{V}(\underline{x},\overline{x}|\alpha)-\mathcal{V}(\underline{x},\underline{x}|\alpha)$. If $\overline{w}=0.5\mathcal{V}(\overline{x},\overline{x}|\alpha)$ and $\underline{w}=0.5\mathcal{V}(\underline{x},\underline{x}|\alpha)$, then the incentive constraints are satisfied with equality (using that under random assignment or aggregate spillovers $\mathcal{V}(\overline{x},\underline{x}|\alpha)=\mathcal{V}(\underline{x},\overline{x}|\alpha)$) and an agent with $\underline{x}$ receive the same positive payoff if the agent hires another $\underline{x}$ or an $\overline{x}$, and similarly for an agent with $\overline{x}$. Hence, these wages along with the interior matching α constitute a competitive equilibrium.

Consider now *ex ante* deterministic assignment of teams (case (1.*ii*) in Section 3.3) where η is PAM.

First, we examine the case where workers are matched according to PAM. The incentive constraints are

$$V(\overline{x},\overline{x}|\overline{x},\overline{x})-\overline{w}\geq V(\overline{x},\underline{x}|\overline{x},\overline{x})-\underline{w} \tag{A.1}$$

$$V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)-\underline{w}\geq V(\underline{x},\overline{x}|\underline{x},\underline{x})-\overline{w}, \tag{A.2}$$

and the necessary condition $\Gamma(1)\geq 0$ is now $V(\overline{x},\overline{x}|\overline{x},\overline{x})+V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)-V(\overline{x},\underline{x}|\overline{x},\overline{x})-V(\underline{x},\overline{x}|\underline{x},\underline{x})\geq 0$. If $V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)\geq 0$, then it is easy to verify that $\overline{w}=\underline{w}+V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)$ and any $\underline{w}\in[0,\min\{V(\underline{x},\underline{x}|\underline{x},\underline{x}),V(\overline{x},\underline{x}|\overline{x},\overline{x})\}]$ satisfy (A.1)–(A.2) and yields positive payoffs to both $\underline{x}$ and $\overline{x}$. If $V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)<0$ and $V(\overline{x},\overline{x}|\overline{x},\overline{x})-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)\geq 0$, then $\overline{w}=V(\overline{x},\overline{x}|\overline{x},\overline{x})-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)$ and $\underline{w}=0$ satisfy all the constraints. And if $V(\underline{x},\overline{x}|\underline{x},\underline{x})-V\left(\underline{x},\underline{x}|\underline{x},\underline{x}\right)<0$ and $V(\overline{x},\overline{x}|\overline{x},\overline{x})-V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)<0$, then $\underline{w}=\overline{w}+V\left(\overline{x},\underline{x}|\overline{x},\overline{x}\right)-V(\overline{x},\overline{x}|\overline{x},\overline{x})$ and any $\overline{w}\in[0,\min\{V(\overline{x},\overline{x}|\overline{x},\overline{x}),V(\underline{x},\overline{x}|\underline{x},\underline{x})\}]$ satisfy all the constraints. Hence, a competitive equilibrium with PAM exists when $\Gamma(1)\geq 0$.

Consider now a first stage matching according to NAM. Then the incentive constraints are

$$V(\overline{x},\underline{x}|\overline{x},\underline{x})-\underline{w}\geq V\left(\overline{x},\overline{x}|\overline{x},\underline{x}\right)-\overline{w}$$

$$V(\underline{x},\overline{x}|\underline{x},\overline{x})-\overline{w}\geq V\left(\underline{x},\underline{x}|\underline{x},\overline{x}\right)-\underline{w},$$

and the necessary condition $\Gamma(0)\leq 0$ becomes $V\left(\overline{x},\overline{x}|\overline{x},\underline{x}\right)+V\left(\underline{x},\underline{x}|\underline{x},\overline{x}\right)-V\left(\overline{x},\underline{x}|\overline{x},\underline{x}\right)-V(\underline{x},\overline{x}|\underline{x},\overline{x})\leq 0$. Since every team competes with a mixed team in this case, and the function V is symmetric in its first and second argument as well as in its third and fourth, it follows that the analysis of the NAM case is analogous to the one above for random assignment and aggregate spillovers: simply replace $\mathcal{V}(x,x'|0)$ by $V(x,x'|\underline{x},\overline{x})$ in the construction of the wages. Hence, a competitive equilibrium with NAM exists in this case if $\Gamma(0)\leq 0$.

Suppose now that $\Gamma(\alpha)=0$ for some $0<\alpha<1$, where $\Gamma(\alpha)=\mathcal{V}(\overline{x},\overline{x}|\alpha)+\mathcal{V}(\underline{x},\overline{x}|\alpha)-\mathcal{V}(\overline{x},\underline{x}|\alpha)-\mathcal{V}(\underline{x},\overline{x}|\alpha)$, and $\mathcal{V}(\overline{x},\overline{x}|\alpha)=\alpha V(\overline{x},\overline{x}|\overline{x},\overline{x})+(1-\alpha)V(\overline{x},\overline{x}|\overline{x},\underline{x})$, $\mathcal{V}(\underline{x},\underline{x}|\alpha)=\alpha V(\underline{x},\underline{x}|\underline{x},\underline{x})+(1-\alpha)V(\underline{x},\underline{x}|\underline{x},\overline{x})$, $\mathcal{V}(\overline{x},\underline{x}|\alpha)=\alpha V(\overline{x},\underline{x}|\overline{x},\overline{x})+(1-\alpha)V(\overline{x},\underline{x}|\overline{x},\underline{x})$, and $\mathcal{V}(\underline{x},\overline{x}|\alpha)=\alpha V(\underline{x},\overline{x}|\underline{x},\underline{x})+(1-\alpha)V(\underline{x},\overline{x}|\underline{x},\overline{x})$. From the incentive constraints of agents of each characteristic, which must hold with equality for them to be willing to randomize, it follows that $\overline{w}-\underline{w}=\mathcal{V}(\overline{x},\overline{x}|\alpha)-\mathcal{V}(\overline{x},\underline{x}|\alpha)=\mathcal{V}(\underline{x},\overline{x}|\alpha)-\mathcal{V}(\underline{x},\underline{x}|\alpha)$. If these differences are positive, then $\overline{w}=\underline{w}+\mathcal{V}(\overline{x},\overline{x}|\alpha)-V(\overline{x},\underline{x}|\alpha)=\underline{w}+\mathcal{V}(\underline{x},\overline{x}|\alpha)-\mathcal{V}(\underline{x},\underline{x}|\alpha)$ and $\underline{w}\in[0,\min\{\mathcal{V}(\underline{x},\underline{x}|\alpha),\mathcal{V}(\overline{x},\underline{x}|\alpha)\}]$ satisfy all the constraints and yield the same positive payoff to an agent with $\underline{x}$ no matter who the agent ends up hiring, and similarly for an agent with $\overline{x}$. If instead these differences are negative, then set $\underline{w}=\overline{w}+V(\overline{x},\underline{x}|\alpha)-\mathcal{V}(\overline{x},\overline{x}|\alpha)=\overline{w}+\mathcal{V}(\underline{x},\underline{x}|\alpha)-\mathcal{V}(\underline{x},\overline{x}|\alpha)$ and $\overline{w}\in[0,\min\{\mathcal{V}(\overline{x},\overline{x}|\alpha),\mathcal{V}(\underline{x},\overline{x}|\alpha)\}]$. In each case, these wages along with matching $\alpha\in(0,1)$ constitute a competitive equilibrium with stochastic matching.

We have thus proven that if either $\Gamma(1)\geq 0$ or $\Gamma(0)\leq 0$, then a competitive equilibrium exists. The only case remaining is $\Gamma(1)<0$ and $\Gamma(0)>0$. Since Γ is continuous in α, it follows from the Intermediate Value Theorem there is an $\alpha\in(0,1)$ such that $\Gamma(\alpha)=0$, so a competitive equilibrium exists by the construction above. □

A.2. *Proof of Proposition 2*

(*i*) The objective function is convex if $A\geq 0$. If it is strictly convex (or linear with $B\neq 0$), then the optimal solution is at a corner, so $\alpha^P\in\{0,1\}$, and which corner depends on whether $0.5(A+B)+C=0.5(\mathcal{V}(\overline{x},\overline{x}|1)+\mathcal{V}(\underline{x},\underline{x}|0))$ is bigger or smaller than $C=0.5(\mathcal{V}(\underline{x},\overline{x}|0)+\mathcal{V}(\overline{x},\underline{x}|0))$, that is, the comparison of the value of the aggregate expected output under PAM and NAM. This reduces to $A+B$ bigger than or less than zero.

(*ii*)–(*iii*) A necessary condition for an interior solution is that $A<0$, so the planner's objective is strictly concave. But this is not sufficient since the solution can still be at a corner. If $B\leq 0$, then the planner's objective peaks at $\alpha^P=0$ and NAM is optimal, while if $B+2A\geq 0$ then it peaks at $\alpha^P=1$ and PAM is optimal.

(*iv*) If $A<0$, $B>0$, and $B+2A<0$, then the planner's objective function is strictly concave and peaks at the interior value $\alpha^P=-B/2A$. Hence, an interior matching is optimal. □

A.3. *The ternary case*

In this section, we will describe the model with three characteristics, derive the incentive constraints that define competitive equilibrium, and set up the planner's problem. The analysis suggests that the main insights obtained in the binary case carry over to this case, especially regarding the possibility of a competitive equilibrium with stochastic matching, as well as multiplicity and inefficiency of competitive equilibrium. We will illustrate the results using a couple of examples similar to the ones presented in Section 2.

Assume that $x\in\{\underline{x},\hat{x},\overline{x}\}$, with $\underline{x}<\hat{x}<\overline{x}$, uniformly distributed (*i.e.* a measure $1/3$ of the agents has characteristic $\underline{x}$, $\hat{x}$, $\overline{x}$, respectively). In this setting, there are four possible (deterministic) matchings to form a measure $1/2$ of teams: μ_1, where a measure $1/6$ of teams have composition $\underline{x}\underline{x}$, $1/6$ have $\hat{x}\hat{x}$, and $1/6$ have $\overline{x}\overline{x}$; μ_2, where a measure $2/6$ of teams have composition $\underline{x}\overline{x}$ and $1/6$ have $\hat{x}\hat{x}$; μ_3, where a measure $2/6$ of teams have composition $\underline{x}\hat{x}$ and $1/6$ have $\overline{x}\overline{x}$; and μ_4, where a measure $1/6$ of teams have composition $\underline{x}\underline{x}$ and $2/6$ have $\hat{x}\overline{x}$.

A stochastic matching is a vector $\pi=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $\alpha_i\in[0,1]$, $i=1,2,3,4$, $\sum_i\alpha_i=1$, where α_i is the fraction of the population that matches according to μ_i, $i=1,2,3,4$. Under π, there is a fraction $(\alpha_1+\alpha_4)/6$ of teams $\underline{x}\underline{x}$, $(\alpha_1+\alpha_2)/6$ with $\hat{x}\hat{x}$, $(\alpha_1+\alpha_3)/6$ with $\overline{x}\overline{x}$, $\alpha_2/6$ with $\underline{x}\overline{x}$, $\alpha_3/6$ with $\underline{x}\hat{x}$, and $\alpha_4/6$ with $\hat{x}\overline{x}$.

Let $\mathcal{V}(x,x'|\pi)$ be the expected match output of a team with composition xx' when the matching is π, and similarly, $\mathcal{V}(x,x'|\mu_i)$ is the corresponding expected output when the matching is μ_i, $i=1,2,3,4$. For example, in the case of random assignment of competing teams, we have,

$$\mathcal{V}(x,x'|\pi)=\frac{\alpha_1+\alpha_4}{3}\mathcal{V}(x,x'|\underline{x},\underline{x})+\frac{\alpha_1+\alpha_2}{3}\mathcal{V}(x,x'|\hat{x},\hat{x})+\frac{\alpha_1+\alpha_3}{3}\mathcal{V}(x,x'|\overline{x},\overline{x})+\frac{2\alpha_2}{3}\mathcal{V}(x,x'|\underline{x},\overline{x})$$
$$+\frac{2\alpha_3}{3}\mathcal{V}(x,x'|\underline{x},\hat{x})+\frac{2\alpha_4}{3}\mathcal{V}(x,x'|\hat{x},\overline{x}) \tag{A.3}$$

while with aggregate spillovers and multiplicatively separable match output, we have $\mathcal{V}(x,x'|\pi)=\ell(\pi)k(x,x')$, where we could allow ℓ to vary with team composition, as in Section 5.1.

To derive a competitive equilibrium, we need to specify a matching and wages $\underline{w}$, $\hat{w}$, and $\overline{w}$, corresponding with hiring a partner with $\underline{x}$, $\hat{x}$, and $\overline{x}$. A competitive equilibrium with deterministic matching μ_i must satisfy the appropriate incentive constraints. For instance, if the matching is μ_1, the incentive constraints are

$$\mathcal{V}(\underline{x},\underline{x}|\mu_1)-\underline{w}\geq\mathcal{V}(\underline{x},\hat{x}|\mu_1)-\hat{w} \tag{A.4}$$

$$\mathcal{V}(\underline{x},\underline{x}|\mu_1)-\underline{w}\geq\mathcal{V}(\underline{x},\overline{x}|\mu_1)-\overline{w} \tag{A.5}$$

$$\mathcal{V}(\hat{x},\hat{x}|\mu_1)-\hat{w}\geq\mathcal{V}(\hat{x},\underline{x}|\mu_1)-\underline{w} \tag{A.6}$$

$$\mathcal{V}(\hat{x},\hat{x}|\mu_1)-\hat{w}\geq\mathcal{V}(\hat{x},\overline{x}|\mu_1)-\overline{w} \tag{A.7}$$

$$\mathcal{V}(\overline{x},\overline{x}|\mu_1)-\overline{w}\geq\mathcal{V}(\overline{x},\underline{x}|\mu_1)-\underline{w} \tag{A.8}$$

$$\mathcal{V}(\overline{x},\overline{x}|\mu_1)-\overline{w}\geq\mathcal{V}(\overline{x},\hat{x}|\mu_1)-\hat{w}. \tag{A.9}$$

and similarly for matchings μ_2, μ_3, and μ_4. As in the binary case under PAM, one can show that with μ_1, $\mathcal{V}(\cdot,\cdot|\mu_1)$ supermodular in (x,x') is necessary and sufficient for the existence of a competitive equilibrium. Similarly, with μ_2 the condition is $\mathcal{V}(\cdot,\cdot|\mu_2)$ submodular in (x,x'). As in the multiplicative example in the binary case, it is easy to construct a similar example with a PAM or NAM equilibrium and also illustrate that both can coexist if the aggregate externality switches the complementarity property of $\mathcal{V}$. We omit the details to avoid repetition.

Regarding a stochastic matching π, agents must be given incentives to randomize and thus the relevant incentive constraints must hold with equality. In particular, a competitive equilibrium with $\pi\gg 0$ requires that all the inequalities above hold as equalities, and this obtains if and only if $\mathcal{V}(\cdot,\cdot|\pi)$ is modular in (x,x'). Indeed, any interior solution $\pi\gg 0$ that solves the system of four equations in four unknowns $(\alpha_i,\ i=1,2,3,4)$

$$\mathcal{V}(\underline{x},\underline{x}|\pi)+\mathcal{V}(\hat{x},\hat{x}|\pi)=2\mathcal{V}(\underline{x},\hat{x}|\pi)$$

$$\mathcal{V}(\underline{x},\underline{x}|\pi)+\mathcal{V}(\overline{x},\overline{x}|\pi)=2\mathcal{V}(\underline{x},\overline{x}|\pi)$$

$$\mathcal{V}(\hat{x},\hat{x}|\pi)+\mathcal{V}(\overline{x},\overline{x}|\pi)=2\mathcal{V}(\hat{x},\overline{x}|\pi)$$

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=1,$$

is a competitive equilibrium with appropriately chosen wages.

For a simple example, assume random assignment of competing teams and also that a team obtains a payoff 1 if it is assigned to an identical team and 0 otherwise. We first claim that $\alpha_1=1$, and thus matching μ_1, is a competitive equilibrium with wages $(\underline{w},\hat{w},\overline{w})=(\frac{1}{6},\frac{1}{6},\frac{1}{6})$. To see this, note that $\mathcal{V}(\cdot,\cdot|\mu_1)$ is strictly supermodular. Even easier, note that all the inequalities (A.4)–(A.9) are slack with the wages assumed (the left side of each is $1/3-1/6=1/6$ and the right side is $-1/6$). Thus, a competitive equilibrium with PAM exists. We next claim that $\pi=(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ along with wages $(\underline{w},\hat{w},\overline{w})=(\frac{1}{12},\frac{1}{12},\frac{1}{12})$ is a competitive equilibrium with stochastic matching. To see this, note that under π the probability of being assigned to a team of the same composition is $1/6$ for any team. Hence, π solves the system of equations above, and the wages (half of the output for each member) ensures that each agent is indifferent regarding whom to hire and thus willing to randomize, and payoffs are positive. As a result, a competitive equilibrium with *stochastic matching* ensues. Since this equilibrium coexists with the PAM one, it follows that there are *multiple* competitive equilibria.

The planner's problem is as follows:

$$\max_{\pi}\frac{1}{2}\left(\frac{\alpha_1+\alpha_4}{3}\mathcal{V}(\underline{x},\underline{x}|\pi)+\frac{\alpha_1+\alpha_2}{3}\mathcal{V}(\hat{x},\hat{x}|\pi)+\frac{\alpha_1+\alpha_3}{3}\mathcal{V}(\overline{x},\overline{x}|\pi)+\frac{2\alpha_2}{3}\mathcal{V}(\underline{x},\overline{x}|\pi)+\frac{2\alpha_3}{3}\mathcal{V}(\underline{x},\hat{x}|\pi)+\frac{2\alpha_4}{3}\mathcal{V}(\hat{x},\overline{x}|\pi)\right),$$

subject to $\alpha_i\in[0,1]$, $i=1,2,3,4$, and $\sum_i\alpha_i=1$. If the objective function is strictly quasiconcave and there is an interior π that solve the first-order conditions with respect to α_i, $i=1,2,3,4$, then this is the optimal matching. In some cases, it is relatively easy to pin down the curvature of the planner's objective function. This is the case when competing teams are assigned randomly and $\mathcal{V}$ is given by (3.1). This is because the objective function becomes a quadratic function in π, and hence it is easy to check if it is concave or convex by evaluating the Hessian. The function is strictly concave if the Hessian is negative definite, and thus if the first-order conditions have an interior solution, then it is the efficient matching. Similarly for corner solutions and the strictly convex case. Another tractable case (both for the analysis of competitive equilibria and for the planner's problem), which leads to a quadratic objective function for the planner, is when there are aggregate spillovers and they are linear and multiplicative, and different for different team composition. For instance, let match output be a product of a function of (x,x') and a linear term that depends on the mass of teams whose composition is not (x,x') (similar to the case in Section 5.1), such as $\mathcal{V}(\underline{x},\underline{x}|\pi)=(1-((\alpha_1+\alpha_4)/3)k(\underline{x},\underline{x})$, etc., or the mass that is (x,x') such as $\mathcal{V}(\underline{x},\underline{x}|\pi)=((\alpha_1+\alpha_4)/3)k(\underline{x},\underline{x})$, etc.

For an illustration of the planner's problem, consider the example above with random matching in which a team obtains a payoff 1 if it is assigned to an identical team and 0 otherwise. The planner's objective function is

$$\frac{1}{2}\left(\left(\frac{\alpha_1+\alpha_4}{3}\right)^2+\left(\frac{\alpha_1+\alpha_2}{3}\right)^2+\left(\frac{\alpha_1+\alpha_3}{3}\right)^2+\frac{4\alpha_2^2}{9}+\frac{4\alpha_3^2}{9}+\frac{4\alpha_4^2}{9}\right),$$

which is clearly strictly convex and thus the optimal matching is at a corner. If $\alpha_1=1$, then the planner obtains $3/18$, while if either α_2, α_3, or α_4 is equal to 1, then the planner obtains $5/18$. Hence, a corner solution with either $\alpha_2=1$,

$\alpha_3 = 1$, or $\alpha_4 = 1$ is optimal.[35] Since in this case there are competitive equilibria with stochastic matching and with PAM, it follows that both are *inefficient*.

In short, this example shows that with three types, multiple competitive equilibria can exist, can be inefficient, and can also entail stochastic matching. Note that the example is generalizable beyond the ternary case.

Besides suggesting that all the insights derived in the binary case extend to the ternary case, the main takeaway from this section is that moving from two to a larger but finite number of characteristics presents mainly combinatorial difficulties without providing new insights.[36] For this reason, we consider in the main body of the paper the binary case and the one with a continuum of characteristics, which affords a neat calculus-based approach to analyse the competitive equilibria of the model.

A.4. *Proof of Proposition 4*

Under these conditions, the planner's objective in the restricted problem is strictly concave in α and it is strictly increasing at $\alpha = 0$ and strictly decreasing at $\alpha = 1$. Thus, in the restricted problem the planner does not choose PAM or NAM. All the more when we allow the planner to choose among all possible matchings. □

A.5. *Endogenous assignment of competing teams*

Assume that, after the first stage where teams form, in the second stage teams *choose* the other team they compete with. One way to think about it is as a competitive equilibrium in the second stage where teams take as given the "price of acquiring a competitor" of a given composition. Intuition suggests that this is now a standard Becker-like matching-among-teams problem, and thus the second-stage competitive equilibrium will be efficient (there was a "missing market" that now has been allowed to open). But then in the first stage agents anticipate the equilibrium sorting pattern in the second stage, and hence they know that whoever they choose as a partner affects the incentives to choose a competing team in the second stage. In this way they can internalize the matching externalities, and this can lead to efficiency in the first stage as well.

Proving this assertion in general is beyond the scope of this article, since it would require to have available a full solution for the planner's problem with a continuum of types. But we can use our binary setup to prove one case in detail and then describe how to prove the other cases (which proceed along the same lines).

As in Section 4.1, there is a measure one of agents, half of them with characteristic $\underline{x}$ and half with $\bar{x}$. Assume that matching in the first stage is stochastic and given by $\alpha \in (0,1)$. That is, at the beginning of the second stage there is a measure of $1/2$ of teams, $\alpha/2$ of them has composition $\bar{x}\bar{x}$, $\alpha/2$ has composition $\underline{x}\underline{x}$, and $1-\alpha$ has composition $\underline{x}\bar{x}$. Assume that the "price" of each of these teams is $\bar{t}$, $\underline{t}$, and $\hat{t}$, respectively, and that each team wants to "acquire" a competitor taking these prices as given. That is, there is now a market in which the teams endogenously choose whom to compete with.

For definiteness, we consider a PAM assignment in the second stage. A team with composition $\bar{x}\bar{x}$ is willing to match with another team of the same composition if and only if

$$2V(\bar{x},\bar{x}|\bar{x},\bar{x}) - \bar{t} \geq V(\bar{x},\bar{x}|\underline{x},\bar{x}) + V(\underline{x},\bar{x}|\bar{x},\bar{x}) - \hat{t} \tag{A.10}$$

$$2V(\bar{x},\bar{x}|\bar{x},\bar{x}) - \bar{t} \geq V(\bar{x},\bar{x}|\underline{x},\underline{x}) + V(\underline{x},\underline{x}|\bar{x},\bar{x}) - \underline{t}. \tag{A.11}$$

The explanation of (A.10) is as follows: the total output that the competing teams generate is the sum of the outputs of each of them. So if both teams have composition $\bar{x}\bar{x}$, then total output is $2V(\bar{x},\bar{x}|\bar{x},\bar{x})$. If instead a team with $\bar{x}\bar{x}$ matches with one with $\underline{x}\bar{x}$, total output is $V(\bar{x},\bar{x}|\underline{x},\bar{x}) + V(\underline{x},\bar{x}|\bar{x},\bar{x})$. A similar explanation applies to the (A.11), which compares the gain of matching with another $\bar{x}\bar{x}$ with that of matching with $\underline{x}\underline{x}$.

For teams with $\underline{x}\bar{x}$ and $\underline{x}\underline{x}$ the incentive constraints are

$$2V(\underline{x},\bar{x}|\underline{x},\bar{x}) - \hat{t} \geq V(\underline{x},\bar{x}|\bar{x},\bar{x}) + V(\bar{x},\bar{x}|\underline{x},\bar{x}) - \bar{t} \tag{A.12}$$

$$2V(\underline{x},\bar{x}|\underline{x},\bar{x}) - \hat{t} \geq V(\underline{x},\bar{x}|\underline{x},\underline{x}) + V(\underline{x},\underline{x}|\underline{x},\bar{x}) - \underline{t} \tag{A.13}$$

$$2V(\underline{x},\underline{x}|\underline{x},\underline{x}) - \underline{t} \geq V(\underline{x},\underline{x}|\bar{x},\bar{x}) + V(\bar{x},\bar{x}|\underline{x},\underline{x}) - \bar{t} \tag{A.14}$$

$$2V(\underline{x},\underline{x}|\underline{x},\underline{x}) - \underline{t} \geq V(\underline{x},\underline{x}|\underline{x},\bar{x}) + V(\underline{x},\bar{x}|\underline{x},\underline{x}) - \hat{t} \tag{A.15}$$

35. For a strictly concave example, suppose instead that a team obtains a payoff 1 if assigned to a competing team of different composition. Then it is easy to show that the unique efficient matching is interior with $\pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

36. To see some of the combinatorial hurdles, note that one needs to calculate all the possible partitions of the population subject to the constraint that all agents with the same characteristic are matched to the same type of partner, so as to preserve equal treatment, and then calculate the distribution of different team compositions. With random matching, however, it is still true that the planner's problem will be a nicely behaved quadratic optimization program, for which efficient algorithms exist.

Proceeding in an analogous way as in Section 4.1 (*e.g.* adding incentive constraints (A.10) and (A.12), etc.), one can show that PAM ensues if and only if the following conditions hold:

$$V(\overline{x},\overline{x}|\overline{x},\overline{x})+V(\underline{x},\overline{x}|\underline{x},\overline{x})\geq V(\overline{x},\overline{x}|\underline{x},\overline{x})+V(\underline{x},\overline{x}|\overline{x},\overline{x})$$

$$V(\underline{x},\overline{x}|\underline{x},\overline{x})+V(\underline{x},\underline{x}|\underline{x},\underline{x})\geq V(\overline{x},\overline{x}|\underline{x},\underline{x})+V(\underline{x},\underline{x}|\overline{x},\overline{x})$$

$$V(\underline{x},\overline{x}|\underline{x},\overline{x})+V(\underline{x},\underline{x}|\overline{x},\overline{x})\geq V(\underline{x},\overline{x}|\underline{x},\underline{x})+V(\underline{x},\underline{x}|\underline{x},\overline{x}),$$

which is a supermodularity condition on $V(\cdot|\cdot)$ on the vectors $(\underline{x},\underline{x})<(\underline{x},\overline{x})<(\overline{x},\overline{x})$.

Assume V satisfies this condition and thus there is PAM assignment in the second stage. Then the condition for a stochastic matching α in the first stage to be part of a competitive equilibrium is that each agent is indifferent between matching with an agent of the same or a different characteristic at wages $\underline{w}$ and $\overline{w}$, *taking into account that the assignment is PAM in the second stage.* Formally,

$$V(\overline{x},\overline{x}|\overline{x},\overline{x})-\overline{w}=V(\underline{x},\overline{x}|\underline{x},\overline{x})-\underline{w}$$

$$V(\underline{x},\underline{x}|\underline{x},\underline{x})-\underline{w}=V(\underline{x},\overline{x}|\underline{x},\overline{x})-\overline{w},$$

from which it follows that a necessary and sufficient condition for a competitive equilibrium with stochastic matching α is that V satisfies

$$V(\overline{x},\overline{x}|\overline{x},\overline{x})+V(\underline{x},\underline{x}|\underline{x},\underline{x})=2V(\underline{x},\overline{x}|\underline{x},\overline{x}).$$

But this is *precisely* the condition for the planner—who can only alter matching in the first stage of team formation and takes the second stage assignment as given—to be indifferent between PAM and NAM and any mixture of the two. Thus, the competitive equilibrium with stochastic matching α is efficient in this case. For an illustration, it is immediate to check that all these conditions are satisfied if, as in the motivating example in Section 2, teams get a payoff 1 if they match with an identical team and zero otherwise.

A similar analysis can be peformed for the other combinations of matchings in the first stage and assignments in the second stage, such as PAM in the first stage when teams are formed and PAM in the second stage when competitors endogenously are assigned, or NAM in the first and in the second stage, etc.[37]

A.6. *Proof of Lemma 1*

We first show that $0<\underline{\kappa}<\hat{\kappa}<\overline{\kappa}$. Note that $\overline{\kappa}>\hat{\kappa}$ if and only if $\overline{X}^{\gamma}\lambda>\hat{X}^{\gamma}(\lambda+(\alpha/2))$, which rearranges to $\overline{X}>(1+(\alpha/(2\lambda)))^{\frac{1}{\gamma}}\hat{X}$. Since $\overline{X}=2\overline{x}$ and $\hat{X}=\underline{x}+\overline{x}$, $\overline{k}>\hat{k}$ if and only if

$$\frac{\underline{x}}{\overline{x}}<\frac{2-(1+\frac{\alpha}{2\lambda})^{\frac{1}{\gamma}}}{(1+\frac{\alpha}{2\lambda})^{\frac{1}{\gamma}}}. \tag{A.16}$$

Since $\lambda\geq 1$ and $\gamma\geq 1$, the right side of the inequality is smallest at $\lambda=\gamma=\alpha=1$, and it equals $1/3$. Similarly, $\hat{\kappa}>\underline{\kappa}$ if and only if $\hat{X}^{\gamma}(\lambda+(\alpha/2))>\underline{X}^{\gamma}(\lambda+1-(\alpha/2))$, which rearranges to

$$\frac{\underline{x}}{\overline{x}}<\frac{1}{2\left(\frac{\lambda+1-\frac{\alpha}{2}}{\lambda+\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}}-1}. \tag{A.17}$$

Since $\lambda\geq 1$ and $\gamma\geq 1$, then the right side is smallest when $\lambda=\gamma=1$ and $\alpha=0$, in which case it becomes $1/3$. Hence, $\underline{\kappa}<\hat{\kappa}<\overline{\kappa}$, and it is obvious that $\underline{\kappa}>0$.

To show that these are equilibrium choices in the second stage, consider the choice of $\underline{\kappa}$ for a team with composition $\underline{X}$. Note that $\underline{\kappa}=\text{argmax}_k(\lambda+1-(\alpha/2))-(k^2/2\underline{X}^{\gamma})$; since the objective is strictly concave in k and $\underline{\kappa}<\hat{\kappa}$, it follows that $\underline{\kappa}$ yields a higher payoff to $\underline{X}$ than $\hat{\kappa}$ in the case where the spillover is $1-(\alpha/2)$. But then all the more $\underline{\kappa}$ dominates $\hat{\kappa}$ when the spillover under $\hat{\kappa}$ is $1-(\alpha/2)-(1-\alpha)<1-(\alpha/2)$. Similarly, $\underline{\kappa}$ is a better choice for $\underline{X}$ than $\overline{\kappa}$ when the spillover is $1-(\alpha/2)$, and thus it continues to dominate it if the spillover under $\overline{\kappa}$ is zero. We have thus shown that $\underline{\kappa}$ is an optimal choice for any team with composition $\underline{X}$.

A similar argument proves that $\hat{\kappa}$ is the optimal choice for any team with composition $\hat{X}$. This is because since $\hat{\kappa}=\text{argmax}_k(\lambda+1-(\alpha/2)-(1-\alpha))-(k^2/2\hat{X}^{\gamma})$, the objective is strictly concave in k, and $\hat{\kappa}<\overline{\kappa}$, it follows that $\hat{\kappa}$ yields a higher payoff to $\hat{X}$ than $\overline{\kappa}$ in the case where the spillover is $1-(\alpha/2)-(1-\alpha)$. But then it also dominates it when the spillover under $\overline{\kappa}$ is zero.

Finally, it is straightforward that the optimal choice for any team with $\overline{X}$ is $\overline{\kappa}$, as the spillover is zero no matter what k the team chooses, and $\overline{\kappa}$ is the unconstrained maximum in this case. □

37. The only subtlety in these "corners" is that agents need to conjecture that there will be a positive mass of every team composition in the second stage to make sense of the "right-hand side" of the incentive constraints in the first stage such as, for example, $V(\overline{x},\overline{x}|\overline{x},\overline{x})-\overline{w}\geq V(\underline{x},\overline{x}|\underline{x},\overline{x})-\underline{w}$ (since if PAM ensues in the first stage there will be no teams of composition $\underline{x}\overline{x}$ in the second). This can be formally justified via perturbations or trembles.

A.7. *Proof of Proposition 5*

The analysis in the text reveals that equilibrium is unique. It is interior if and only if $\Gamma(0)>0$ and $\Gamma(1)<0$. The expressions for $\Gamma(0)$ and $\Gamma(1)$ are given by

$$\Gamma(0) = \frac{A^2\overline{X}^{\gamma}\lambda^2}{2}\left(1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{\lambda}\right)^2-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\right)$$

$$\Gamma(1) = \frac{A^2\overline{X}^{\gamma}\lambda^2}{2}\left(1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{2\lambda}\right)^2-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{2\lambda}\right)^2\right).$$

Note that $\Gamma(0)>0$ for all $\gamma\geq 1$ and $\underline{x}/\overline{x}<1$. To see, this it suffices to write the last term of $\Gamma(0)$ as $(1/2)((1+(\underline{x}/\overline{x}))/2)^{\gamma}<1$. Regarding $\Gamma(1)$, it is necessary that the negative term offsets the two positive terms. This holds if $1\leq\gamma<1+2(\log(1+(1/2\lambda))/\log 2)$ and $\underline{x}/\overline{x}$ sufficiently small. Note that when $\underline{x}/\overline{x}=0$, $\Gamma(1)<0$ under the stated condition on γ. By continuity it holds for $\underline{x}/\overline{x}$ sufficiently small, proving the assertion. Because $\Gamma(\alpha)$ is strictly decreasing in α, there exists a unique $\alpha\in(0,1)$ such that $\Gamma(\alpha)=0$. This allocation (the matching α) plus wages given by $\overline{w}=0.5\mathcal{V}(\overline{X}|\alpha)$ and $\underline{w}=0.5\mathcal{V}(\underline{X}|\alpha)$ constitute the unique competitive equilibrium.

To show that the equilibrium value α is strictly increasing in γ, let us write the equilibrium condition including γ as an argument, that is, $\Gamma(\alpha^*(\gamma),\gamma)=0$. It follows from (5.24) that this is continuously differentiable in each argument; moreover, $\Gamma_{\alpha}(\alpha^*(\gamma),\gamma)<0$. Therefore, $\alpha^*_{\gamma}(\gamma)=-\Gamma_{\gamma}(\alpha^*(\gamma),\gamma)/\Gamma_{\alpha}(\alpha^*(\gamma),\gamma)$, and this is strictly positive if and only if $\Gamma_{\gamma}(\alpha^*(\gamma),\gamma)>0$. Differentiating (5.24) with respect to γ and evaluating it at α^* yields

$$\Gamma_{\gamma}(\alpha^*(\gamma),\gamma)=\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(\log\frac{\underline{x}}{\overline{x}}\right)\left(1+\frac{1-\frac{\alpha^*(\gamma)}{2}}{\lambda}\right)^2+2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{\alpha^*(\gamma)}{2\lambda}\right)^2\left(\log 2-\log\left(1+\frac{\underline{x}}{\overline{x}}\right)\right),$$

where the first term is negative while the second is positive. Consider the limit of this expression as $\underline{x}/\overline{x}\to 0$. Since α^* converges to a number strictly between 0 and 1 as $\underline{x}/\overline{x}$ goes to zero, the second term converges to

$$2^{1-\gamma}\left(1+\frac{\lim_{\frac{\underline{x}}{\overline{x}}\to 0}\alpha^*(\gamma)}{2\lambda}\right)^2\log 2>0.$$

Similarly, since α^* converges to a number strictly between 0 and 1 as $\underline{x}/\overline{x}$ goes to zero, it follows that the convergence of the first term depends on the limit of

$$\lim_{\frac{\underline{x}}{\overline{x}}\to 0}\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\log\frac{\underline{x}}{\overline{x}},$$

which is of the $0\cdot\infty$ type. Passing $(\underline{x}/\overline{x})^{\gamma}$ to the denominator we can transform it into a $0/0$ expression. Using L'Hôpital's rule we obtain the following limit

$$\lim_{\frac{\underline{x}}{\overline{x}}\to 0}\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\log\frac{\underline{x}}{\overline{x}}=\lim_{\frac{\underline{x}}{\overline{x}}\to 0}\frac{\frac{1}{\frac{\underline{x}}{\overline{x}}}}{-\gamma\left(\frac{\underline{x}}{\overline{x}}\right)^{-(\gamma+1)}}=\lim_{\frac{\underline{x}}{\overline{x}}\to 0}\left(-\frac{1}{\gamma}\right)\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}=0.$$

Hence, the first term in Γ_{γ} converges to zero as $\underline{x}/\overline{x}$ goes to zero. As a result, for $\underline{x}/\overline{x}$ sufficiently small, the equilibrium α is strictly increasing in γ, completing the proof of the proposition. □

A.8. *Derivation of within- and between-firm variance*

The within-firm variance is an average of the variances within each firm, so

$$\begin{aligned}
\mathrm{Var}[w|\alpha^\star] &= \frac{1}{2}\left(\frac{\alpha^\star}{2}\mathrm{Var}[w|\overline{X}]+\frac{\alpha^\star}{2}\mathrm{Var}[w|\underline{X}]+(1-\alpha^\star)\mathrm{Var}[w|\hat{X}]\right)\\
&= \frac{1}{2}(1-\alpha^\star)\mathrm{Var}[w|\hat{X}]\\
&= \frac{1}{2}(1-\alpha^\star)\left(\frac{1}{2}\left(\underline{w}-\frac{\underline{w}+\overline{w}}{2}\right)^2+\frac{1}{2}\left(\overline{w}-\frac{\underline{w}+\overline{w}}{2}\right)^2\right)\\
&= \frac{1}{2}(1-\alpha^\star)\left(\frac{\overline{w}-\underline{w}}{2}\right)^2\\
&= \frac{A^4\lambda^4}{128}(1-\alpha^\star)\left(\overline{X}^{\gamma}-\underline{X}^{\gamma}\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2,
\end{aligned}$$

where the second equality uses $\mathrm{Var}[w|\overline{X}] = \mathrm{Var}[w|\underline{X}] = 0$, as these teams consists of homogeneous agents and thus members get paid the same wage, and the rest follows from simple algebra.

Since the mean wage is simply $(\underline{w}+\overline{w})/2$, the variance between firms is:

$$\begin{aligned}\mathrm{Var}[w_i+w_j|\alpha^\star] &= \frac{1}{2}\left(\frac{\alpha^\star}{2}\left(2\overline{w}-\frac{\underline{w}+\overline{w}}{2}\right)^2+\frac{\alpha^\star}{2}\left(2\underline{w}-\frac{\underline{w}+\overline{w}}{2}\right)^2+(1-\alpha^\star)\left(\underline{w}+\overline{w}-\frac{\underline{w}+\overline{w}}{2}\right)^2\right)\\
&= \frac{1}{2}\left(\frac{\alpha^\star}{2}\left(\frac{3\overline{w}-\underline{w}}{2}\right)^2+\frac{\alpha^\star}{2}\left(\frac{3\underline{w}-\overline{w}}{2}\right)^2+(1-\alpha^\star)\left(\frac{\underline{w}+\overline{w}}{2}\right)^2\right)\\
&= \frac{A^4\lambda^4}{128}\left(\frac{\alpha^\star}{2}\left(3\overline{X}^\gamma-\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2+\frac{\alpha^\star}{2}\left(\overline{X}^\gamma-3\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right.\\
&\quad\left.+(1-\alpha^\star)\left(\overline{X}^\gamma+\underline{X}^\gamma\left(1+\frac{1-\frac{\alpha^\star}{2}}{\lambda}\right)^2\right)^2\right),\end{aligned}$$

where the first equality is the definition of the variance of the sum of wages, and the rest follows by replacing $\overline{w}$ and $\underline{w}$ and algebraic manipulation.

A.9. *Knowledge spillovers: the planner's problem*

Assume there is an equilibrium in the second stage as described in Lemma 1. Then the planner solves

$$\max_\alpha \frac{A^2}{4}\left(\frac{\alpha}{2}\overline{X}^\gamma\lambda^2+\frac{\alpha}{2}\underline{X}^\gamma\left(\lambda+1-\frac{\alpha}{2}\right)^2+(1-\alpha)\hat{X}^\gamma\left(\lambda+\frac{\alpha}{2}\right)^2\right).$$

The objective function is strictly concave in α if $\lambda > 0.25$. To show this, note that, except for a constant, its derivative is

$$\lambda^2\overline{X}^\gamma+\underline{X}^\gamma\left(\lambda+1-\frac{\alpha}{2}\right)\left(\lambda+1-\frac{3}{2}\alpha\right)-2\hat{X}^\gamma\left(\lambda+\frac{\alpha}{2}\right)\left(\lambda-1+\frac{3}{2}\alpha\right),$$

and thus the second derivative is, after some algebra,

$$\underline{X}^\gamma\left(-2(\lambda+1)+\frac{3}{2}\alpha\right)-\hat{X}^\gamma(4\lambda-1+3\alpha).$$

The first term is strictly negative, and so is the second if $\lambda > 0.25$, making the objective is strictly concave in α.

Rewrite the derivative of the objective function as follows:

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1+\frac{1}{\lambda}-\frac{\alpha}{2\lambda}\right)\left(1+\frac{1}{\lambda}-\frac{3}{2\lambda}\alpha\right)-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1+\frac{\alpha}{2\lambda}\right)\left(1-\frac{1}{\lambda}+\frac{3}{2\lambda}\alpha\right).$$

The efficient matching is interior if and only if this derivative is strictly positive at $\alpha=0$ and strictly negative at $\alpha=1$. At $\alpha=0$ it is given by $\lambda^2\overline{X}^\gamma+\underline{X}^\gamma(\lambda+1)^2-2\hat{X}^\gamma\lambda(\lambda-1)$, which is strictly positive if

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1+\frac{1}{\lambda}\right)^2-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1-\frac{1}{\lambda}\right)>0. \tag{A.18}$$

This is strictly positive for any $\lambda\le 1$, and thus also for λ in a neighbourhood of 1. Alternatively, it holds for $\underline{x}/\overline{x}$ small, and either $\gamma>1$ or λ close to one. In turn, at $\alpha=1$, the derivative of the objective function is given by $\lambda^2\overline{X}^\gamma+\underline{X}^\gamma\left(\lambda^2-0.25\right)-2\hat{X}^\gamma(\lambda+0.5)^2$, which is strictly negative if

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1-\frac{1}{4\lambda^2}\right)-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^\gamma\left(1+\frac{1}{2\lambda}\right)^2<0, \tag{A.19}$$

and this holds if $\underline{x}/\overline{x}$ is sufficiently small and $\gamma<1+2(\log(1+(1/2\lambda))/\log 2)$.

Hence, the planner's problem solution α^p is interior under mild conditions on primitives that satisfy (A.18)–(A.19). It is a valid solution if in addition it satisfies (A.16)–(A.17).

We claim that (A.16)–(A.19) hold if λ and γ are close to one (i.e. in a neighbourhood of one) and $\underline{x}/\overline{x}<1/3$. To see this, assume that $\gamma=\lambda=1$. Then conditions (A.16) and (A.17) reduce to $\underline{x}/\overline{x}<1/3$. Condition (A.18) is satisfied since it holds for any $\lambda\le 1$. Finally, condition (A.19) reduces to $-5-6(\underline{x}/\overline{x})<0$. This shows that the claim is true at $\gamma=\lambda=1$, and since the inequalities are strict, it follows that the claim holds in a neighbourhood of one for each parameter, thereby completing the proof. *In fact, to satisfy all the conditions so far it suffices that* $\lambda \geq 1/(2^{\gamma-1}-1) > 1$, $\gamma < 2$, *and* $\underline{x}/\overline{x}$ *sufficiently small.*

In short, there exists a second stage equilibrium as described above, such that the planner's optimal matching in the first stage, given by α^p, is interior. Moreover, since the planner's objective is strictly concave under the parametric assumptions made, it follows that the interior α^p is unique.

It remains to show that the planner's optimal α^p strictly increases in γ. The optimal α^p is the relevant root of the derivative of the objective function equal to zero (it is a quadratic). That is, it solves[38]

$$1+\left(\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{1}{\lambda}-\frac{\alpha^p}{2\lambda}\right)\left(1+\frac{1}{\lambda}-\frac{3}{2\lambda}\alpha^p\right)-2^{1-\gamma}\left(1+\frac{\underline{x}}{\overline{x}}\right)^{\gamma}\left(1+\frac{\alpha^p}{2\lambda}\right)\left(1-\frac{1}{\lambda}+\frac{3}{2\lambda}\alpha^p\right)=0. \tag{A.20}$$

We claim that the efficient α^p *is strictly increasing in* γ *for all* $\lambda \geq 1$ *and* $\gamma \geq 1$ *so long as* $\underline{x}/\overline{x}$ *is sufficiently small.* To prove it, note that the second term of (A.20) is zero at $\underline{x}/\overline{x}=0$ and the third term strictly decreases in γ. Hence, it follows from the strict concavity of the objective function that the efficient α^p strictly increases in γ.

To see one case in closed form, assume that $\underline{x}=0$ and $\lambda=1$. Then the quadratic becomes:

$$1-2^{1-\gamma}\left(1+\frac{\alpha^p}{2}\right)\frac{3}{2}\alpha^p=0,$$

which rearranges to

$$(\alpha^p)^2+2\alpha^p-\frac{2}{3}2^{\gamma}=0.$$

The relevant root is

$$\alpha^p=-1+\sqrt{1+\frac{2}{3}2^{\gamma}},$$

which satisfies $\alpha^p>0$, $\alpha^p<1$ so long as $\gamma<\log 4.5/\log 2 \cong 2.17$, and it is increasing in γ. Since the result holds for $\underline{x}=0$ in strict form, it also holds for $\underline{x}$ sufficiently small.

Intuitively, the efficient α^p for this example is different from the equilibrium $\alpha^{\star}$ derived in the text, so competitive equilibrium is inefficient. Because the efficient α^p is smaller than that in equilibrium, the *extent* of positive sorting in the competitive equilibrium is too high. The planner chooses fewer firms with PAM, and as a result, the equilibrium between-firm inequality is too high.

A.10. *Proof of Proposition 6*

The only part that is not proven in the text is that a competitive equilibrium with PAM exists. The sufficient conditions on V for PAM stated in Section 4.2 do not hold, but we can show directly from the maximization problem of each agent that, if a is bigger than a threshold, then it is globally optimal to hire a partner of the same characteristic when he conjectures that the equilibrium in the market exhibits PAM. An agent with characteristic x facing a wage function $w(x)=w(0)+((4\beta)/(9b))\big((a-\nu)((x^2)/2)+\beta((x^4)/4)\big)$ solves

$$\max_{x'}\left(\frac{(a-\nu+2\beta xx'-\beta x^2)^2}{9b}-\frac{4\beta}{9b}\left((a-\nu)\frac{x'^2}{2}+\beta\frac{x'^4}{4}\right)-w(0)\right).$$

The first-order condition for an interior maximum is

$$\frac{4\beta}{9b}x(a-\nu+2\beta xx'-\beta x^2)-\frac{4\beta}{9b}\left((a-\nu)x'+\beta x'^3\right)=0,$$

which clearly holds at $x'=x$, so this is a critical point. Taking the second derivative of the objective function, we obtain $(4/9)(\beta/b)(2\beta x^2-(a-\nu)-3\beta x'^2)$, which is negative if $a>\nu+2\beta\overline{x}^2$. Thus, under this parametric condition the objective function is strictly concave, and as a result $x'=x$ is a global maximum.[39] □

We asserted in footnote 32 that the planner prefers PAM to NAM or a convex combination of them. We will prove a slightly more general result that holds for any symmetric and submodular function c; in particular, it holds for $c(x,x')=\nu-\beta xx'$. We will consider the restricted planner's problem where he chooses any convex combination between

38. One can solve it explicitly in Mathematica since it is a quadratic formula in α, and confirm the asserted monotonicity property.

39. If one instead assumes a general symmetric, twice continuously differentiable, strictly decreasing, convex, and strictly supermodular marginal cost function c, then one can show the existence of a PAM equilibrium for large enough a and negative enough c_{12}. Since this does not add much to the analysis, the details are omitted.

PAM and NAM. Using the profit function of each firm, we first show that the expression for A' in Proposition 4 is $A' = -(4/9b)\int_0^1 (c(x,x)-c(x,\mu_-(x)))^2 dF(x) < 0$. To see this, note that

$$\mathcal{V}(x,x|\mu_+) = \frac{(a-c(x,x))^2}{9b}$$

$$\mathcal{V}(x,\mu_-(x)|\mu_+) = \frac{(a-2c(x,\mu_-(x))+c(x,x))^2}{9b}$$

$$\mathcal{V}(x,\mu_-(x)|\mu_-) = \frac{(a-c(x,\mu_-(x)))^2}{9b}$$

$$\mathcal{V}(x,x|\mu_-) = \frac{(a-2c(x,x)+c(x,\mu_-(x))^2}{9b}.$$

Inserting these equations into A' and simplifying them yields the expression above. As a result, the planner's objective function in the restricted problem is strictly concave in α and thus the solution is either a corner or an interior α. To find out where the solution lies, we compute B' from Proposition 4, which is equal to

$$B' = \frac{1}{9b}\left(2a\left(\int_0^1 c(x,\mu_-(x))dF(x) - \int_0^1 c(x,x)dF(x)\right) + \int_0^1 \left(5c^2(x,x)+3c^2(x,\mu_-(x)) - 8c(x,x)c(x,\mu_-(x))\right)dF(x)\right).$$

Since c is strictly submodular, the first term is positive, and thus $B' > 0$ for a large enough. As a result, it follows from the quadratic form of the planner's objective function that the solution to the restricted problem is either interior or PAM. As in Proposition 2, which case ensues depends on the sign of $B' + 2A'$, whose expression is

$$\begin{aligned} B' + 2A' = \frac{1}{9b}\Bigg(&2a\left(\int_0^1 c(x,\mu_-(x))dF(x) - \int_0^1 c(x,x)dF(x)\right) \\ &+ \int_0^1 \left(5c^2(x,x)+3c^2(x,\mu_-(x)) - 8c(x,x)c(x,\mu_-(x))\right)dF(x) \\ &-8\int_0^1 (c(x,x)-c(x,\mu_-(x)))^2 dF(x)\Bigg). \end{aligned}$$

It is clear that if a is large enough then $B' + 2A' > 0$ and thus the optimal solution for the planner in the restricted problem is PAM. Although we have taken into account only firm profits, the same result holds if we add to it consumer surplus in each sector, which is also given by $(2/(9b))\int_0^1 (a-c(x,x))^2 dF(x)$, which is the same as the sum of the profits of the firms in all the sectors (recall that in each sector there are two firms).

A.11. *Spillovers from a patent race*

To illustrate the mechanics of our model with a continuum of types and aggregate spillovers, we now analyse a patent race with spillovers and construct a PAM equilibrium. We assume that x is distributed uniformly on $[0,1]$, and that team composition (x,x') enters match output as the sum of the characteristics, denoted by $X = x+x'$. Since we do not focus on the comparative statics of complementarities in this case, we set $\gamma = 1$. The uniform distribution of x induces a distribution of team composition X, which we denote by G. Under PAM, the measure one half of pairwise teams is distributed uniformly on $[0,2]$, since $X = 2x$, $x \in [0,1]$, and thus $G(X) = X/2$.[40]

In stage two, each team makes an investment decision, a choice of k. Output is a function of the distribution of k in the economy. What is different here is that we assume that the spillover effect is increasing in the rank that the investment of a team has in the distribution of k. This could be due, for example, to a patent race or a first-mover advantage.[41] The problem of a team with X when matching is μ is

$$\mathcal{V}(X|\mu) = \max_{k\geq 0}\left(AH(k|\mu)k - \frac{k^2}{2X}\right), \tag{A.21}$$

where the spillover function $H(\cdot|\mu)$ is the cdf of k in the economy when the matching is μ, and represents the spillover effects in this economy.[42] Note that the objective function is strictly supermodular in (k,X), and hence the optimal

40. Similarly, under NAM all teams formed consist of a pair $(x,1-x)$ and hence all teams have $X = x+1-x = 1$ and are identical, that is, $G(X)$ is degenerate at $X = 1$.

41. We use the simplest possible formulation because it permits a closed-form solution. For a more general formulation of spillover effects of this sort in a different context, see Eeckhout and Jovanovic (1998).

42. The spillover function does not have to be a cdf and could be a more general function that depends on the distribution of investments and teams formed. Making it a cdf simplifies the equilibrium analysis.

solution, denoted by $\kappa^*(X)$ for each X and where we have omitted μ as an argument to simplify the notation, is increasing in X, strictly so when it is interior. Using the first-order condition of the optimization problem of a team with composition X, it must be the case that, for each X,

$$H(\kappa^*(X)|\mu)+\kappa^*(X)H'(\kappa^*(X)|\mu)=\frac{\kappa^*(X)}{AX}. \tag{A.22}$$

Using that κ^* is monotone, equilibrium in the second stage demands that the H be consistent with the distribution of teams formed in the first stage, given by G. Formally, for all $k\geq 0$ we must have

$$H(k|\mu)=\mathbb{P}[\kappa^*(X)\leq k]=\mathbb{P}[X\leq \kappa^{*-1}(k)]=G(\kappa^{*-1}(k)),$$

plus the boundary conditions $H(0|\mu)=0$ and $H(\sup\kappa^*(X)|\mu)=1$.

Assume that in the first stage we have PAM, and thus $G(X)=X/2$. Then $H(k)=\kappa^{*-1}(k)/2$ for all investment levels k in the support of κ^*. Setting $k=\kappa^*(X)$ in (A.22), and thus $X=\kappa^{*-1}(k)=2H(k)$, and using the equilibrium condition $H(k)=\kappa^{*-1}(k)/2$, we obtain the following ordinary differential equation under PAM:

$$H(k|\mu_+)+kH'(k|\mu_+)=\frac{k}{A2H(k|\mu_+)} \iff H^2(k|\mu_+)+kH(k|\mu_+)H'(k|\mu_+)=\frac{k}{2A}. \tag{A.23}$$

Solving this equation and using the boundary condition $H(0|\mu_+)=0$, we obtain the equilibrium distribution H of investments in the second stage of the problem, given by

$$H(k|\mu_+)=\left(\frac{k}{3A}\right)^{\frac{1}{2}}, \tag{A.24}$$

which is equal to 0 at $k=0$ and equals 1 at $k=3A$. Inserting (A.24) into the first-order condition (A.22) we obtain the equilibrium investment function κ^* in the second stage

$$\kappa^*(X)=\frac{3}{4}AX^2, \tag{A.25}$$

which is strictly increasing in X and equals 0 at $X=0$ and $3A$ at $X=2$, as consistency requires. Finally, we can insert (A.24)–(A.25) into the objective function of problem (A.21) to obtain the following match output function:

$$\mathcal{V}(X|\mu_+)=\frac{3}{32}A^2X^3. \tag{A.26}$$

Since this is strictly supermodular in (x,x') (it depends on its sum $X=x+x'$ and $\mathcal{V}(\cdot|\mu_+)$ is strictly convex in X), we can now rationalize the conjecture of each team in the second stage that the prevailing matching is PAM. Under PAM output is divided equally among team members given the symmetry of the problem, and thus μ_+ along with $w(x)=(3/64)A^2(2x)^3$ for all x constitute a competitive equilibrium in the first stage. This completes the construction of a competitive equilibrium with PAM in this economy, where the distribution of investments in the second stage crucially depends on the formation of teams in the first stage, the general theme of this article.[43]

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REFERENCES

ARROW, K. and HAHN, F. (1971), *General Competitive Analysis* (San Francisco, CA: Holden-Day).

BANDO, K. (2012), "Many-to-one Matching Markets with Externalities among Firms", *Journal of Mathematical Economics*, **48**, 14–20.

BARTH, E., BRYSON, A., DAVIS, J. C. *et al.* (2014), "It's Where You Work: Increases in Earnings Dispersion across Establishments and Individuals in the US", (Discussion Paper, National Bureau of Economic Research).

BECKER, G. (1973), "A Theory of Marriage I", *Journal of Political Economy*, **81**, 813–846.

43. If instead agents conjecture NAM, then $H(\cdot|\mu_-)$ is degenerate at some value of k since all teams have composition $X=1$. One can show that all teams choosing $\kappa^*=A$ is a second-stage equilibrium with $\mathcal{V}(1|\mu_-)=A^2/2$. It then follows that welfare under PAM, given by $(1/2)\int_0^2\mathcal{V}(X|\mu_+)(1/2)dX=(3/32)A^2$, is lower than under NAM, given by $(1/2)(A^2/2)$. Since PAM is dominated by at least another matching, it follows that the equilibrium constructed above is inefficient.

BENGURIA, F. (2015), "Inequality Between and within Firms: Evidence from Brazil" (Available at SSRN 2694693).
BENHABIB, J., PERLA, J. and TONETTI, C. (2017), "Reconciling Models of Diffusion and Innovation: A Theory of the Productivity Distribution and Technology Frontier" (Discussion Paper, National Bureau of Economic Research).
CARD, D., HEINING, J. and KLINE, P. (2013), "Workplace Heterogeneity and the Rise of West German Wage Inequality," *The Quarterly Journal of Economics*, **128**, 967–1015.
CHADE, H., EECKHOUT, J. and SMITH, L. (2017), "Sorting through Search and Matching Models in Economics," *Journal of Economic Literature*, **55**, 493–544.
CHE, Y.-K. and GALE, I. (2003), "Optimal Design of Research Contests," *American Economic Review*, **93**, 646–671.
CHE, Y.-K. and YOO, S.-W. (2001), "Optimal Incentives for Teams," *American Economic Review*, **91**, 525–541.
CHEN, B. (2016), "Matching Prior to Cournot Competition: An Assignment Game with Externalities" (SSRN).
DE LOECKER, J. and EECKHOUT, J. (2017), "The Rise of Market Power and the Macroeconomic Implications" (UCL mimeo).
EECKHOUT, J. and JOVANOVIC, B. (1998), "Inequality" (Discussion Paper, National Bureau of Economic Research).
EECKHOUT, J. and JOVANOVIC, B. (2002), "Knowledge Spillovers and Inequality," *American Economic Review*, **92**, 1290–1307.
FISHER, J. and HAFALIR, I. (2016), "Matching with Aggregate Externalities," *Mathematical Social Sciences*, **81**, 1–7.
HAFALIR, I. (2008), "Stability of Marriage with Externalities," *International Journal of Game Theory*, **37**, 353–369.
JOVANOVIC, B. and ROB, R. (1989), "The Growth and Diffusion of Knowledge," *The Review of Economic Studies*, **56**, 569–582.
KÖNIG, M. D., LORENZ, J. and ZILIBOTTI, F. (2016), "Innovation vs. Imitation and the Evolution of Productivity Distributions," *Theoretical Economics*, **11**, 1053–1102.
KOOPMANS, T. and BECKMANN, M. (1957), "Assignment Problems and the Location of Economic Activities," *Econometrica*, **25**, 53–76.
KREMER, M. and MASKIN, E. (1996), "Wage Inequality and Segregation by Skill" (NBER Working Paper).
LEGROS, P. and NEWMAN, A. (2007), "Beauty is a Beast, Frog is a Prince: Assortative Matching with Nontransferabilities," *Econometrica*, 75, 1073–1102.
LOIOLA, E., de Abreu, N., Bonaventura-Netto, P. *et al.* (2007), "A Survey for the Quadratic Assignment Problem", *European Journal of Operations Research*, **176**, 657–690.
LUCAS, R. E. and Moll, B. (2014), "Knowledge Growth and the Allocation of Time," *Journal of Political Economy*, **122**, 1–51.
LUCAS, R. J. (1988), "On the Mechanics of Economic Development," *Journal of Monetary Economics*, **22**, 3–42.
MAS-COLELL, A., WHINSTON, M. D. and GREEN, J. R. (1995), *Microeconomic Theory* (New York: Oxford University Press).
PERLA, J. and TONETTI, C. (2014), "Equilibrium Imitation and Growth", *Journal of Political Economy*, **122**, 52–76.
PYCIA, M. and YENMEZ, B. (2017), "Matching with Externalities" (UCLA mimeo).
ROMER, P. M. (1986), "Increasing Returns and Long-Run Growth", *Journal of Political Economy*, **94**, 1002–37.
SASAKI, H., and TODA, M. (1996), "Two-Sided Matching Problems with Externalities", *Journal of Economic Theory*, **70**, 93–108.
SONG, J., PRICE, D. J., GUVENEN, F. *et al.* (2015), "Firming Up Inequality" (Discussion Paper, National Bureau of Economic Research).
VLACHOS, J., LINDQVIST, E. and HAKANSON, C. (2015), "Firms and Skills: The Evolution of Worker Sorting!" (Discussion Paper, Stockholm).