Topics in Labor Markets

Jan Eeckhout

2015–2016

Introducing the Topic

- Labor markets: principal ingredient in applied research
- Main aspects:
    - determination and distribution of wages
    - allocation of workers to jobs
    - unemployment
- Study the theoretical underpinnings for analyzing labor markets:
    1. allocation process of skilled workers to jobs of different productivity to explain wages: matching
    2. market frictions as an equilibrium phenomenon to explain unemployment: search

Heterogeneity/Diversity in Economics

- Heterogeneity/diversity is hallmark of economic exchange
- Identical agents $\Rightarrow$ no trade (often: buyers vs. sellers)
- Here: even within class of buyers, workers, firms: different preferences and/or endowments
    - House prices depend on characteristics of the occupants, location and the dwelling itself
    - Assets in stock market depend on many characteristics, most notably mean and variance
    - Labor markets: salaries vary with experience, skill of worker, and productivity of job
- Trade: $\neq$ prices for different goods, from supply & demand

Centralized Trade

- The Pioneers (1975 Nobel prize)
  - Kantorovich (1939): Optimal allocation of resources: linear programming solution to optimized production
  - Koopmans: general equilibrium; prices allocate resources
- Market Design and Implementation: 2012 Nobel prize
  - Lloyd Shapley
  - Al Roth
- Matching:
  - (Non-)divisibilities, 1/2 sides, with/without price competition
  - Examples: VC/startup, worker/firm, marriage
  - Applications without prices: Boston school choice, NRMP (Roth), On-line dating, Kidney Exchange,...
  - With prices: e.g., IPO, stock market, Spectrum Auctions,...

Centralized ⇒ Decentralized Trade

- Central. trade: Walr. auctioneer; market place (order book)
- Decentralized trading: market and information frictions
- Search can explain:
  - lengthy duration of trade: *equilibrium* unemployment, time-to-sell in housing market
  - price/wage dispersion for homogenous products
- Random search:
  1. trading partners meet
  2. determine price

  $\Rightarrow$ Nash bargaining inefficient (Hosios 1990)
- Directed search. Reverse order:
  1. firms commit to price
  2. workers choose with whom to trade

  $\Rightarrow$ frictions from coordination; constr. efficient (Moen 1997)

Centralized ⇒ Decentralized Trade

- Like frictionless trade, gains from search due to heterog.
- Heterogeneity on both sides
- Complementarities: force towards PAM
- Decentralized trading: force towards NAM
- Provide cross-sectional "insurance": maximize probability high types match; minimize probability low types match
- Therefore: stronger than supermodularity to induce PAM

What is Assortative Matching?

- Likes match with likes: looks, size, ...
  - PAM: Positive Assortative Matching
  - NAM: Negative Assortative Matching

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- Likes match with likes: looks, size, ...
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  - NAM: Negative Assortative Matching
- From biology: Assortative Mating on phenotype (any observable trait). Trade-off:
  1. Stronger (sexual) selection (PAM) $\Rightarrow$ specialization $\uparrow$ may perform better in *given* environment
  2. NAM $\Rightarrow$ genetic diversity allows species to adapt in *changing* environment – intertemporal insurance
- Natural selection: PAM is disruptive; NAM is stabilizing
- Infamous examples:
  1. lumper potato disease $\Rightarrow$ famine Ireland
  2. Recently: PAM among computer scientists in Silicon Valley $\Rightarrow$ Asperger syndrome (autism) $\uparrow$

Assortativeness in Economics

- Likes match with likes. Economic evidence:
  - Jobs-workers, coworkers, marriage,...
- Underlying determinants:
  - Technology? Common preferences? Preference for the best? Due to trading (same time, same place)? ...
  - Complementarities between different types
- Steven Pinker's definition of love

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  - I "equilibrium" you
- Why does it matter?
  - Indicates the value of common characteristics,...
  - ...and loss from mismatch
  - Example Unemployment Insur.: incentives to find "right" job

Outline of the Lectures

I Frictionless Matching

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II Random Search and Sorting

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III Directed Search and Sorting

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IV Further Topics

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I Frictionless Matching

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Matching and Uncertainty

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IV Further Topics

- Matching and Uncertainty
- Search and Risk Aversion

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- Matching and Uncertainty
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- Matching with Externalities

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- Matching and Uncertainty
- Search and Risk Aversion
- Matching with Externalities
- Labor Mobility and Size Distributions

Topics in Labor Markets

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I. Frictionless Matching

Two-sided Matching
NTU

- How does matching differ from standard markets:
  1. Centralized trade, no Walrasian auctioneer: no price signal
  2. preferences over agents, not goods
  3. indivisibilities
- Direct application:
  - On-line dating (eHarmony.com, OkCupid.com,...)
  - Market design NIMP: ACP assigns residents to hospitals
  - Kidney Exchange
  - School Choice: Boston, New York,...
  - ...
- Initial analysis: Gale and Shapley (1962):
  1. pose the problem;
  2. they provide an algorithm for the solution;
  3. show existence;

One-to-one matching

- Two disjoint sets $\mathcal{W} = \{w_1, ..., w_p\}$ and $\mathcal{M} = \{m_1, ..., m_n\}$
- Match in pairs, allow for the possibility of being single
- Agents have preferences over the members of the other sex: ordered list (complete and transitive):

$$P(m) = w_1, w_3, [m, w_p], ..., w_2$$

  where $[x, y]$: weak preferences. Similar for women: $P(w)$.
- Denote $\mathbf{P} = \{P(m_1), ..., P(w_1), ...\}$ the preference profile. A marriage market is then denoted by $(\mathcal{W}, \mathcal{M}, \mathbf{P})$.
- A particular men-to-women allocation, called matching $\mu(x)$:

Definition

A matching μ is a one to one correspondence from $\mathcal{W} \cup \mathcal{M}$ onto itself ($\mu^2(x) = x$) such that if $\mu(m) \neq m$ then $\mu(m) \in \mathcal{W}$ and if $\mu(w) \neq w$ then $\mu(w) \in \mathcal{M}$

One-to-one matching

- A matching μ is *blocked by an individual* k if k prefers being single to being matched with $\mu(k)$, i.e. $k \succ_k \mu(k)$
- A matching μ is called individually rational if each agent in μ is acceptable (i.e. μ is not blocked by any individual agent).
- A matching μ is *blocked by a pair of agents* (m, w) if $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

Definition

A matching μ is *stable*, if it is not blocked by any individual or any pair of agents.

Theorem

(Gale and Shapley 1962). A Stable matching exists for every marriage market.

One-to-one matching

- Proof uses the *Deferred Acceptance Algorithm (DAA)*. Starts with one side of the market making proposals (say men):

  1. a. Each man proposes to his first choice (if acceptable ones)
     b. Each woman “holds” the most preferred
     ...

  K. a. Any man rejected at step $k-1$ makes a new proposal to his most preferred acceptable mate who hasn't rejected him yet (make no proposal if no acceptable choices remain)
     b. each woman holds most preferred offer to date, rejects rest
     ...

  K+L STOP when no further proposals are made and match any woman to the man whose proposal she is holding.

- Weak preferences: break ties arbitrarily (e.g. alphabetical,...)
- With finite set of men, women, this algorithm is finite, and hence always stops

One-to-one matching

- This algorithm gives rise to a stable matching
- Suppose not, m can do better, i.e. m prefers w to his current match $\mu(m)$. Then:
  1. $w \succ_m \mu(m)$
  2. m must have proposed to w before proposing to $\mu(m)$
  3. m must have been rejected by w
  4. as a result, $\mu(w) \succ_w m$
  5. no blocking pair
  6. match is stable

One-to-one matching

- Example. Consider $(\mathcal{W}, \mathcal{M}, \mathbf{P})$ where

$$\begin{array}{ll} P(m_1) = w_1, w_2, w_3, w_4 & P(w_1) = m_2, m_3, m_1, m_4, m_5 \\ P(m_2) = w_4, w_2, w_3, w_1 & P(w_2) = m_3, m_1, m_2, m_4, m_5 \\ P(m_3) = w_4, w_3, w_1, w_2 & P(w_3) = m_5, m_4, m_1, m_2, m_3 \\ P(m_4) = w_1, w_4, w_3, w_2 & P(w_4) = m_1, m_4, m_5, m_2, m_3 \\ P(m_5) = w_1, w_2, w_4, m_5 & \end{array}$$

- Then using the DAA:

w_1	w_2	w_3	w_4	(m_i)

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\end{array}
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- Then using the DAA:

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m_1, m_4, m_5			m_2, m_3	
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m_1, m_4, m_5			m_2, m_3	
m_1	m_5	m_3	m_4, m_2	
m_1	m_2, m_5	m_3	m_4	

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$$

- Then using the DAA:

w_1	w_2	w_3	w_4	(m_i)
m_1, m_4, m_5			m_2, m_3	
m_1	m_5	m_3	m_4, m_2	
m_1	m_2, m_5	m_3	m_4	
m_1	m_2	m_3	m_4	m_5

- The stable matching is

$$
\mu_M = \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{array}
$$

One-to-one matching

- Similarly, when women make offers, the stable matching:

$$\mu_W = \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_2 & m_3 & m_4 & m_1 & m_5 \end{array}$$

- Implications from this example:
  1. In general the set of stable matchings is not a singleton
  2. All m weakly prefer μ_M to μ_W, and the opposite for women; i.e. for all $m : \mu_M \succeq_m \mu_W$ and for all $w : \mu_W \succeq_w \mu_M$

  ⇒ There is a conflict between the two sides of the market as to who can make the offer!

Theorem

(Gale and Shapley). When all men and women have strict preferences, there always exists an M-optimal stable matching, and a W-optimal stable matching. Furthermore, the matching μ_M produced by the DAA with men proposing is the M-optimal stable matching. The W-optimal stable matching is the matching μ_W produced by the DAA when women propose.

Two-sided Matching

Sketch of Proof

Terminology: w is achievable for m if there is some stable matching μ such that $\mu(m) = w$

- Inductive step k. Suppose no m has been rejected by an achievable w, and at k, w rejects m and holds on to some other $m' \Rightarrow w$ is not achievable for m
- Now consider μ with $\mu(m) = w$ and $\mu(m')$ achievable for m'. Cannot be stable: by inductive step, (m', w) is blocking pair
- Let $\mu \succ_M \mu'$ denote *all* men like μ at least as well as μ', with at least one strict. Then $\succ_M$ is a *partial* order on the set of matchings, representing the common preferences of the men. Similarly, $\succ_F$ common preference of women

One-to-one matching

Theorem

(Knuth) When all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings: if μ and μ' are stable matchings, then all men like μ at least as well as μ' if and only if all women like μ' at least as well as μ. That is, $\mu \succ_M \mu'$ if and only if $\mu' \succ_W \mu$.

- From the definition of stability.
- The best outcome for one side of the market is the worst for the other.

Lattice Property

- Preliminaries. Set L endowed w. partial order $\geq$; and $X \subset L$
- $a \in L$ is the upperbound of X if $a \geq x, \forall x$
- $\sup X$ least upper bound of X; $\inf X$ greatest lower bound
- Denote by the binary relations "sup" of any two elements $x \vee y$ ("join") and "inf" of any two elements $x \wedge y$ ("meet")
- For any 2 matchings μ, μ', and for all m, w define $\lambda = \mu \vee_M \mu'$ as function that assigns each man his more preferred of the two matches; each woman her less preferred:

$$\begin{aligned}
\lambda(m) &= \mu(m) \text{ if } \mu(m) \succ_m \mu'(m) \text{ and} \\
\lambda(m) &= \mu'(m) \text{ otherwise}
\end{aligned}$$

$$\begin{aligned}
\lambda(w) &= \mu(w) \text{ if } \mu(m) \prec_w \mu'(w) \text{ and} \\
\lambda(w) &= \mu'(w) \text{ otherwise}
\end{aligned}$$

- Define $\nu = \mu \wedge_M \mu'$ analogously, by reversing the preferences

Lattice Property

Theorem

(Lattice Theorem – Conway). When all preferences are strict, if μ and μ' are stable matchings, then the functions $\lambda = \mu \vee_M \mu'$ and $\nu = \mu \wedge_M \mu'$ are both matchings. Furthermore, both are stable.

- Think of λ: ask men to point to preferred mate from 2 stable matchings, women to less preferred mate. Then theorem says:
  1. that no two men will point to the same woman (this follows from the stability of μ and μ'.
  2. Every woman points back at the man pointing at her (not immediate to prove)
  3. the resulting match is stable (because we compare across stable matchings μ and μ')
- The fact that operations $\vee_M$ and $\wedge_M$ produce a stable matching from a pair of stable matchings implies that set of stable matchings has an algebraic structure called a lattice.

Lattice Property

Definition

A lattice is a partially ordered set L, any 2 of whose elements x and y have a sup (i.e. $x \vee y$) and an inf (i.e. $x \wedge y$). A lattice is complete when each of its subsets X has a sup and an inf in L.

A lattice is distributive iff

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \end{aligned}$$

$\forall x, y, z \in L$.

Theorem

(Conway). When all preferences are strict, the set of stable matchings is a distributive lattice under the common order of man, dual to the common order of women.

Lattice Property

Example

- Consider the following marriage market $(\mathcal{W}, \mathcal{M}, \mathbf{P})$ where

$$\begin{array}{ll}
P(m_1) = w_1, w_2, w_3, w_4 & P(w_1) = m_4, m_3, m_2, m_1 \\
P(m_2) = w_2, w_1, w_4, w_3 & P(w_2) = m_3, m_4, m_1, m_2 \\
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P(m_4) = w_4, w_3, w_2, w_1 & P(w_4) = m_1, m_2, m_3, m_4
\end{array}$$

- Then using the DAA:

$$\mu_M = \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ m_1 & m_2 & m_3 & m_4 \end{array}$$

and

$$\mu_W = \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ m_4 & m_3 & m_2 & m_1 \end{array}$$

Lattice Property

Example

- There are 10 stable matchings:

	w_1	w_2	w_3	w_4
μ_M	m_1	m_2	m_3	m_4
μ_2	m_2	m_1	m_3	m_4
μ_3	m_1	m_2	m_4	m_3
μ_4	m_2	m_1	m_4	m_3
μ_5	m_3	m_1	m_4	m_2
μ_6	m_2	m_4	m_1	m_3
μ_7	m_3	m_4	m_1	m_2
μ_8	m_4	m_3	m_1	m_2
μ_9	m_3	m_4	m_2	m_1
μ_W	m_4	m_3	m_2	m_1

$\Rightarrow$

$$\begin{aligned} \mu_2 \wedge_M \mu_3 &= \mu_4 \\ \mu_5 \wedge_M \mu_6 &= \mu_7 \\ \mu_8 \wedge_M \mu_9 &= \mu_W \end{aligned} \quad \text{and} \quad \begin{aligned} \mu_2 \vee_M \mu_3 &= \mu_M \\ \mu_5 \vee_M \mu_6 &= \mu_4 \\ \mu_8 \vee_M \mu_9 &= \mu_7 \end{aligned}$$

Lattice Property

- Now consider the “ranking” of stable matches relative to *any* (including) non-stable matches.

Theorem

Weak Pareto optimality for the men: There is no individually rational matching μ (stable or not) such that $\mu \succ_m \mu_M$ for all $m \in \mathcal{M}$

- Proof uses DAA and by contradiction

Uniqueness

- How relevant is this multiplicity of stable matchings. We show a uniqueness theorem that provides a sufficient condition
- An example. Vertical heterogeneity, i.e. common preferences of each member of a sex over the other sex:

$$\begin{array}{ll} P(m_1) = w_1, w_2, w_3 & P(w_1) = m_1, m_2, m_3 \\ P(m_2) = w_1, w_2, w_3 & P(w_2) = m_1, m_2, m_3 \\ P(m_3) = w_1, w_2, w_3 & P(w_3) = m_1, m_2, m_3 \end{array}$$

- Then from the DAA

$$\mu_M = \begin{array}{ccc} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{array} = \mu_W$$

Uniqueness

Theorem

Consider two ordered sets $\mathcal{W} = (X_i)$ *and* $\mathcal{M} = (x_i)$. *If the preference profile satisfies*

$$\begin{aligned} \forall X_i &\in \mathcal{W} : x_i \succ_{X_i} x_j, \forall j > i \\ \forall x_i &\in \mathcal{M} : X_i \succ_{x_i} X_j, \forall j > i \end{aligned}$$

then there is a unique stable matching $\mu^*(X_i) = x_i,\ \forall i$

- Proof. Suppose there exists a stable matching $\mu' \neq \mu^*$, i.e. with for some i $\mu'(X_i) = x_k, k \neq i$. Then from stability, there exists some $j \neq k$ such that $\mu'(x_j) = X_l, l \neq j$. Let

$$\begin{aligned} \lambda &= \min\{i : \mu'(X_i) = x_k, k \neq i\} \\ \gamma &= \min\{j : \mu'(x_j) = X_l, l \neq j\} \end{aligned}$$

  Since $\mu^*(X_\lambda) = x_\gamma$, it follows that $\lambda = \gamma$.

Uniqueness

- Then $\mu'(X_i) = x_k$ implies $\lambda < k$ and $\mu'(x_\lambda) = X_l$ implies $\lambda < l$. Now under preferences as above, it follows that

$$\begin{array}{ccc} x_\lambda & \succ_{X_\lambda} & x_k \\ X_\lambda & \succ_{x_\lambda} & X_l \end{array}$$

  so that X_λ and x_λ form a blocking pair against μ', and hence μ' is not a stable matching. A contradiction.
- Intuition
  1. Starting at agents m and w with index $i = 1$ we can assign those two agents (no blocking pair).
  2. Now, $i = 2$, who may like 1 more but cannot get them $\Rightarrow$ will match given preferences
  3. This unravels all the way down
  4. Note that there is no restriction on the relative ranking of any two men k, l for a woman i as long as k, l are either "above" or "below" i.

Uniqueness

- Vertical Heterogeneity:

$$\begin{aligned} \forall X_i &\in \mathcal{W} : x_k \succ_{X_i} x_j, \forall k < j \\ \forall x_i &\in \mathcal{M} : X_k \succ_{x_i} X_j, \forall k < j \end{aligned}$$

- Horizontal Heterogeneity:

$$\begin{aligned} \forall X_i &\in \mathcal{W} : x_i \succ_{X_i} x_j, \forall j \\ \forall x_i &\in \mathcal{M} : X_i \succ_{x_i} X_j, \forall j \end{aligned}$$

- Sufficient condition, not necessary. Counter example:

$$\begin{array}{ll} P(m_1) = w_3, w_1, w_2, w_4 & P(w_1) = m_2, m_1, m_3, m_4 \\ P(m_2) = w_4, w_4, w_3, w_1 & P(w_2) = m_1, m_2, m_3, m_4 \\ P(m_3) = w_1, w_3, w_2, w_4 & P(w_3) = m_2, m_3, m_4, m_1 \\ P(m_4) = w_3, w_4, w_2, w_1 & P(w_4) = m_3, m_4, m_1, m_2 \end{array}$$

- Note this profile does not satisfy preference condition since $w_3 \succ_{m_1} w_1$ and $m_2 \succ_{w_1} m_1$

Strategic Behavior

- So far: preference orderings are common knowledge
- If preferences are private information, stable matching is strategy-proof if $\exists$ no incentive to misrepresent preferences
- An example (same as before):

$$\begin{array}{ll}
P(m_1) = w_1, w_2, w_3, w_4 & P(w_1) = m_2, m_3, m_1, m_4, m_5 \\
P(m_2) = w_4, w_2, w_3, w_1 & P(w_2) = m_3, m_1, m_2, m_4, m_5 \\
P(m_3) = w_4, w_3, w_1, w_2 & P(w_3) = m_5, m_4, m_1, m_2, m_3 \\
P(m_4) = w_1, w_4, w_3, w_2 & P(w_4) = m_1, m_4, m_5, m_2, m_3 \\
P(m_5) = w_1, w_2, w_4, m_5 &
\end{array}$$

- with stable matching

$$\mu_M = \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{array}$$

Strategic Behavior

- Observe that w_1 matches to m_1, her third choice
- Consider preferences $\mathbf{P}'$, in which all agents except w_1 state their preferences as before, but w_1 misrepresents:

$$P'(w_1) = m_2, m_3, m_4, m_5, m_1$$

in which case w_1 is better off:

$$\mu'_M = \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_3 & m_1 & m_2 & m_4 & m_5 \end{array}$$

- Misrepresenting may pay, how can we assure mechanisms such that honesty works? Incentives in the DAA? Welfare?

Theorem

(Impossibility Theorem – Roth). No stable matching mechanism exists for which stating the true preferences is a dominant strategy for every agent.

Strategic Behavior

Theorem

If preferences are strict, and there is more than one stable matching, then at least one agent can profitably misrepresent his or her preferences, assuming the others tell the truth. (This agent can misrepresent in such a way as to be matched to his or her most preferred achievable mate under the true preferences at every stable matching under the mis-stated preferences.)

Theorem

(Dubins and Freedman; Roth) The mechanism that yields the M-optimal stable matching makes it a dominant strategy for each man to state his true preferences. (Similar for W-optimal.)

Corollary

If set of stable matchings is unique, then DAA makes it dominant strategy for each man and woman to state true preferences

- More general matching environment (and iff): Sönmez (1999)

Other Matching Problems

Roommate Problem

- Allocating freshmen to rooms in a dorm;
  - from one joint set
  - multiple agents may be assigned
- Main issue: existence not guaranteed. Example (match pairs):

$$\begin{array}{ll} P(a) = b, c, d & P(c) = a, b, d \\ P(b) = c, a, d & P(d) = \text{any} \end{array}$$

- All candidate matchings blocked

$$\mu_1 = \frac{c \quad a}{b \quad d} \quad \text{blocked by } (c, a)$$

$$\mu_2 = \frac{a \quad d}{b \quad c} \quad \text{blocked by } (b, c)$$

$$\mu_3 = \frac{b \quad a}{d \quad c} \quad \text{blocked by } (a, b)$$

Other Matching Problems

Many-to-one Matching

- Firms and workers; colleges and students;...
- Not simply matching worker to job several times where there is complementarity between job and worker
- Key is the complementarity/subsitutability between workers
- E.g. b, c complements $\Rightarrow a \succ_i b$ and $\{b, c\} \succ_i \{a, c\}$
- Gross Substitutes: see below

NTU – Applications

- The Labor Market for Medical Interns
    - Roth, A. 1984, "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory" *Journal of Political Economy*
    - Roth, A. 1986, "On the Allocation of Residents to Rural Hospitals: A General Property of Two-sided Matching Markets," *Econometrica*
    - Roth, A. 1991, "A Natural Experiment in the Organization of Entry Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom," *American Economic Review*
    - Roth, A. and E. Peranson, 1999, "The Redesign of the Matching Market for American Physicians," *American Economic Review*

NTU – Applications

- School Choice
  - Ergin, H. and T. Sönmez, 2006, "Games of School Choice under the Boston Mechanism," *Journal of Public Economics*
  - Abdulkadiroglu, A. and T. Sönmez, 2003, "School Choice: A Mechanism Design Approach," *American Economic Review*
    Abdulkadiroglu, A., P. Pathak, A. E. Roth, and T. Sönmez: 2005, "The Boston Public School Match," *American Economic Review P&P*
- Kidney Exchange
  - Roth, A. E., T. Sönmez, and U. Ünver, 2004 "Kidney Exchange," *Quarterly Journal of Economics*
  - Roth, A. E., T. Sönmez, and U. Ünver: 2005a, "A Kidney Exchange Clearinghouse in New England," *American Economic Review P&P*
  - Roth, A. E., T. Sönmez, and U. Ünver: 2005b, "Pairwise Kidney Exchange," *Journal of Economic Theory*

Two-sided Matching

TU – Introducing Wages

- Less attractive agents may compensate the more attractive one to form a match
- Labor market: wage
- But also non-monetary transfers:
  - Services: transfer with services (cleaning for roommates, child care in marriage,...)
  - Presents
  - In traits when multidimensional: she is attractive but smokes, he is rich but has a temper,...

Two-sided Matching

TU – Introducing Wages

- From ordinal to cardinal preferences: need to assign valuations
- Example. Preference order (with associated utility):

$$\begin{array}{ll} P(x_1) = y_1(5), y_2(3), y_3(2) & P(y_1) = x_2(3), x_1(2), x_3(1) \\ P(x_2) = y_1(3), y_2(2), y_3(1) & P(y_2) = x_1(2), x_2(1), x_3(0) \\ P(x_3) = y_1(7), y_2(2), y_3(1) & P(y_3) = x_1(4), x_2(3), x_3(1) \end{array}$$

- There are two stable equilibrium allocations:

$$\mu(x_1, x_2, x_3) = \quad (y_1, y_2, y_3) \text{ and } (y_2, y_1, y_3)$$

with payoffs

$$\begin{array}{c} (5,2,1) \\ (3,1,1) \end{array} \quad \text{and} \quad \begin{array}{c} (3,3,1) \\ (2,2,1) \end{array}$$

Two-sided Matching: TU

- Allowing for transfers, we can write this as

$$f(x,y) = \begin{pmatrix} 5+3 & 3+2 & 2+1 \\ 2+3 & 3+2 & 1+3 \\ 7+1 & 2+0 & 1+1 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 3 \\ 5 & 5 & 4 \\ 8 & 2 & 2 \end{pmatrix}$$

Two-sided Matching: TU

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- Allocation different from any of those under NTU!
- Stability: payoffs (w_1, w_2, w_3) and (π_1, π_2, π_3) must satisfy:

$$\begin{array}{llllll} w_1 + \pi_1 \geq 8 & w_1 + \pi_2 = 5 & w_1 + \pi_3 \geq 3 & & 0 \leq w_1 - w_3 \leq 3 \\ w_2 + \pi_1 \geq 5 & w_2 + \pi_2 \geq 5 & w_2 + \pi_3 = 4 & \Rightarrow & 0 \leq w_2 - w_1 \leq 1 \\ w_3 + \pi_1 = 8 & w_3 + \pi_2 \geq 2 & w_3 + \pi_3 \geq 2 & & 2 \leq w_2 - w_3 \leq 3 \end{array}$$

e.g. $w_1 = 3, w_2 = 4, w_3 = 1$ and $\pi_1 = 7, \pi_2 = 2, \pi_3 = 0$

- Unique allocation, continuum prices (discrete outside option)

Two-sided Matching: TU

- 2 disjoint sets $\mathcal{X}$ (m workers) and $\mathcal{Y}$ (n firms)
- Payoffs: $\forall (x,y) \in \mathcal{X} \times \mathcal{Y} : \exists f(x,y) \in \mathbb{R}^+$
- An assignment game is then completely defined by $(\mathcal{X}, \mathcal{Y}, f)$

Definition

A feasible assignment for $(\mathcal{X}, \mathcal{Y}, f)$ is a matrix $\mu = (\mu(x,y))$ (of zeros and ones) that satisfies:

$$\begin{aligned} \sum_x \mu(x,y) &\leq 1 \\ \sum_y \mu(x,y) &\leq 1 \\ \mu(x,y) &\geq 0 \end{aligned}$$

- Then $\mu(x,y) = 1$ if (x,y) match and $\mu(x,y) = 0$ otherwise

Two-sided Matching: TU

Definition

A feasible assignment is optimal for $(\mathcal{X}, \mathcal{Y}, f)$ if, for all feasible assignments μ', $\sum_{x,y} f(x,y)\mu(x,y) \geq \sum_{x,y} f(x,y)\mu'(x,y)$

- Example.

$$f(x,y) = \begin{pmatrix} 10 & 12 & 7 \\ 6 & 8 & 2 \\ 5 & 5 & 9 \end{pmatrix}$$

then

$$\mu_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Both optimal: $\sum_{x,y} f(x,y)\mu(x,y) = 27$ in both cases

Two-sided Matching: TU

Definition

The pair of vectors (w, π), with $w \in \mathbb{R}^m$ and $\pi \in \mathbb{R}^n$ is called a feasible payoff for $(\mathcal{X}, \mathcal{Y}, f)$ if there is a feasible assignment μ such that

$$\sum_{x \in \mathcal{X}} w(x) + \sum_{y \in \mathcal{Y}} \pi(y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} f(x, y) \cdot \mu(x, y)$$

Definition

A feasible outcome $((w, \pi), \mu)$ is stable if

1. $w(x) \geq 0, \pi(y) \geq 0$ (individual rationality);
2. $w(x) + \pi(y) \geq f(x, y), \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$ (no blocking pair)

Optimal Assignment

We can then show the following theorem:

Theorem

(Shapley and Shubik). Let $(\mathcal{X}, \mathcal{Y}, f)$ be assignment game. Then:

1. *set of stable outcomes and core of $(\mathcal{X}, \mathcal{Y}, f)$ are same;*
2. *the core of $(\mathcal{X}, \mathcal{Y}, f)$ is the (nonempty) set of solutions of the dual LP of the corresponding assignment problem.*

Corollary

If x is an optimal assignment, then it is compatible with any stable payoff (w, π)

Corollary

If $((w, \pi), \mu)$ is a stable outcome, then μ is an optimal assignment.

From $\sum_{x \in \mathcal{X}} w(x) + \sum_{y \in \mathcal{Y}} \pi(y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} f(x, y) \cdot \mu(x, y)$ and stability: $w(x) + \pi(x, y) \geq f(x, y)$, for all (x, y).

Lattice Property

- Core with partial order $\succeq_{\mathcal{X}}$ forms a complete lattice (dual with ordering $\succeq_{\mathcal{Y}}$).
- A trivial example. 2 men, 2 women, with payoff matrix

$$f(x,y) = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} \Rightarrow \text{ unique allocation } \quad \mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Set of payoffs that are a stable outcome must satisfy

$$1 \leq w(1) - w(2) \leq 2$$
$$w(1), w(2) \geq 0.$$

$\Rightarrow$ $\exists$ continuum of equilibria. For example E_1: $(w(1), w(2)) = (1, 0)$, and E_2: $(w(1), w(2)) = (2, 1)$

$$E_2 \succeq_{\mathcal{X}} E_1$$
$$E_1 \succeq_{\mathcal{Y}} E_2$$

Assortative Matching

The Basic Model

- Assignment Game
- Worker type x, Γ (uniform)
- Job type y, Υ (uniform)
- Output $f(x, y) \geq 0$
- Common rankings: $f_x > 0$ and $f_y > 0$
- Cross-partials f_{xy}: key for monotone matching
- Examples:

$$f^{+}(x, y) = \alpha x^{\theta} y^{\theta} \quad \text{and} \quad f^{-}(x, y) = \alpha x^{\theta}(1 - y)^{\theta} + g(y),$$

The Basic Model

Equilibrium

- Assignment of workers to firms: $\mu(x) = y$ (note change of notation from matrix, where $\mu(x, y)$ was defined on $\mathcal{X} \times \mathcal{Y}$)
- Wage schedule: $w(x)$
- Profit schedule: $\pi(y)$
- Stable Equilibrium: μ and payoffs such that $\forall x, y$:

$$
\begin{aligned}
w(x) + \pi(y) &\geq f(x, y) \\
w(x) + \pi(\mu(x)) &= f(x, \mu(x))
\end{aligned}
$$

The Basic Model

- As a competitive equilibrium (TU: core = GE). Given wage schedule $w(x)$, firm maximization:

$$\max_x f(x,y) - w(x)$$

- FOC:

$$f_x(x,y) - \frac{\partial w(x)}{\partial x} = 0$$

- Let $w^\star(x)$ be the equilibrium wage of worker x

$$w^\star(x) = \int_0^x f_x(\tilde{x}, \mu(\tilde{x}))d\tilde{x} \ + w_0,$$

- Profits:

$$\pi^\star(y) = \int_0^y f_y(\mu^{-1}(\tilde{y}), \tilde{y})d\tilde{y} \ - w_0$$

The Basic Model

Assortativeness: SOC

- What is the equilibrium allocation μ? Follows from SOC:

$$f_{xx}(x,y) - w_{xx}(x) < 0$$

- w_{xx}? $\Rightarrow$ derivative of FOC at $y = \mu(x)$:

$$f_{xx}(x,\mu(x)) + f_{xy}(x,\mu(x))\frac{d\mu(x)}{dx} - w_{xx}(x) = 0$$

- SOC is satisfied provided (μ differentiable)

$$f_{xy}(x,\mu(x))\frac{d\mu(x)}{dx} > 0.$$

- PAM: $\frac{d\mu(x)}{dx} > 0$ if $f_{xy} > 0$ NAM: $\frac{d\mu(x)}{dx} < 0$ if $f_{xy} < 0$

Supermodularity

- Supermodularity

$$f(x_2, y_2) + f(x_1, y_1) \geq f(x_2, y_1) + f(x_1, y_2)$$

- $f(x, y)$ differentiable: $f_{xy}(x, y) \geq 0$
- Stronger Degree of SM: g concave $\Rightarrow g \circ f$ supermodular:

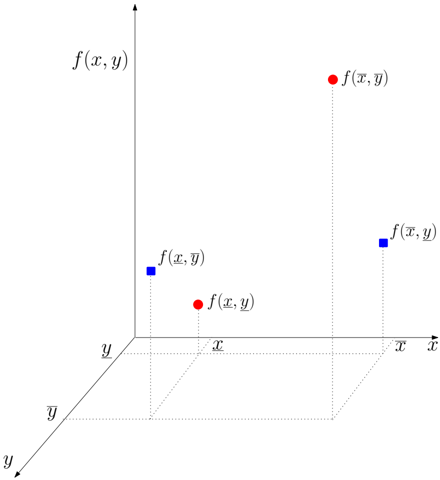
$$g \circ f(x_2, y_2) + g \circ f(x_1, y_1) \geq g \circ f(x_2, y_1) + g \circ f(x_1, y_2)$$

- $f(x, y)$ differentiable:

$$\frac{\partial^2 g(f(x, y))}{\partial x \partial y} \geq 0 \quad \iff \quad \frac{f_{xy}(x, y) f(x, y)}{f_x(x, y) f_y(x, y)} \geq -\frac{g''(f(x, y)) f(x, y)}{g'(f(x, y))}$$

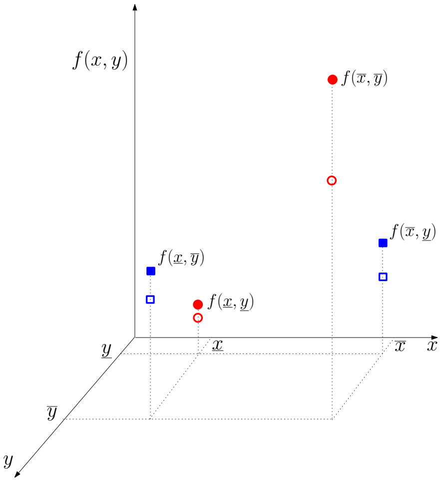
- RHS: Arrow-Pratt measure of the transform g
- Examples:
  1. $g(f)$ linear $\Rightarrow$ RHS=0
  2. $g(f) = log(\cdot) \Rightarrow$ RHS=1
  3. $g(f) = \sqrt[n]{f} \Rightarrow$ RHS$=1 - n^{-1}$

Supermodularity



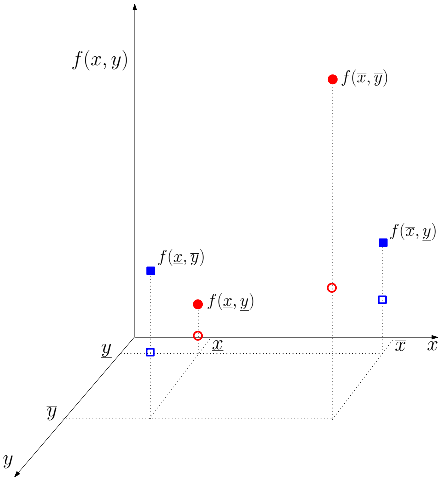
Supermodular: $f_{xy} > 0$

Supermodularity



$$g \circ f\text{-sup.:} \quad f_{xy} > -\frac{g''(f))f}{g'(f)}\frac{f_x f_y}{f}$$

Supermodularity



log f-sup.: $\quad f_{xy} > 1 f_x f_y / f$

Examples of Supermodular Functions

- Function of sum: $V(x+y)$
  - V convex $\Rightarrow$ supermodular
  - V concave $\Rightarrow$ submodular

  Examples:
  - $(x+y)^{\alpha}$: SM for $\alpha > 1$; neither root- nor log-SM for $\alpha < 2$
  - $(x+y)^{\alpha}$: root-SM for $\alpha > 2$, never log-SM
  - β^{x+y} : log-SM
- max $-$ min Operators:
  - $V(x,y) = \max\{x,y\}$: weakly log-SBM
  - $V(x,y) = \min\{x,y\}$: weakly log-SPM (e.g. "O-Ring")

Examples of Supermodular Functions

max operator is weakly SBM

- Need to verify that:

$$V(\overline{x}, \overline{y})V(\underline{x}, \underline{y}) \leq V(\overline{x}, \underline{y})V(\underline{x}, \overline{y})$$

  where $V(x, y) = \max\{x, y\}$
- Let $\overline{x} > \underline{x}$ and $\overline{y} > \underline{y}$. Then 6 possibilities:
  1. $\overline{x} > \overline{y} > \underline{x} > \underline{y} \quad \Rightarrow \quad \overline{x} \cdot \underline{x} < \overline{x} \cdot \overline{y}$

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Supermodularity is sufficient for PAM, not necessary.

- Workers: $\mathcal{X} = \{1,2,3\}$;   Firms: $\mathcal{Y}\{1,2,3\}$
- Match surplus function: $f(x,y) = x \cdot y$; observe $f_{xy}(x,y) > 0$
- Stable Equilibrium: allocation and payoff vector
- Supermodularity:

$$\Delta = [f(i,j) + f(i+1,j+1)] - [f(i,j+1) + f(i+1,j)] > 0$$

$$f(x,y) = \begin{pmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \sum f(x,\mu(x)) = 14$$

Supermodularity and Matching

Local Supermodularity

Supermodularity is sufficient for PAM, not necessary.

- Workers: $\mathcal{X} = \{1, 2, 3\}$; Firms: $\mathcal{Y}\{1, 2, 3\}$
- Match surplus function: $f(x, y) = x \cdot y$; observe $f_{xy}(x, y) > 0$
- Stable Equilibrium: allocation and payoff vector
- Supermodularity:

$$\Delta = [f(i, j) + f(i+1, j+1)] - [f(i, j+1) + f(i+1, j)] > 0$$

$$f(x, y) = \begin{pmatrix} 9 & 6 & 4 \\ 6 & 5 & 2 \\ 4 & 2 & 1 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \text{and} \quad \sum f(x, \mu(x)) = 15$$

Supermodularity and Matching

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But. For *any* distribution, supermodularity is necessary.

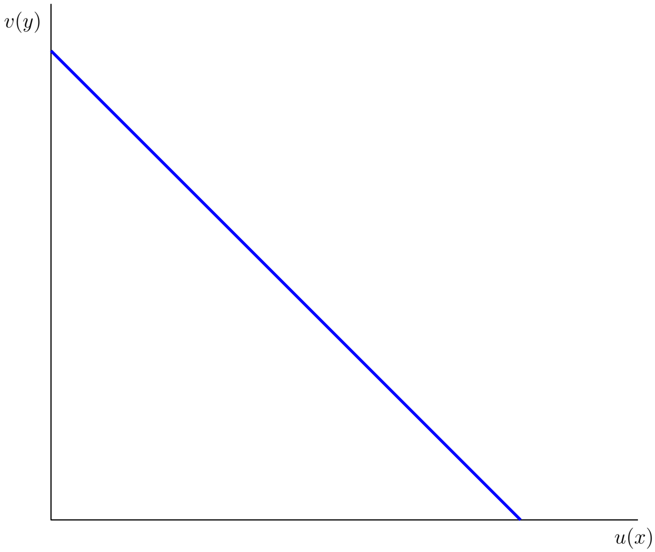
Supermodularity and Matching

Generalized Increasing Differences

- The assignment game is very special: linear Pareto frontiers for matched pairs
- Agents may be risk averse; there may be moral hazard,...
- What are the properties of the assignment when the pairwise frontiers are non-linear?
- Legros-Newman (2007): Generalized Increasing Differences

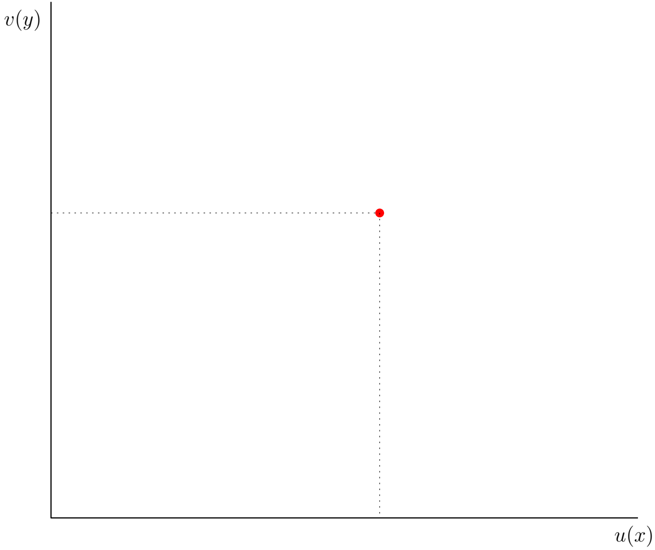
Supermodularity and Matching

Generalized Increasing Differences



Supermodularity and Matching

Generalized Increasing Differences



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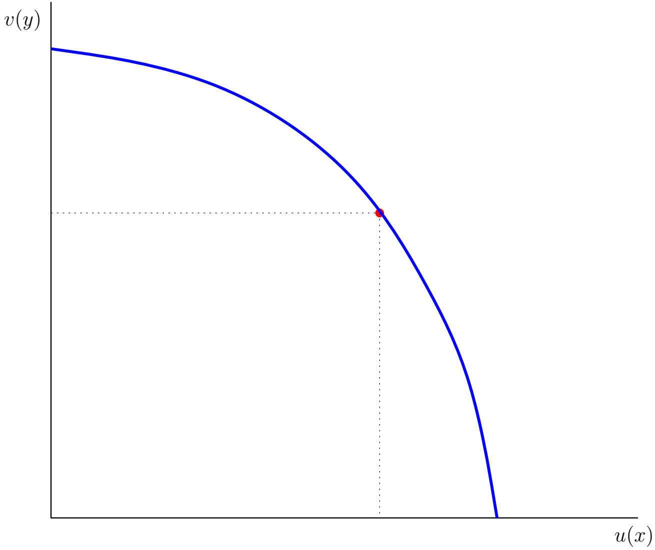

Supermodularity and Matching

Generalized Increasing Differences

- Things change if *type-dependent* preferences u_x, v_y
- Pareto frontier: $v_y = \phi(x, y, u_y)$ and $\psi(x, y, v_x) = \phi^{-1}(v_y)$
- Legros-Newman (2007). GID (discrete). Equivalent for continuous types (where $\phi_3 < 0$)

$$\phi_{12} > \frac{\phi_1}{\phi_3}\phi_{23}$$

- GID $\Rightarrow$ Positive Assortative Matching
- Key: relative slope of frontiers (by pair), not concavity

- The firm problem: choose x to maximize $\phi(x, y, \psi(x))$ given $\psi(x)$, utility ("wage") of worker x; $\phi_3 < 0$. The FOC

$$\phi_1(x, y, \psi) + \phi_3(x, y, \psi)\psi'(x) = 0$$

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$\Rightarrow$ SOC satisfied provided $\phi_{12}\mu' + \phi_{23}\mu'\psi' > 0$

Supermodularity and Matching

Generalized Increasing Differences

- Use FOC to substitute for ψ'

$$\mu' \left[\phi_{12} - \frac{\phi_1}{\phi_3} \phi_{23} \right] > 0$$

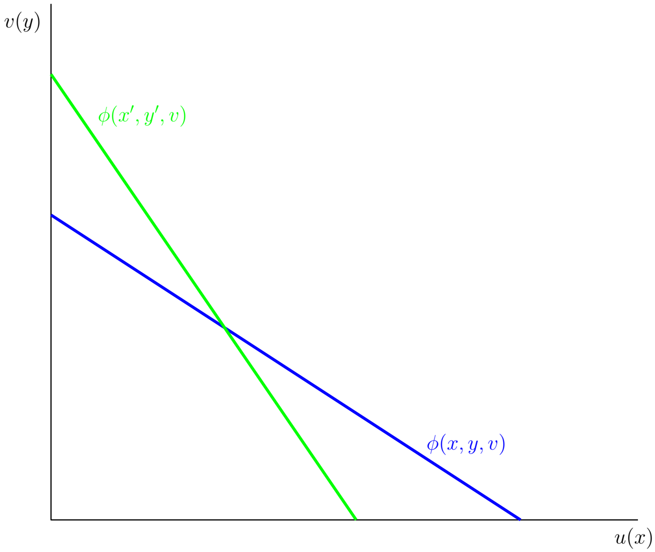
- There is positive assortative matching, i.e., $\mu' > 0$ provided

$$\phi_{12} > \frac{\phi_1}{\phi_3} \phi_{23}$$

- Note equivalent condition: $\psi_{12} > \frac{\psi_2}{\psi_3} \psi_{13}$

Supermodularity and Matching

Generalized Increasing Differences



Applications

Who matches with whom?

- Common preferences: $u(w(x))$ and $v(\pi(y))$: non-linear, but no changes
- Risk aversion and Uncertainty: will the risk averse match with the risk neutral/loving?
- Principal–Agent relations: need non-linear preferences for standard P/A model; what if P,A differ in skill, risk aversion,...? Who matches with whom?
- Household public goods
- References: Legros-Newman, Chiappori-Reny, Serfes, Ackerberg-Botticini,...

Large Firms

- Many to one matching w/ transfers: Kelso-Crawford (1982)
- Workers $i = 1, ..., m$ and firms $j = 1, ..., n$
- Utility worker i at firm j with salary s_i is $u_{ij}(s_i)$
- $\forall i, \exists$ vector $\sigma^i \equiv (\sigma_{i1}, ..., \sigma_{in})$ where σ_{ij} lowest salary i would accept at firm j (= value of unemployment $u_{i0}(0)$)
- For firm j and subset C of workers, $Y^j(C)$ is firm's income
    1. $Y^j(\emptyset) = 0$ (production requires workers)
    2. $Y^j(C \cup \{i\}) - Y^j(C) > \sigma_{ij}$ for any C which does not contain i (marginal contribution > value unemployment)
- A *matching* is a set of disjoint partnerships $\{j, C^j\}$
- An *outcome* (μ, π, s) is a matching μ and for each $\{j, C^j\}$ an allocation: $Y^j(C^j) = \pi_j + \sum_{i \in C_j} s_j$
- (μ, π, s) is *individually rational* if $s_i \geq \sigma_{i\mu(i)}$ and $\pi_j \geq 0$
- Salaries are modeled as discrete variables (pennies)

Large Firms

- An individually rational outcome (μ, π, s) is a core allocation unless there is a firm j, a subset of workers C and a vector r of salaries r_i for all workers i in C such that

$$\begin{aligned} \pi_j &< Y^j(C^j) - \sum_{i \in C_j} r_j \\ u_{i\mu(i)} &< u_{ij}(r_i) \end{aligned}$$

  for all i in C.
- If these two inequalities are satisfied for some (j, C, r), then the outcome (μ, π, s) is *blocked* by (j, C, r)
- Core may be empty

Large Firms

Empty Core – Example

- 2 firms, j, k, 2 workers $1, 2$. Workers' utility equals their salary
- Firms' income $Y^j(C)$ and $Y^k(C)$ for subsets of workers is:

$$\begin{array}{ll} Y^j(\{1\}) = 4 & Y^k(\{1\}) = 8 \\ Y^j(\{2\}) = 1 & Y^k(\{2\}) = 5 \\ Y^j(\{1,2\}) = 10 & Y^k(\{1,2\}) = 9 \end{array}$$

- The only matchings at which no worker is unemployed are

$$\begin{array}{ll} \mu_1 = \{j, \{1,2\}\}, \{k\} & \mu_3 = \{j\}, \{k, \{1,2\}\} \\ \mu_2 = \{j, \{1\}\}, \{k, \{2\}\} & \mu_4 = \{j, \{2\}\}, \{k, \{1\}\} \end{array}$$

- All are blocked. For example, μ_1 is not stable. Blocked if:
  1. $s_j(1) < 8$ since k is willing to offer up to 8.
  2. if $s_j(2) < 5$.
- Observe: j earns more from 1 and 2 than the sum from each worker separately, i.e. complementarity $\Rightarrow$ j employs 1 at $s_j(1) > 4$ only if 2 is also employed

Large Firms

Gross Substitutes

- Defintion of GS:
  - Let $M^j(s^j) = \arg\max_C \pi^j(C; s^j)$
  - Consider 2 vectors $s^j, \tilde{s}^j$ and $T^j(X^j) \equiv \{i | i \in C^j \text{ and } s^j = \tilde{s}^j\}$
  - Then $\forall j$, if $C^j \in M^j(S^j)$ and $\tilde{s}^j \geq s^j$, then:

$$\exists \tilde{C}^j \in M^j(\tilde{s}^j) \text{ such that } T^j(C^j) \subseteq \tilde{C}^j$$

- All workers must be gross substitutes to each firm: increases in other workers' salaries $\not\Rightarrow$ withdraw an offer from a worker whose salary has not risen
- additive separability of prod. technol. $\Rightarrow$ GS, but more general
- GS is sufficient, not necessary
- Also relevant in package auctions (see Hatfield and Milgrom)

Sorting in Large Firms

- Background:
  - Matching: one-to-one (e.g. Becker 1973) $\rightarrow$ extensive margin
  - Macro / Labor / Trade / Urban / Devel: intensive margin
  - Intensive Margin $\Rightarrow$ Firm Size

- Trade-Off: better workers vs. more workers
  - managerial time: "span of control": Sattinger 75, Lucas 78
  - assignment of land, of "distance", of assets...

Sorting in Large Firms

- Goals:
  1. Capture factor intensity in tractable manner (no peer effects)
  2. Sorting condition: complementarity quality vs. quantity
  3. Characterize firm size, assignment, wages
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- Economic Relevance
  1. Characterizing production technology across industries: Walmart vs. mom-&-pop store; consulting and law firms;...
  2. Misallocation debate: output difference across economies
     - Firm heterogeneity in productivity $\rightarrow$ differences in K, p, A (Restuccia-Rogerson (08), Hsieh-Klenow (10),...)
     - Intensive margin *and* heterogeneity
     - Also worker heterogeneity $\Rightarrow$ skill (mis)allocation and human capital distribution matter

Intensive and Extensive Margin

- Output for given worker type in firm y with resources r:

$$F(x, y, l, r)$$

Trade-off between better workers (x) and more (l) workers

- Firm chooses $x_1, x_2, ...$ and $l_1, l_2 ...$ and for each intensity $r_1, r_2, ...$
- Total output of firm y:

$$F(\underbrace{x_1, y}_{quality}, \underbrace{l_1, r_1}_{quantity}) + F(x_2, y, l_2, r_2) + \cdots$$

where $r_1 + r_2 + \cdots = 1$

The Model

- *Population*
  - Workers of type $x \in X = [\underline{x}, \overline{x}]$, distribution $H^w(x)$
  - Firms of types $y \in Y = [\underline{y}, \overline{y}]$, distribution $H^f(y)$
- *Production of firm y* $F(x, y, l_x, r_x)$
  - l_x workers of type x,r_x fraction of firm's resources
  - F increasing in all, concave in last two arguments
  - F constant returns to scale in last two arguments

$\Rightarrow$ Denote: $f(x, y, \theta) = rF\left(x, y, \frac{l}{r}, 1\right)$, where $\theta = \frac{l}{r}$

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- *Preferences*
  - transferable utility (additive in output goods and numeraire)

Literature

Special Cases

- Becker 73: $l_{ji} = r_{ij} \quad \rightarrow \quad F(x, y, \min\{l, r\}, \min\{l, r\})$
- Sattinger 75: $l_{ji} \leq \frac{r_{ij}}{t(x_i, y_i)} \quad \rightarrow \quad F = \min\left\{l, \frac{r}{t(x,y)}\right\}$
- Garicano 00: $l \leq \frac{r}{t(x)} \quad \rightarrow \quad F = y \min\left\{l, \frac{r}{t(x)}\right\}$
- Lucas 78: Worker input independent of skill $F = yg(l)$
- Rosen 74: more general; existence

  (also, Kelso-Crawford 82, Cole-Prescott 97, Gul-Stacchetti 99, Milgrom-Hatfield 05)
- Roy 51: $l_{ji} = r_{ij}$ & no factor intensity
- Roy 51+CES: particular functional form for decreasing return
- Frictional Markets: one-on-one matching, competitive search

  (Shimer-Smith 00, Atakan 06, Mortensen-Wright 03, Shi 02, Shimer 05, Eeckhout-Kircher 10)

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Hedonic wage schedule $w(x)$ taken as given.

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- *Feasible Resource Allocation*:

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- *Feasible Resource Allocation*:
  - $\mathcal{R}(x, y, \theta)$: resources to any $x' \leq x$ by any $y' \leq y$ with $\frac{l_{x'}}{r_{x'}} \leq \theta$.
    1. Resource feasibility $[\mathcal{R}(y|X, \Theta) \leq H^f(y)\ \forall y]$
    2. Worker feasibility $[\int_{\theta \in \Theta} \int_{x' \leq x} \theta d\mathcal{R}(\theta, x'|Y) \leq H^w(x)\ \forall x]$
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- *Competitive Equilibrium* is a tuple $(w, \mathcal{R})$ s.t.
  1. Optimality Cond. $[(x, y, \theta) \in \text{supp}\mathcal{R}$ only if it satisfies $(\star)]$
  2. Market Clearing $[\int \theta d\mathcal{R}(\theta|x, Y) \leq h^w(x),$ "$=$" if $w(x) > 0, \forall x]$

Assortative Matching

Definition (Assortative Matching)

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Proposition (Condition for PAM)

A necessary condition to have equilibria with PAM is that

$$F_{12}F_{34} \geq F_{23}F_{14}$$

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holds along the equilibrium path. The reverse inequality entails NAM.

- Necessary and sufficient for *any* distribution of x, y.

Assortative Matching

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- Interpretation ($F_{34} > 0$ by assumption):

  1. $F_{12} > 0$: bet. manag. produce more w/ bet. workers (Becker)
  2. $F_{23} > 0$: bet. manag., larger span of control (as in Lucas)
  3. $F_{14} > 0$: bet. workers produce more w/ manag. time (school?)

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- Quantity-quality trade-off by firm y with resources r:

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- Examples: technological differences across industries, establishments
  1. Walmart vs. mom-&-pop store: low x, high y, high θ, $\theta' < 0$

     $\Rightarrow$ $F_{23} > 0$, $F_{14} > 0$, F_{12} not too large $\Rightarrow$ NAM
  2. Law firm, Mgt Consulting: high x, high y, low θ, $\theta' > 0$

     $\Rightarrow$ $F_{14} > 0$, $F_{23} > 0$, F_{12} large $\Rightarrow$ PAM

Sketch of Proof of PAM-Condition

Assume PAM allocation with resources on $(x, \mu(x), \theta(x))$. Must be optimal, i.e., maximizes:

$$\max_{x,\theta} f(x, \mu(x), \theta) - \theta w(x).$$

First order conditions:

$$\begin{aligned} f_\theta(x, \mu(x), \theta(x)) - w(x) &= 0 \\ f_x(x, \mu(x), \theta(x)) - \theta(x) w'(x) &= 0 \end{aligned}$$

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The Hessian is

$$Hess = \begin{pmatrix} f_{\theta\theta} & f_{x\theta} - w'(x) \\ f_{x\theta} - w'(x) & f_{xx} - \theta w''(x) \end{pmatrix}.$$

Second order condition requires $|Hess| \geq 0$:

$$f_{\theta\theta}[f_{xx} - \theta w''(x)] - (f_{x\theta} - w'(x))^2 \geq 0$$

Differentiate FOC's with respect to x, substitute:

$$-\mu'(x)[f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_x/\theta] \geq 0$$

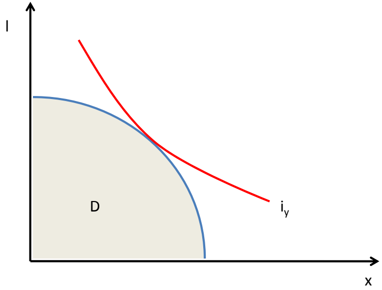
Positive sorting means $\mu'(x) > 0$, requiring $[\ \cdot\ ] < 0$ and after rearranging:

$$F_{12} F_{34} \geq F_{23} F_{14}$$

$F_{12}F_{34} > F_{23}F_{14}$: Graphical

Budget Set: $D = \{(x, l) | lw(x) \leq M\}$

Iso-output Curve: $i_y = \{(x, l) | F(x, y, l, 1) = \Pi\}$



Slope of Iso-output Curve: $\frac{\partial l}{\partial x} = -\frac{F_1(x,y,l,1)}{F_3(x,y,l,1)}$.

Fix $F_{23} > 0$ and consider better firm:

- If $F_{12} \simeq 0$, higher y has flatter slope (numerator is constant).
- If $F_{12} \gg 0$, then higher y will have steeper slope.

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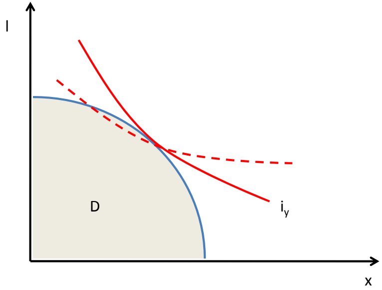

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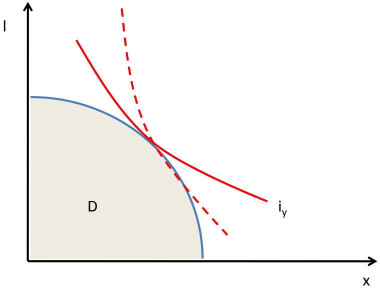

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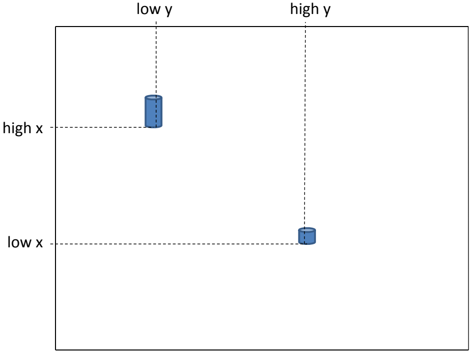
Efficiency: Gains from "Re-sorting"

Assume $F_{12}F_{34} > F_{23}F_{14}$ but negative sorting. Then improved output after re-sorting.



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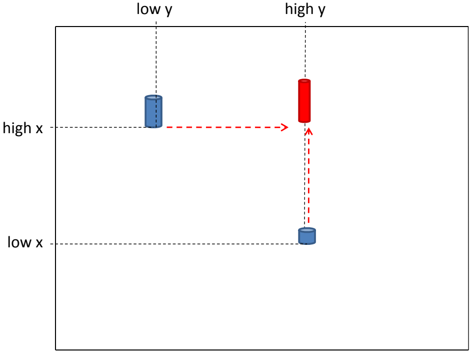

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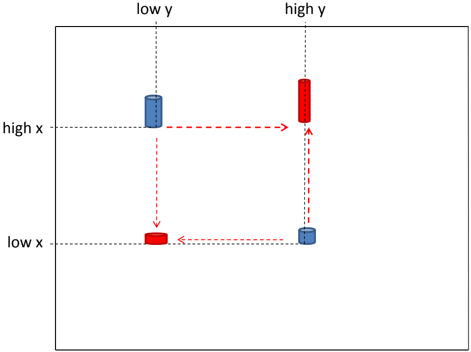

Special Cases

Efficiency Units of Labor

- Skill "=" Quantity: $F(x, y, l, r) = \tilde{F}(y, xl, r) \quad \Rightarrow \quad F_{12}F_{34} = F_{23}F_{14}$

Special Cases

Efficiency Units of Labor

- Skill "=" Quantity: $F(x, y, l, r) = \tilde{F}(y, xl, r) \quad \Rightarrow \quad F_{12}F_{34} = F_{23}F_{14}$

Multiplicative Separability

- $F(x, y, l, r) = A(x, y)B(l, r)$ sorting if $\frac{AA_{12}}{A_1 A_2} \frac{BB_{12}}{B_1 B_2} \geq 1$
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Spacial sorting in mono-centric city:

- $F(x,y,l,r) = l(xg(y) + v(r/l)) \Rightarrow$ higher earners in center.

Firm Size, Assignment, Wages

Proposition

Under assortative matching (symmetric distributions of x, y):

$$PAM \quad : \quad \theta'(x) = \frac{F_{23} - F_{14}}{F_{34}}; \quad \mu'(x) = \frac{1}{\theta(x)} \quad ; \quad w'(x) = \frac{F_1}{\theta(x)},$$

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θ' from FOC: $f_\theta = w(x)$ and $f_x/\theta = w'$, diff. and subst. μ'.

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Corollary

Under assortative matching, better firms hire more workers if and only if along the equilibrium path

$$F_{23} > F_{14} \text{ under PAM, and} \qquad -F_{23} < F_{14} \text{ under NAM.}$$

Firm Size, Assignment, Wages

Proposition

Under assortative matching $\qquad \mathcal{H}(x) = \frac{h_w}{h_f}$

$$PAM \quad : \qquad \theta'(x) = \frac{\mathcal{H}(x)F_{23} - F_{14}}{F_{34}}; \quad \mu'(x) = \frac{1}{\theta(x)}\mathcal{H}(x); \quad w'(x) = \frac{F_1}{\theta(x)},$$

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Firm Size under PAM

$F_{23} > F_{14}$

- Firm size increasing depends on relative strength of
  1. F_{23}: span of control
  2. F_{14}: resource intensity of labor
- If marginal impact of output from firm y' span of control is larger than worker x's marginal impact of resources $\Rightarrow$ high productivity firms are larger

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- Special case: Lucas 78

General Capital, Monopolistic Competition

- General Capital:
  - $F(x, y, l, r) = \max_k \hat{F}(x, y, l, r, k) - ik$; Sorting cond. on max

General Capital, Monopolistic Competition

- General Capital:
  - $F(x, y, l, r) = \max_k \hat{F}(x, y, l, r, k) - ik$; Sorting cond. on max
- Monopolistic Competition in the Output Market:
  - consumers have CES preferences with substitution ρ
  - sales revenue of firm y: $\chi F(x, y, l, 1)^\rho$
  - Sorting condition

$$\left[\rho \tilde{F}_{12} + (1-\rho)(\tilde{F})\frac{\partial^2 \ln \tilde{F}}{\partial x \partial y}\right]\left[\rho \tilde{F}_{34} - (1-\rho) l \tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial l^2}\right]$$
$$\geq \left[\rho \tilde{F}_{23} + (1-\rho)\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial y \partial l}\right]\left[\rho \tilde{F}_{14} + (1-\rho)\left(l\tilde{F}_{13} - l\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial x \partial r}\right)\right].$$

  - independent of χ
  - our condition under $\rho = 1$, log-sm when production linear in l.

Wrap up on Large Firms

The model:

- Lay out a matching model with factor intensity
- Derive tractable sorting condition ($F_{12}F_{34} \geq F_{14}F_{23}$)
- Characterize equilibrium firm size, assignment and wages
- Search frictions: relation unemployment, skill and firm size

Wrap up on Large Firms

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Economic Relevance & Applications in trade/macro/labor...:

- Mismatch debate: worker heterogeneity matters
- Comparative statics: impact of aggregate fluctuations
- Empirical: How does unemployment change across skills/firm size?

Topics in Labor Markets

Jan Eeckhout

2015–2016

II. Random Search and Sorting

Search Frictions

- Centralized trade has strong information requirement
- Even if observe all information, need coordination
- Introduce decentralized trade $\Rightarrow$ search frictions
- True in all trade environments (even on stock exchanges – market microstructure)
- Result in labor markets: unemployment
- Study unemployment as an equilibrium phenomenon

Search Frictions

- Information problem: cannot simply broadcast the information and trade efficiently
- Sources of Frictions?
  - Time to find (searching): dynamic
  - Inspection, need time to ascertain quality: heterogeneity
  - Coordination failure, all turn up at the same location: strategic
  - Need to trade now, cannot wait for all in mechanism
  - ...
- Focus on:
  - Random search
  - Directed search
  - Market segregation
- How does it affect match formation and sorting? Wages?

Search Frictions

The Classical Models

- Sampling from a distribution of wages: McCall (partial equilibrium); Rotschild
- Equilibrium unemployment: Mortensen-Pissarides
- On the job search: Burdett-Mortensen

Assortative Matching With Search Frictions

(Loading movie.mp4)

Search Frictions

No Transfers

- McNamara-Collins (93), Morgan (95), Burdett-Coles (97), Eeckhout (99), Bloch-Ryder (00), Smith (06)
- x, y from disjoint sets $[0, 1]$ (density of unmatched $u(x)$), assume "cloning"; ρ arrival rate; symmetric problem
- $f(x, y)$ utility (non-transferable!) to x given match with y. With probability δ the match is dissolved
- $v(x)$: x's expected value of being unmatched, $v(x|y)$ the analogous value from being matched with y.
- In an interval dt, a single's Bellman equation is:

$$v(x) = \frac{1}{1+rdt}\left[\left(\rho dt \int_{\Omega(x)} u(y)dy\right)\mathbb{E}[\max\{v(x|y), v(x)\}|y \in \Omega(x)] + \left(1 - \rho dt \int_{\Omega(x)} u(y)dy\right) v(x)\right]$$

$$\Rightarrow \quad rv(x) = \rho \int_{\Omega(x)} \max\{v(x|y) - v(x), 0\} u(y)dy$$

Search Frictions

No Transfers

- $v(x|y) \geq v(x) \iff f(x,y) \geq rv(x)$: accept all types y such that $y \geq a(x)$, or $A(x) = [0, a(x)]$

$\Rightarrow$ $\Omega(x) = \{y | x \geq a(y)\}$

- Insert $v(x|y)$ and the threshold $a(x)$ into $v(x)$:

$$rv(x) = \frac{\int_{[a(x),1]\cap\Omega(x)} f(x,y)u(y)dy}{\psi + \int_{[a(x),1]\cap\Omega(x)} u(y)dy}$$

  where $\psi = (r+\delta)/\rho$ is a measure of the frictions in the model.

- An equilibrium: threshold function a and density u s.t. $a(x)$ is optimal for each x given u, and flow equation for every x:

$$\delta\left(g(x) - u(x)\right) = \rho u(x) \int_{[a(x),1]\cap\Omega(x)} u(y)dy.$$

Search Frictions

No Transfers

- Let us *assume* for now that $\Omega(x) = [0, b(x)]$ for all x, and that b is an increasing function of x. Then:

$$rv(x) = \max_{a \in [0,1]} \frac{\int_a^{b(x)} f(x,y)u(y)dy}{\psi + \int_a^{b(x)} u(y)dy},$$

- First-order condition (differentiating wrt a):

$$\psi = \int_a^{b(x)} \left(\frac{f(x,y)}{f(x,a)} - 1 \right) u(y)dy.$$

- Fixed search cost, similar:

$$c = \int_a^{b(x)} \left(f(x,y) - f(x,a) \right) u(y)dy,$$

Search Frictions

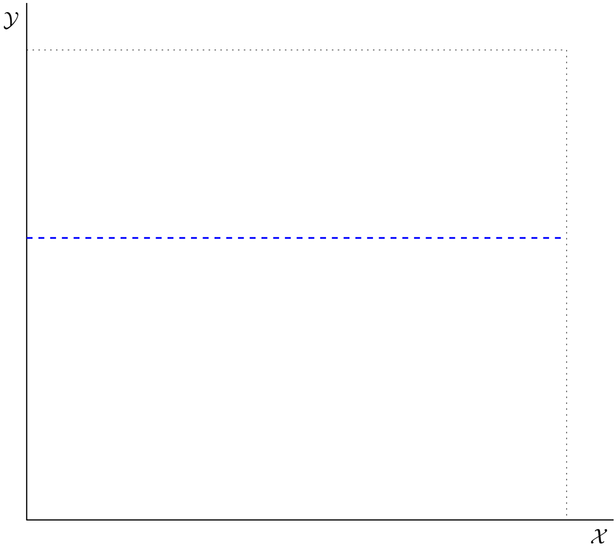
No Transfers

- Result: if $f(x, y)$ multiplicatively separable, then $a(x) = a$
- Let $f(x, y) = f_1(x) f_2(y)$. Then solution:

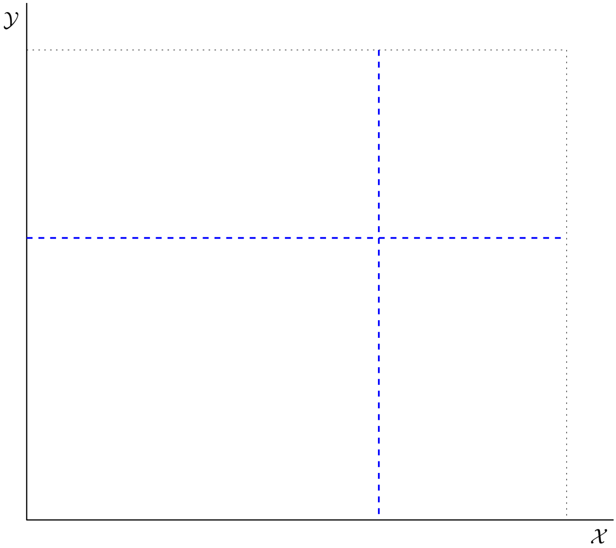
$$\psi f_1(x) f_2(a) - \int_{\Omega} \left[f_1(x) f_2(y) - f_1(x) f_2(a) \right] u(y) dy = 0$$

$\Rightarrow$ a is independent of x: class formation

Class Formation



Class Formation



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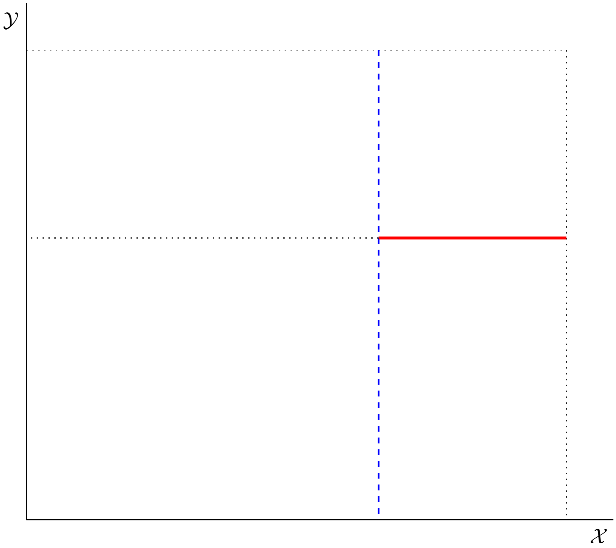

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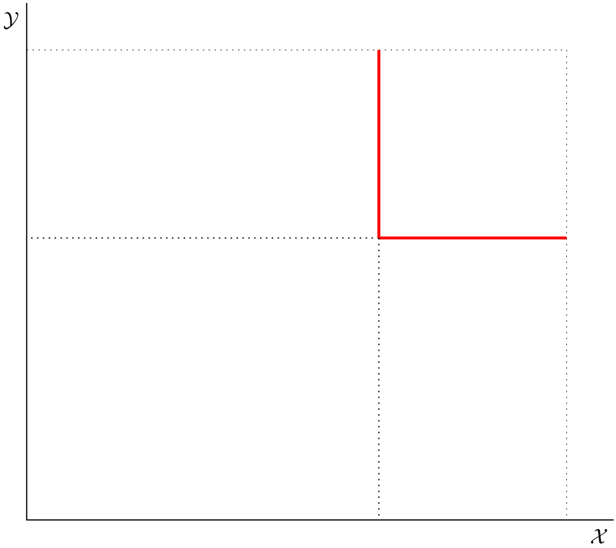

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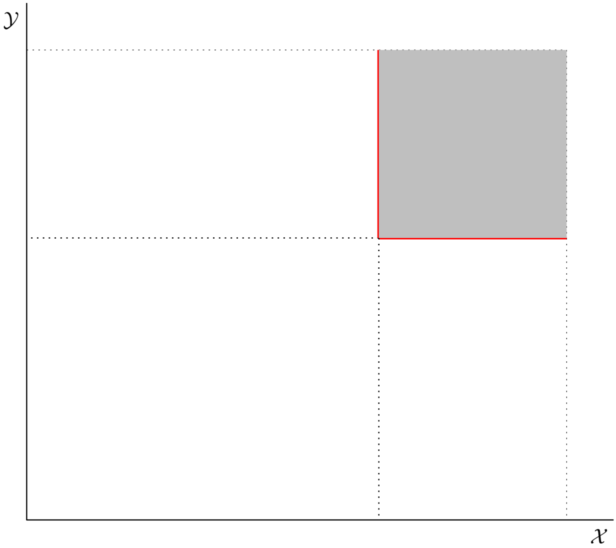

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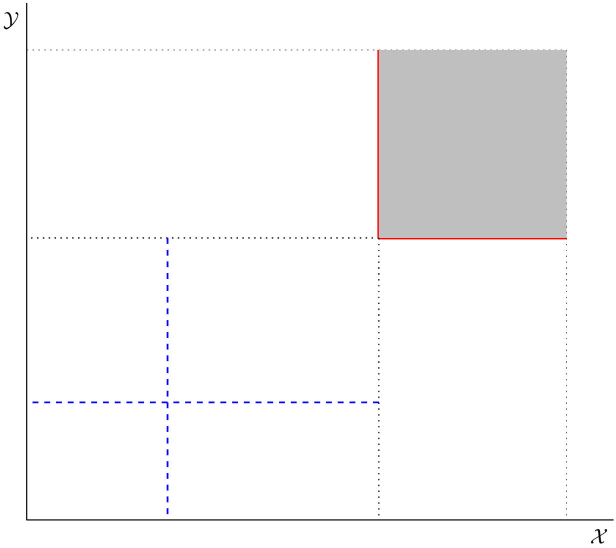

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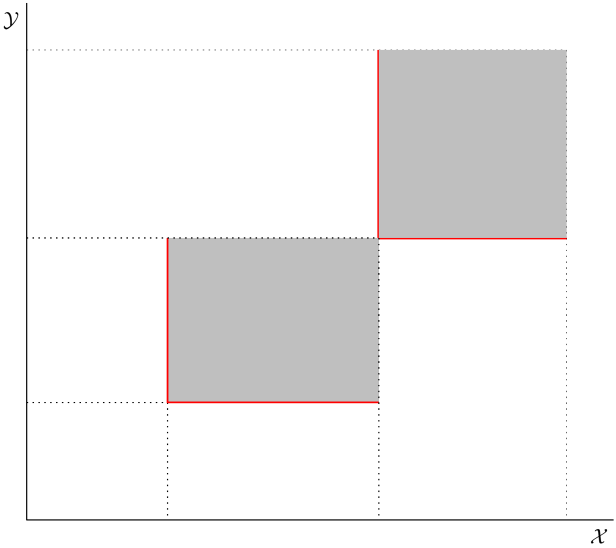

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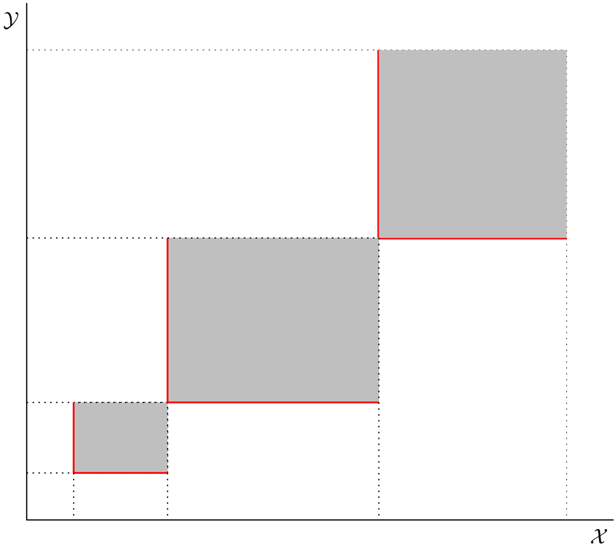

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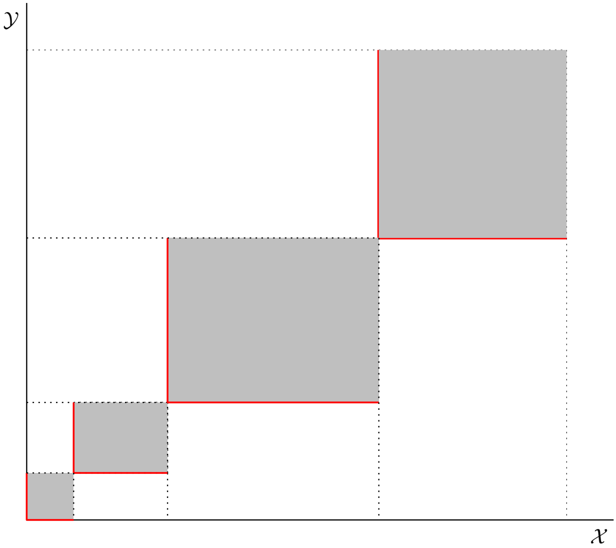


Search Frictions
No Transfers

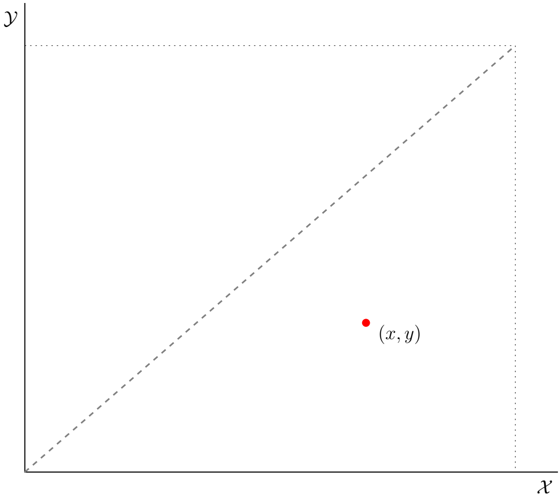
- Characterization of equilibrium allocation: PAM/NAM
- Smith (2006): symmetric (or one-sided), $f(x, y) = f(x, y)$
- If $f(x, y)$ log-supermodular, then PAM:

$$f(x_2, y_2)f(x_1, y_1) > f(x_2, y_1)f(x_1, y_2) \ \text{ for } \ x_2 > x_1, y_2 > y_1$$

- With equality: class formation, weak PAM

PAM – Definition

- Acceptance set Ω s.t. if (x, y) are matched, then also (x, x) and (y, y)



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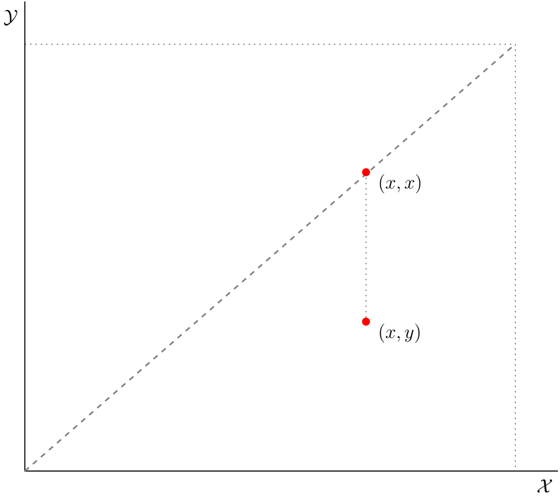

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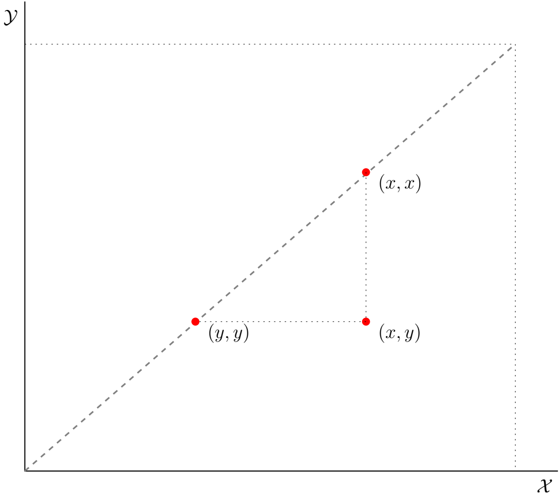

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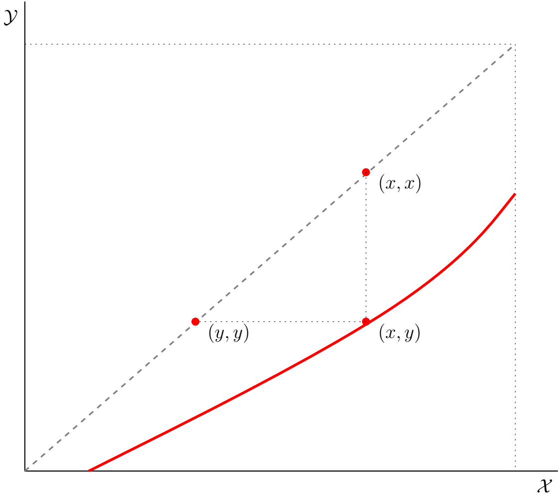

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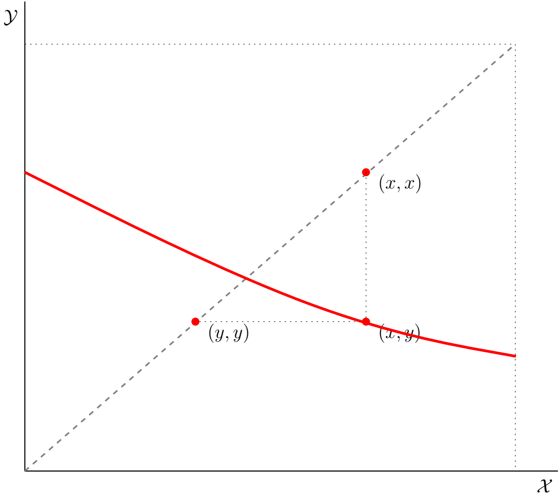
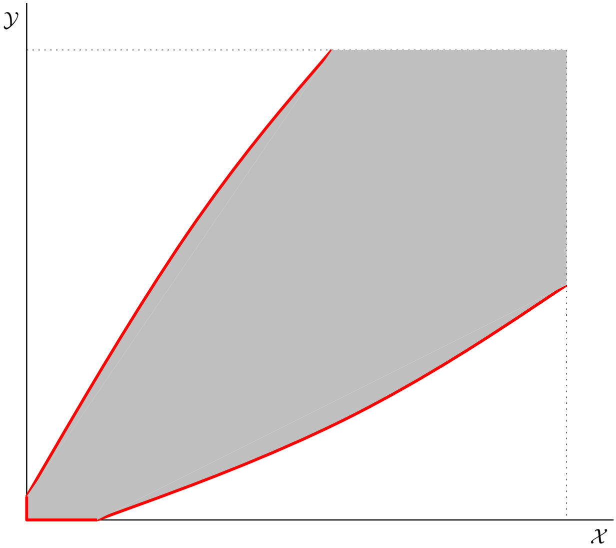


LOG-SUPERMODULARITY ⇒ PAM



Multiple Stationary Equilibria

- So far, population of singles was fixed: "clones"
- But, not realistic: different types match at different rates
- Matching rate is endogenous: depends on strategy of others
- Endogenous distribution of singles

$\Rightarrow$ multiple stationary equilibria (Burdett-Coles (1997))

Multiple Stationary Equilibria

- Burdett-Coles example, use their notation
- Model:
  - 2 types: $x_H > x_L > 0$. Utility $u = x$.
  - α : meeting probability;
  - δ : probability of death;
  - β : measure of m/w entering the market.
  - Exogenous inflow from distribution $F_w = F_m = F(x)$

$$F(x) = \begin{cases} 0, & \text{if } x < x_L \\ 1 - \pi, & \text{if } x_L \leq x < x_H \\ 1, & \text{if } x \geq x_H \end{cases}$$

  - Let λ be the class size, and y is the reservation value (cf. ϕ)
- Let $N(t)$: total # single agents $N(t) = N_H + N_L$
- Denote $\eta(t)$ the % of singles of type H, then $N_H = \eta N$

Multiple Stationary Equilibria

I. Single Class Equilibirium

- H's marry L's
- The law of motion (where $\dot{N}(t)$ is the time derivative):

$$\begin{aligned} \dot{N}(t) &= \beta - (\alpha+\delta)N(t) \\ \dot{N}_H(t) &= \beta\pi - (\alpha+\delta)N_H(t) \end{aligned}$$

with

$$\dot{\eta} = \frac{\dot{N}_H}{\dot{N}}; \quad \eta = \frac{N_H}{N}$$

so that

$$\dot{\eta} \;=\; \frac{\beta\pi - (\alpha+\delta)N_H(t)}{\beta - (\alpha+\delta)N(t)} = \frac{\beta}{N(t)}(\pi - \eta).$$

- In a steady state, $\dot{N}(t) = 0$ so that

$$\beta - (\alpha+\delta)N(t) = 0 \quad \Rightarrow \quad N(t) = \frac{\beta}{\alpha+\delta}$$

Multiple Stationary Equilibria

I. Single Class Equilibirium

- Likewise for $\dot{N}_H(t) = 0$

$$N_H(t) = \frac{\beta\pi}{\alpha + \delta}$$

$$\Rightarrow \quad \eta = \pi$$

- H's accept L's $\Rightarrow$ reservation type $y(1) \leq x_L$ and class size is $\lambda_1 = 1$ and $\eta = \pi$
- We can write

$$y = \frac{\alpha}{r + \delta + \alpha} \left(\pi x_H + (1 - \pi) x_L \right) \leq x_L$$

$$\Rightarrow \quad \alpha \leq \frac{r + \delta}{\pi} \frac{x_L}{x_H - x_L}$$

Multiple Stationary Equilibria

II. Elitist Equilibrium

- H's only marry H's
- Laws of motion

$$\begin{aligned}\dot{N}(t) &= \beta - (\alpha\left(\eta^2 + (1-\eta)^2\right) + \delta)N(t)\\ \dot{N}_H(t) &= \beta\pi - (\alpha\eta + \delta)N_H(t)\end{aligned}$$

where η^2 is the probability that 2 high types meet

- In a steady state implies $\dot{N}(t) = 0$ and $\dot{N}_H(t) = 0$ or

$$\overline{N} = \frac{\beta}{\alpha\left(\eta^2 + (1-\eta)^2\right) + \delta} \quad \text{and} \quad \overline{N}_H = \frac{\beta\pi}{\alpha\eta + \delta}$$

- Since $\eta = \frac{N_H}{N}$, it follows that

$$\overline{\pi} = \frac{\overline{\eta}(\alpha\overline{\eta} + \delta)}{\alpha\left(\overline{\eta}^2 + (1-\overline{\eta})^2\right) + \delta}$$

Multiple Stationary Equilibria

II. Elitist Equilibrium

- We show that $\lambda_1 = \overline{\eta}(\pi)$ and $\lambda_2 = 1 - \overline{\eta}(\pi)$ can indeed be an equilibrium. This requires that

$$y = \frac{\alpha \overline{\eta}(\pi) x_H}{r + \delta + \alpha \overline{\eta}(\pi)} > x_L \quad \Rightarrow \quad \alpha > \frac{r + \delta}{\overline{\eta}(\pi)} \frac{x_L}{x_H - x_L}$$

$\Rightarrow$ Multiple equilibria iff

$$\alpha \in \left[\frac{r + \delta}{\overline{\eta}(\pi)} \frac{x_L}{x_H - x_L}, \frac{r + \delta}{\pi} \frac{x_L}{x_H - x_L} \right]$$

- Observe that:
  1. $\overline{\eta}(\pi) > \pi$: only if $\pi < 0.5$: # singles of H type increases as the the probability of match is lower
  2. as $x_H - x_L$ increases, the range for permissible α decreases.

Multiple Stationary Equilibria

- Multiplicity important for policy: Diamond (1982) provides rationale for Keynesian demand management
- But, Diamond needs IRTS in the matching function $M(u, v)$: $2\times$ population $\Rightarrow$ more than $2 \times \#$ matches
- IRTS, obvious, like network externalities
- But there is evidence of CRTS: $M(u, v) = v \cdot m\left(\frac{v}{u}, 1\right)$; see Petrongolo-Pissarides
- Here: generate multiplicity with CRTS, but based on *s*election

Search Frictions

Transfers

- Now: can transfer utility between matched partners (bribe partner to accept!)
- Big implication: before there could be disagreement in acceptance decision
- Now, transfers make everyone agree: if the surplus over continuation is positive $\Rightarrow$ match
- Transfers? Obvious in labor and goods market: there are prices; marriage: money? roses? washing dishes? child care?...

Search Frictions

Transfers

- Shimer and Smith (2000): TU + Nash Bargaining: upon meeting, value of match surplus is split
- Value function

$$(r+\delta)V(x) = \rho \int_\Omega [v(x|y) - v(x)]\, u(y)dy$$

where $rv(x|y) = f(x,y) + \kappa\,[v(x) - v(x|y)]$, or equivalent ly

$$(r+\kappa)\,[v(x|y) - v(x) + v(y|x) - v(y)] = f(x,y) - rv(x) - rv(y)$$

- Total surplus $s(x,y)$: output net of continuation values
- Form match if $v(x|y) - v(x) + v(y|x) - v(y) \geq 0$.
- Nash bargaining

$$v(x|y) - v(x) = v(y|x) - v(y) = \frac{1}{2}\frac{1}{\delta + r}\,[f(x,y) - rv(x) - rv(y)]$$

Search Frictions

Transfers

- Substitute in Bellman equation:

$$rv(x) = \frac{1}{2}\frac{\rho}{\delta + r}\int_{\Omega(x)} \max\{f(x,y) - rv(x) - rv(y), 0\}\, u(y)dy$$

- With $\psi = (r+\kappa)/\rho$ is a measure of the frictions:

$$rv(x) = \frac{\int_{[a(x),1]\cap\Omega(x)} [f(x,y) - rv(y)]u(y)dy}{2\psi + \int_{[a(x),1]\cap\Omega(x)} u(y)dy}$$

- The first order condition for the optimal a implies

$$\psi = \frac{1}{2}\int_{a}^{b(x)} \left(\frac{f(x,y) - rv(y)}{f(x,a) - rv(a)} - 1\right) u(y)dy.$$

- Observe:
  1. no explicit solution for $v(\cdot)$
  2. existence: hairy fixed point problem from law of motion

Search Frictions

Transfers

- Shimer-Smith (2000). PAM if

$$f_{xy} > 0, \quad (\log f_x)_{xy} > 0, \quad (\log f_{xy})_{xy} > 0$$

- With additional assumption of monotonicity $f_x, f_y > 0$, Eeckhout-Kircher (2010) show that these conditions imply

$$(\log f)_{xy} > 0$$

- Log-supermodularity $\Rightarrow$ PAM
- Shimer and Smith: more general, existence proof (hard due to endogeneity of distribution of singles)

Search Frictions

Constant search costs

- Morgan(95), Chade(01), Atakan(06), Eeckhout-Kircher(10)
- Before, search cost is opportunity cost of time (e.g. when search is time consuming), proportional to value
- When search obtains swiftly, the appropriate measure of search costs is money rather than time
- Value function ($\alpha = 0$):

$$V = -c + \int_{\mathcal{A}} u(x, \tilde{y}) dF_y(\tilde{y}) + \int_{\neg\mathcal{A}} V dF_y(\tilde{y})$$

or given acceptance if $u(x, \phi) = V$

$$c = \int_{\mathcal{A}} \left[u(x, \tilde{y}) - u(x, \phi) \right] dF_y(\tilde{y})$$

Search Frictions

Constant search costs

- Now a sufficient condition for assortative matching is that $u(x, y)$ is supermodular

$$c = \int_{\mathcal{A}} [u(x, \tilde{y}) - u(x, \phi)] \, dF_y(\tilde{y})$$

  compared to discounting

$$\frac{r}{\beta} = \int_{\mathcal{A}} \left[\frac{u(x, \tilde{y})}{u(x, \phi)} - 1 \right] dF_y(\tilde{y})$$

- General search cost $c(x)$: whenever $c'(x) > 0$ need stronger-than-supermodularity in order to obtain Assortative Matching
- Reason: search cost $\uparrow$ for higher types $\Rightarrow$ less "picky"

Taking Stock

- In order to obtain PAM:
  - No frictions: Supermodularity
  - Discounting: Log-supermodularity
  - General cost: degree supermodularity proportional to $c'(x)$
- Intuition: Opportunity cost is higher for higher types $\Rightarrow$ choose faster acceptance (lower marginal type)

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- Random Search = Dumb search?
  - No information about types or prices: strong assumption!
  - Endogenous market segmentation: Jacquet and Tan (JPE 2007), (with match makers: Bloch and Ryder 2000)
  - Prices allocate resources: directed/competitive search

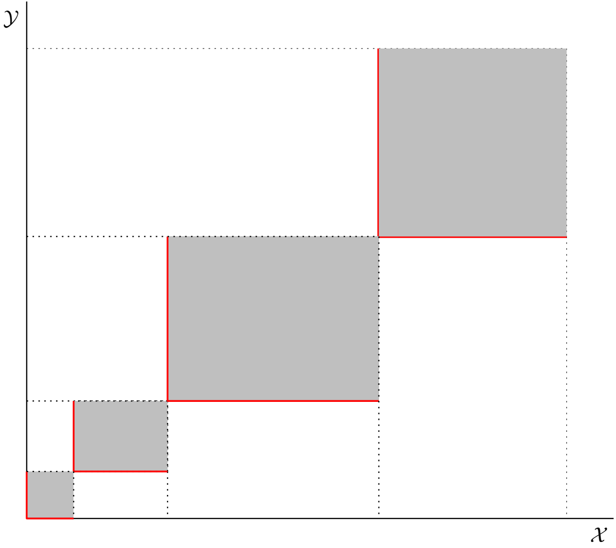
Market Segmentation

Jacquet and Tan (2007)

- Class formation: why not set up a market place for each class
- Advantage: do not have to meet the lower types you reject *and* higher types who reject you
- Disadvantage: none given constant returns to matching
- Now trade-off changes: can be more picky
- Induction: obtain perfect, frictionless matching allocation?

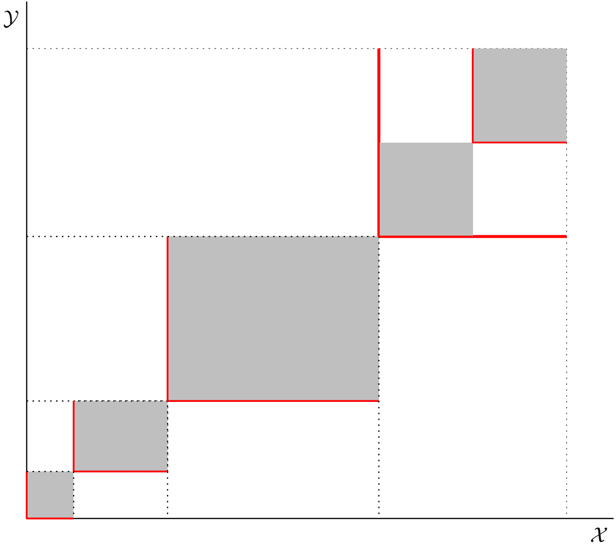
Market Segmentation

Form Subclasses



Market Segmentation

Form Subclasses



Market Segmentation

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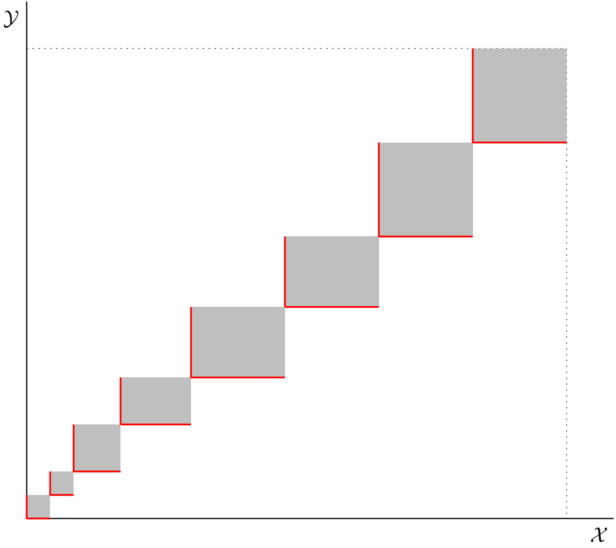
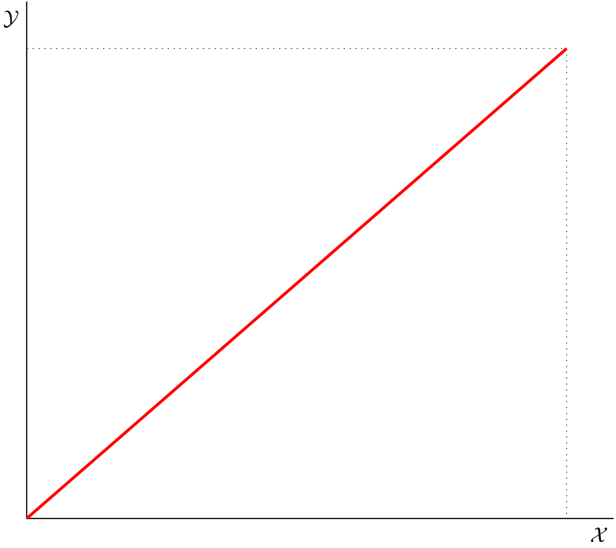


Market Segmentation

Frictionless?



Market Segmentation

- Induction: obtain perfect, frictionless matching allocation?
- No, when entry into the market is unrestricted

⇒ Public good component due to non-excludability

- High type cannot commit not to accept a match with a slightly lower type
- Still class formation in equilibrium
- Optimal allocation: zero measure, continuum of markets without mismatch

Identifying Sorting

How can we exploit variation due to search frictions (mismatch) to infer information about the degree of complementarities?

Identifying Sorting

How can we exploit variation due to search frictions (mismatch) to infer information about the degree of complementarities?

1. Do more productive workers work in more prod. jobs?
   - Positive exercise: learn about production / search process
2. Is sorting important? How big is it?
   - Normative exercise: matters for policy (depends on complementarities)

Identifying Sorting: sign and strength

- Constraint: use wage data only (most precise measure of job productivity) and matched employer-employee data
- Objective a minimalist, stylized model (assignment model – Becker (1973)) that allows us to show:
    1. Identifying the *sign* (1.) is impossible
       Reason: Workers get mainly paid by their marginal product

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  2. Identifying the *strength* (2.) is possible
     Choices reveal how big complementarities/substitutes are.
  3. Cannot be done with "standard" fixed-effect method
- Use of output/profit data possible, but mostly available at firm level; per individual worker difficult

  (Haltiwanger et al. (1999), van den Berg and van Vuuren (2003), Mendes, van den Berg, Lindeboom (2007))

  $\Rightarrow$ Need at least a theory of the firm

The fixed Effects Regression

- Evidence from fixed effects regressions (Abowd, Kramarz, and Margolis (1999), Abowd et al (2004),....):

$$\log w_{it} = a_{it}\beta + \delta_i + \psi_{j(i,t)} + \varepsilon_{it}$$

  where:
  - a_{it}: time varying observables of workers
  - δ_i : worker fixed effect
  - $\psi_{j(i,t)}$: fixed effect of firm (at which i works at t)
  - ε_{it}: orthogonal residual

- Correlation of δ_i and ψ_j between matched pairs is taken as an estimate of the degree of sorting
- Repeatedly established: zero or negative correlation $\Rightarrow$ no complementarities in the production technology?

Approach

- Characterize wages in the frictionless model
- Extend to search frictions $\Rightarrow \exists$ mismatch in equilibrium
- Derive analytically what we can learn from wage data

Relates to recent literature:

- Gautier, Teulings (2004, 2006)
  - Second-order approximation to steady-state; assumes PAM
- Lopes de Melo (2008), Lise, Meghir, Robin (2008), Bagger-Lentz (2008)
  - Simulated search models with strong complementarities give nonetheless small or negative fixed effect estimates
- Structural model of Abowd, Kramarz, Lengermann, Perez-Duarte (2009):
  - "test a simple version of Becker's matching model"
  - *assume* a split of output: $\beta f(x, y)$
  - is *inconsistent* with Becker's (1973) equilibrium wages

Findings

From wage data alone:

1. No identification of sign of sorting from wages:
   - on frictionless equilibrium allocation
   - off-equilibrium set
   - economy with frictions (constant costs)
2. Fixed effects pick up neither *sign* nor *strength*
3. BUT we can identify strength
   This is economically more meaningful than sign
4. Discussion: discounting, type-dependent search costs [some, (small) identification], more general technologies...

The Model

Players and Production

- Worker type x, distributed according to Γ (uniform)
- Job type y, distributed according to Υ (uniform)
- Output $f(x, y) \geq 0$
- Common rankings: $f_x > 0$ and $f_y > 0$
- Cross-partials either always positive ($f \in \mathcal{F}^+$ if $f_{xy} > 0$) or always negative ($f \in \mathcal{F}^-$ if $f_{xy} < 0$): monotone matching
- Examples of production functions we will use:

$$
\begin{aligned}
f^+(x, y) &= \alpha x^\theta y^\theta + h(x) + g(y), \\
f^-(x, y) &= \alpha x^\theta (1 - y)^\theta + h(x) + g(y),
\end{aligned}
$$

where $g(\cdot)$ and $h(\cdot)$ are increasing functions.

The Frictionless Model

On the equilibrium path

- Assignment of workers to firms: $\mu(x) = y$ (worker x to firm y)
- Wage schedule: $w(x)$
- Profit schedule: $\pi(y)$
- Equilibrium: μ and payoffs such that $\forall x, y$:

$$\begin{aligned} w(x) + \pi(y) &\geq f(x, y) \\ w(x) + \pi(\mu(x)) &= f(x, \mu(x)) \end{aligned}$$

The Frictionless Model

Becker's Result

- Firm maximization:

$$\max_x f(x, y) - w(x)$$

- FOC:

$$f_x(x, y) - \frac{\partial w(x)}{\partial x} = 0$$

- Let $w^\star(x)$ be the equilibrium wage of worker x

$$w^\star(x) = \int_0^x f_x(\tilde{x}, \mu(\tilde{x}))d\tilde{x} \; + w_0,$$

- Profits: $\pi^\star(y) = \int_0^y f_y(\mu^{-1}(\tilde{y}), \tilde{y})d\tilde{y} \; - w_0$

The Frictionless Model

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- PAM if f supermodular $(f_{xy} > 0) \Rightarrow \mu(x) = x$ (from the SOC)
- NAM if f submodular $(f_{xy} < 0) \Rightarrow \mu(x) = 1 - x$

The Frictionless Model

Cannot identify PAM/NAM

Proposition (1)

For any $f^+ \in \mathcal{F}^+$ that induces PAM there exists a $f^- \in \mathcal{F}^-$ that induces NAM with identical equilibrium wages $w^\star(x)$.

Proof.

$$\begin{aligned} w^{\star,+}(x) &= \int_0^x f_x^+(\tilde{x},\tilde{x})d\tilde{x} \; + w_0 \\ w^{\star,-}(x) &= \int_0^x f_x^-(\tilde{x},1-\tilde{x})d\tilde{x} \; + w_0 \end{aligned}$$

Sufficient: $f_x^+(\tilde{x},\tilde{x}) = f_x^-(\tilde{x},1-\tilde{x})$.

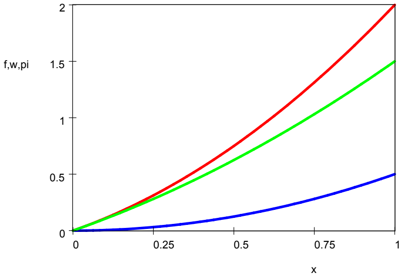
Define: $f^-(x,y) = f^+(x,1-y)$ on $[0,1]^2$

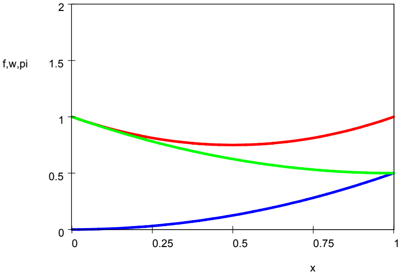
Need: f^- increasing in y. If f_y^- is bounded, add linear term. If not, $g(y)$ increases faster than $-f^+(x,1-y)$ $\square$

The Frictionless Model

Example with $\alpha = +/-1, \theta = 1$

- Wages: $w(x, \mu(x)) = \frac{x^2}{2}$
- Derived from $f^+ = xy + y$ and $f^- = x(1-y) + y$
- But $\pi^{\star,+}(y) = \frac{y^2}{2} + y$
  $\pi^{\star,-}(y) = y + \frac{(1-y)^2}{2}$, and $\pi^{\star,-}(x) = 1 - x + \frac{x^2}{2}$





The Frictionless Model
No Identification of PAM/NAM

- Based on wage data alone, we cannot "know" which are the good jobs (higher ranked y)
- The good worker matches with the most attractive firm
- Under NAM, the bad firm is more attractive because it pays higher wages

The Frictionless Model

Off the Equilibrium Allocation

Off-equilibrium wages between x and y (not matched):
("Trembles" to such wages yield independent variation).

- Equilibrium requires $w(x,y) \in W(x,y)$:

$$\begin{aligned} f(x,y) - w(x,y) &\leq \pi(\mu(x), y) \\ w(x,y) &\leq w(x, \mu(x)) \end{aligned}$$

- Examples: Bargaining split, firms or worker optimal wage

The Frictionless Model

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- Examples: Bargaining split, firms or worker optimal wage

Proposition (2)

For any $f^+ \in \mathcal{F}^+$ with PAM there exists $f^- \in \mathcal{F}^-$ with NAM and identical set of equilibrium wages $W^+(x,y) = W^-(x,1-y)$.

Mismatch due to Search Frictions

Two Stage Search Process:

1. First, costless random meeting stage
   - one round of pairwise random meetings
   - if match is formed: wage as split of surplus over waiting
2. Second, if not matched: costly competitive matching
   - pay search cost c each
   - get matched according to the competitive assignment
   - production at end

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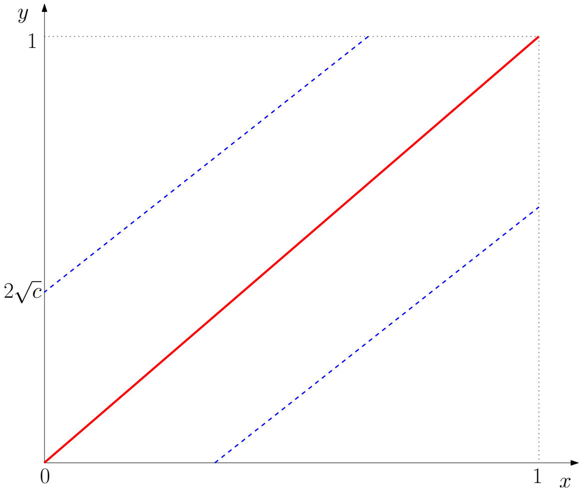
   - get matched according to the competitive assignment
   - production at end

- For simplicity assume symmetry
  - $f_{xy}(x, y) = f_{xy}(y, x)$ for $f \in \mathcal{F}^+$
  - $f_{xy}(x, y) = f_{xy}(1 - y, 1 - x)$ for $f \in \mathcal{F}^-$
- Second stage payoffs: $w(x, \mu(x)) - c$ and $\pi(\mu^{-1}(y), y) - c$
- First stage: Match provided

$$f(x, y) - (w^\star(x) + \pi^\star(y) - 2c) \geq 0$$

Mismatch due to Search Frictions

The Example: $\theta = 1$



Mismatch due to Search Frictions

Wages

$$
\begin{aligned}
w(x,y) &= \frac{1}{2}\left[f(x,y) - w(x,\mu) - \pi(\mu^{-1},y) + 2c\right] + w(x,\mu) - c \\
&= \frac{1}{2}\left[f(x,y) + w(x,\mu(x)) - \pi(\mu^{-1}(y),y)\right]
\end{aligned}
$$

Mismatch due to Search Frictions

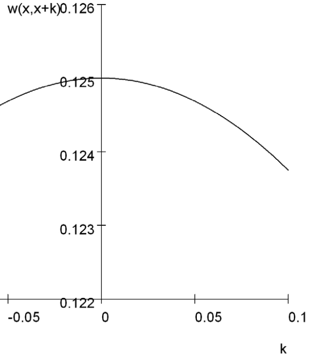
Proposition (3)

For any $f \in \mathcal{F}^+$ that induces PAM there exists a $f \in \mathcal{F}^-$ that induces NAM with identical equilibrium wages $w^\star(x)$.

- From wages alone we cannot identify the sign of f_{xy}
- Here: we aim to identify the strength of f_{xy} (i.e. $|f_{xy}|$)

Mismatch due to Search Frictions

Lemma: *(Bliss Point) Wages $w(x, y)$ are non-monotone in y.*



- Example. Mediocre lawyer in top firm: paid less than in mediocre firm. Top firm must forego higher future profit
- Obvious in model of competition (Becker), also in infinite horizon search models (see Gautier and Teulings (2006))

Mismatch due to Search Frictions

Inconclusive Firm Fixed Effect

Decompose wage process:

$$w(x,y) = \delta(x) + \psi(y) + \varepsilon_{xy}, \tag{1}$$

Unbiased δ and ψ (integrate over y and x, respectively)

$$\delta(x) = \int_{B(x)} [w(x,y) - \psi(y)]\, d\Upsilon(y|x), \tag{2}$$

$$\psi(y) = \int_{A(y)} [w(x,y) - \delta(x)]\, d\Gamma(x|y), \tag{3}$$

Firm fixed effect δ is constant if Ψ is constant:

$$\psi(y) = \underbrace{\int_{A(y)} [w(x,y) - w_{av}(x)]\, d\Gamma(x|y)}_{=:\Psi(y)} + \int_{A(y)} \int_{B(x)} \psi(\tilde{y}) d\Upsilon(\tilde{y}|x) d\Gamma(x|y) \tag{4}$$

Inconclusive Firm Fixed Effect

Proposition (4)

The firm fixed effect is ambiguous. It is zero under uniform distributions and $f(x,y) = \alpha xy + h(x) + g(y)$.

- The firm effect Ψ is

$$\Psi(y) = \int_{y-K}^{y+K} \left[w(x,y) - w_{av}(x) \right] d\Gamma(x|y)$$

- Assuming a long panel: $w_{av}(x) = \int_{x-K}^{x+K} w(x,y) d\Upsilon(y|x)$

- Show that $\Psi' \gtrless 0$

$$\begin{aligned} \Psi'(y) &= \int_{y-K}^{y+K} \frac{\partial w(x,y)}{\partial y} \gamma(x|y) dx \\ &\quad + \left(w(y+K,y) - w_{av}(y+K) \right) \gamma(y+K|y) \\ &\quad - \left(w(y-K,y) - w_{av}(y-K) \right) \gamma(y-K|y) \end{aligned}$$

- First effect: change in matched type (Beckerian effect)
- Second effect: change in set of matched partners
- Both effects: ambiguous, often opposite sign, zero under uniform

Identifying the Strength of Sorting

Without Knowing the Sign

Proposition (5)

We can identify strength of sorting, i.e., cross-partial $|f_{xy}|$.

Two parts:

1. Use *wage gap* to identify the cost of search c
2. Use *range* of matched types to identify $|f_{xy}|$

1. Wage Gap

- Maximum wage in panel: identify type (optimal = max):

$$\overline{w}_k = \max_{t \in \{1,\ldots,T\}} w_k^t$$

- $\Omega_W(\overline{w})$: distribution of maximum wages ($\Omega_F(\overline{w})$ for firms)
- Identify search by wage gap(where $\underline{w}_x = \min_{t \in \{1,\ldots,T\}} w_x^t$):

$$c = \overline{w}_x - \underline{w}_x,$$

Identifying the Strength of Sorting

Without Knowing the Sign

2. Range of Matched Types

- Search loss $L(x, y)$ due to mismatch:

$$
\begin{aligned}
L(x,y) &= f(x,y) - \int_0^x f_x(\tilde{x}, \mu(\tilde{x}))d\tilde{x} - \int_0^y f_y(\mu^{-1}(\tilde{y}), \tilde{y})d\tilde{y} \\
&= -\int_{\mu^{-1}(y)}^x \int_{\mu^{-1}(\tilde{y})}^x |f_{xy}(\tilde{x}, \tilde{y})| d\tilde{x} d\tilde{y} \\
&= -\int_y^x \int_{\tilde{y}}^x |f_{xy}(\tilde{x}, \tilde{y})| d\tilde{x} d\tilde{y} \quad \text{(for PAM)}
\end{aligned}
$$

- Search decision: $L(x, \underline{y}(x)) = -2c$.
- This functional equation identifies $|f_{xy}|$: compares variation in matching sets $(x - \underline{y}(x))$ to variation in wage $(2c)$
- If wage variation high, matching sets small $\Rightarrow$ large loss from mismatch, i.e. the cross-partial large

Identifying the Strength of Sorting

Without Knowing the Sign

- More structure (example): constant cross-partial α, then

$$-L(x, y) \quad = \quad |\alpha|(x^{\theta} - \underline{y}(x)^{\theta})^2 = 4c$$

use data on observed pairs x, y to estimate α, θ

Identifying the Strength of Sorting

Without Knowing the Sign

- More structure (example): constant cross-partial α, then

$$
\begin{aligned}
-L(x,y) &= |\alpha|(x^\theta - \underline{y}(x)^\theta)^2 = 4c \\
&\Leftrightarrow\; x = \left(2\,(c/|\alpha|)^{1/2} - \underline{y}(x)^\theta\right)^{1/\theta}
\end{aligned}
$$

use data on observed pairs x, y to estimate α, θ

Identifying the Strength of Sorting

Without Knowing the Sign

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  use data on observed pairs x, y to estimate α, θ

- Total loss from search (mismatch minus perfect matching):

$$\mathcal{G} = \int_0^1 \int_0^1 L(x,y)dxdy = -|\alpha| \frac{\theta^2}{(2\theta+1)(\theta+1)^2}.$$

Type-Dependent Search Costs

Discounting – Shimer-Smith (2000)

Result: Non-monotone wages also under discounting

- Discount factor β. Technology $f^+(x,y) = xy$
- 1st period wages (surplus matching (split) + value waiting):

$$\begin{aligned} w^+(x,y) &= \frac{1}{2}\left[xy - \beta\frac{x^2}{2} - \beta\frac{y^2}{2}\right] + \frac{1}{2}\beta\frac{x^2}{2} \\ &= \frac{1}{2}xy + \beta\frac{x^2}{4} - \beta\frac{y^2}{4} \end{aligned}$$

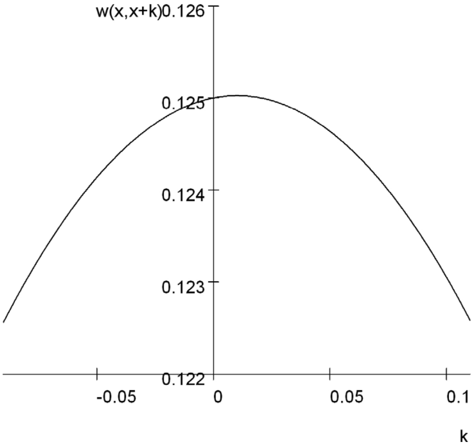
- Match if surplus is positive. [Matching set $A(y) = \left[\underline{K}y, \overline{K}y\right]$, $K = \beta^{-1} \pm \sqrt{\beta^{-2} - 1}$.]
- Under NAM technology, $f^-(x,y) = -xy + y$

$$w^-(x,y) = \frac{1}{2}x\tilde{y} + \beta\frac{x^2}{4} - \beta\frac{\tilde{y}^2}{4} + \frac{1}{2}(1-\beta)(1-\tilde{y})$$

- $w^+ \approx w^-$ small when $\beta \approx 1$: some, but small sign ident.
- Wage is also inverted U-shaped

Mismatch due to Search Frictions

Non-monotone Wages under Discounting



Non-monotonicities arise generally

General Type-Dependent Search Costs

Non-monotonicities with general search costs:

$$f(x,y) - (w^\star(x) + \pi^\star(y) - c(x) - c(y)) \geq 0.$$

Discounting: $c(y) = (1-\beta)\pi^\star(x)$
Differing arrival rates: $c(y) = (1-\alpha(y)\beta)\pi^\star(x)$

Wages are non-monotonic (whenever $c'(y) \leq y$):

$$\begin{aligned} w(x,y) &= \frac{1}{2}xy + \frac{1}{4}x^2 - \frac{1}{4}y^2 - \frac{1}{2}c(x) + \frac{1}{2}c(y) \\ \Rightarrow \qquad \partial w/\partial y &= \frac{1}{2}x - \frac{1}{2}y + c'(y) \end{aligned}$$

- Non-monotonicities arise *always* when higher types reject some lower types (because then workers obtain their continuation value at the highest and lowest type willing to match)
- Even with OJS (fixed entry cost, then type realized): No opportunity cost for worker, but usually the firm cannot search while matched, and some matches are not formed.

Further Identification

Local Complementarity

- Given wage equation (where γ is general bargaining share):

$$w(x,y) = \gamma\left[f(x,y) - w(x,\mu) - \pi(\mu^{-1},y) + 2c\right] + w(x,\mu) - c$$

$\Rightarrow$

$$w_{xy}(x,y) = \gamma f_{xy}(x,y)$$

- Any $(x,y),(x',y')$ with $x \neq x'$ and $y \neq y'$ $\Rightarrow$ cross-partial is

$$\frac{w(x',y') - w(x,y') - (w(x',y) - w(x,y))}{\gamma[x'-x][y'-y]}$$

Further Identification

Local Complementarity

- Frictionless model: optimal $y = \mu(x)$ s.t. $w_y(x, \mu(x)) = 0$
- Concavity conveys information:

$$w_{yy}(x, \mu(x)) = -\frac{w_{xy}(x, \mu(x))}{\mu'(x)}$$

- Even with search frictions (provided costs are constant):

$$w_{yy}(x, x) = w_{xy}(x, x) = \gamma f_{xy}(x, x),$$

- Can capture complementarities locally provided search costs are constant and the cross-partial is constant
- Gautier-Teulings use a second order approximation with quadratic technologies
- Does not work with varying costs:

$$w_{yy}(x, x) = \gamma \left[f_{xy}(x, x) + k''(x) \right]$$

Infinite Horizon

- Assume symmetry and equal splits; stationary distribution of unmatched $G(\cdot)$
- Output $f(x,y)$; payoff is $-c$ if no match
- $v(x), v(y)$ the (identical) value functions of a type x and y:

$$v(x) = \int_{\mathcal{M}(x)} w(x,y)dG(y) + \int_{y \notin \mathcal{M}(x)} dG(y)[v(x) - c]$$

- Surplus of a match: $s(x,y) = f(x,y) - [v(x) + v(y) - 2c]$.
- Marginal type $\underline{y}$: $f(x, \underline{y}(x)) - v(x) - v(\underline{y}(x)) = -2c$.
- Wage:

$$\begin{aligned} w(x,y) &= \frac{s(x,y)}{2} + v(x) - c \\ &= \frac{1}{2}[f(x,y) - v(x) - v(y) + 2c] + v(x) - c. \end{aligned}$$

Infinite Horizon

- Again: non-monotonic wage schedule (both at high and at low marginal type, $s(x, y) = 0$) and $w = v(x) - c$
- Sign of cross-partial not identified
- Recover cost of search? Let $\underline{w}(x) = v(x) - c$ be lowest wage and $\mathbb{E}w(x)$ be the average wage, then from value function

$$v(x) = \pi \mathbb{E}w(x) + (1 - \pi)\underline{w}(x)$$

  where $\pi = \text{Prob}\{\mathcal{M}\}$
- Then

$$c = [\mathbb{E}w(x) - \underline{w}]\,\pi.$$

Wrap Up

- We cannot identify the sign of sorting from wage data
- We can identify the strength: economically relevant
- Standard fixed effects get neither sign nor strength

- Discussion
  1. Identifying sign: attributing profit or output data
  2. More general technologies: horizontal vs vertical diff
  3. Different reasons for mismatch (e.g. productivity shocks)
  4. Type-Dependent Search Costs (e.g. discounting)
  5. On-the-job Search

Identification Questions

- Identification not based on prices?
- In marriage markets: identifying preferences (Chiappori,...)
- Production characteristics?
  - Direct measures of output; team production
  - Football teams: can observe output measures (but estimation is complicated since output is a function of rival)
  - Can we measure the impact of substituting one player for another?

Topics in Labor Markets

Jan Eeckhout

2015–2016

III. Directed Search

Motivation

- Role of *search frictions* in the classic *assignment problem* when there is *price competition*. Complementarities are common in:
  - labor market, housing market, business partnerships, product markets,...

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  - price for each type combination: $p(x, y)$
  - perfect trade. concern: important trade imperfections

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- Our approach: decentralized price competition
  - trading probability per price-type combination: $\lambda(x, y, p)$
  - higher λ: higher trade prob. for sellers but lower for buyers
  - price competition, absent centralized market clearing

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  - trading probability per price-type combination: $\lambda(x, y, p)$
  - higher λ: higher trade prob. for sellers but lower for buyers
  - price competition, absent centralized market clearing
- Shimer and Smith (2000) [Atakan 2006]: random search
  - no information about prices and types, imperfect trade
  - Concern: No information is a strong assumption

Motivation

- We uncover a natural economic explanation for the forces that govern the matching patterns (when good types match with good types?)
  - New conditions for positive / negative sorting: root-supermodularity
  - Economic Forces: Complementarities in Match-Value vs Search Technology

Motivation

- Two key aspects to matching:
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  (2) The probability (speed) of trade ("trading-security"):

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  (2) The probability (speed) of trade ("trading-security"):

  –AM even with some supermodularity: nowhere *root*-sm

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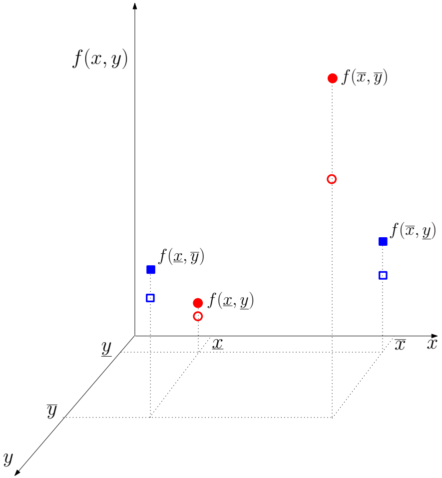
Root vs Log-supermodularity



Root-Supermodular: $f_{xy} > \frac{n-1}{n} f_x f_y / f$

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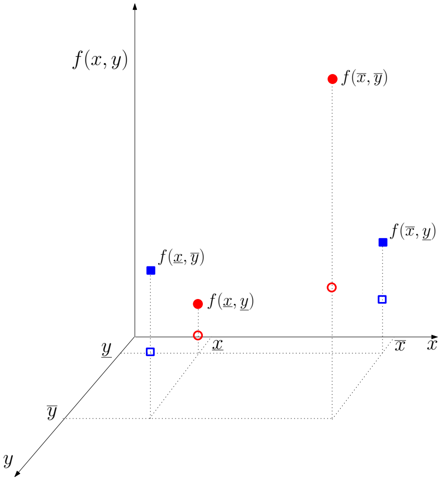


Log-Supermodular: $\quad f_{xy} > 1 f_x f_y / f$

Related Literature

Decentralized Price Competition

Peters (1984,1991,1997a,2000), Moen (1997), Acemoglu, Shimer (1999a,b), Burdett, Shi, Wright (2001), Shi (2001), Mortensen, Wright (2002), Rocheteau, Wright (2005), Galenianos, Kircher ('06), Kircher ('07), Delacroix, Shi ('06),....

General Matching Function

Random search (Pissarides 1984, 1985; Petrongolo and Pissarides 2001); Comp. Search (Moen 1997,...), Dir. Search (Menzio 2007)

Assortative Matching

Becker (1973) , Burdett, Coles (1997),..., Shimer, Smith (2000)

Market Games and Walrasian Outcomes

Shubik ('73), Shapley and Shubik ('77),..., Rubinstein and Wolinsky ('85), Gale ('86), Atakan ('06), Lauermann ('07)

Competing Auctions - Ex post Screening

McAfee ('93), Peters ('97b), Shimer ('05), Eeckhout & Kircher ('08)

The Model

- *Players*
  - Measure $S(\bar{y})$ sellers: observable types $y \in [\underline{y}, \bar{y}]$ dist $S(y)$
  - Measure 1 buyers: private type $x \in [\underline{x}, \bar{x}]$ i.i.d. from $B(x)$
  - Unit demands and supplies

- *Payoffs* of trade between (x, y) at price p:
  - Buyer: utility $f(x, y) - p$
  - Seller: profit p
  - No trade: payoffs normalized to zero

The Model

The extensive form

2 stage extensive form:

1. Sellers post prices: $G(y, p)$ seller distribution of (y, p)
2. Buyers observe G and choose y, p (or $\emptyset$)
   - $H(x, y, p)$ buyer distribution over (x, y, p).
   - If buyer meets such a seller, he gets the good and pays p

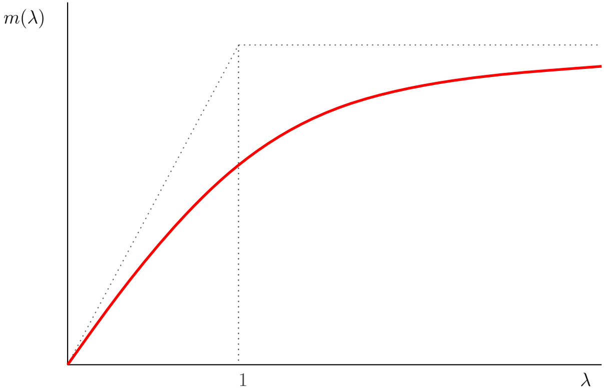
Matching Technology:

- Primitive: total number of matches $M(b, s)$ (CRTS)
- Let $\lambda = b/s$ be buyer-seller ratio (depends on (y, p))
- Matching probability $m(\lambda) = M(b, s)/s$

  Seller: $m(\lambda)$; Buyer: $q(\lambda) = m(\lambda)/\lambda$
- $m' > 0$, $q' < 0$, $m, q \in [0, 1]$, $m'' < 0$ (with decr. elasticity)

The Model

Matching Function



The Model

Matching Function

Different interpretations of $m(\lambda(y, p))$

1. anonymous (symmetric) strategies (buyer miscoordination)
2. spacial separation (Acemoglu 1997)
3. market makers providing trading platforms (Moen 1997)

Examples of Matching Function

1. anonymous strategies [urn-ball]: $m_1(\lambda) = 1 - e^{-\lambda}$
2. fraction $1 - \beta$ buyers get lost: $m_2(\lambda) = 1 - e^{-\beta\lambda}$
3. random on island [telegraph-line]: $m_3(\lambda) = \lambda/(1 + \lambda)$

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Number of matches: $M(b, s) = sM(\frac{b}{s}, 1) = sm(\lambda)$

Payoffs and Optimal Decisions given G and H

Queue length $\lambda_{GH}(y,p) = dH_{\mathcal{X}\mathcal{Y}}/dG$ on equilibrium path

1. Seller payoffs: $\pi(y,p,G,H) = m(\lambda_{GH}(y,p))p$
2. Buyer payoffs: $u(x,y,p,G,H) = q(\lambda_{GH}(y,p))\,[f(x,y) - p]$

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Complete queue length (Subgame Perfection "off-equilibrium-path", Acemoglu and Shimer (1999b))

$$\lambda_{GH}(y, p) = \sup\left\{\lambda \in \mathbb{R}_+ : \exists x; q(\lambda)\,[f(x, y) - P] \geq \max_{(y', p') \in \operatorname{supp} G} u(x, y', P', G, H)\right\}$$

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Definition

An equilibrium is a pair of trading distributions (G, H) such that:

(i) Seller Optimality: $(y, p) \in \mathrm{supp}G$ only if p maximizes 1. for y;

(ii) Buyer Optimality: $(x, y, p) \in \mathrm{supp}H$ only if (y, p) maximizes 2. for x.

Assortative Matching

Assignment Function

Definition

Assortative: $\exists$ monotone function ν such that points $(x, \nu(x))$ have full measure under $H_{\mathcal{X}\mathcal{Y}}$.

- $\nu(x)$ is the seller type with whom x wants to trade

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Assortative: $\exists$ monotone function ν such that points $(x, \nu(x))$ have full measure under $H_{\mathcal{X}\mathcal{Y}}$.

- $\nu(x)$ is the seller type with whom x wants to trade
- $\mu(y)$ is the buyer type that wants to trade with seller y ($\mu = \nu^{-1}$)
- positive assortative (PAM, $+$AM): μ strictly increasing
- negative assortative (NAM, -AM): μ strictly decreasing

Assortative Matching

Main Insights

- n-root-supermod needed to overcome NAM ($n = \frac{1}{\alpha}; n \geq 1, a \in [0,1]$)
- n equals elasticity of substitution in matching
- n results simple (efficiency) trade-off
    - complementarities in production
    - complementarities in search technology

Illustration of -AM

Private Values

Illustration of -AM

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1. . The quality of the match.
2. . The probability (speed) of trade.

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- Frictions: Equilibrium is –AM
- High value buyer pays high p to avoid no-sale
  ("trading-security motive")
- Low type seller is more interested in price than prob.
  (so low seller types provide trading security for buyers)

Illustration of -AM

Private Values

- With private values: single crossing
- Buyers' indifference curves in 2-dimensional plane

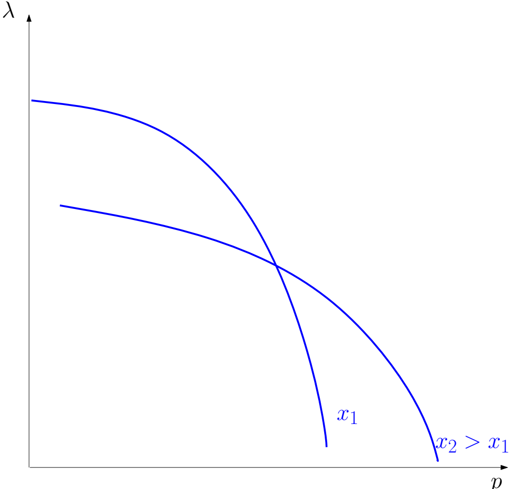

Illustration of -AM

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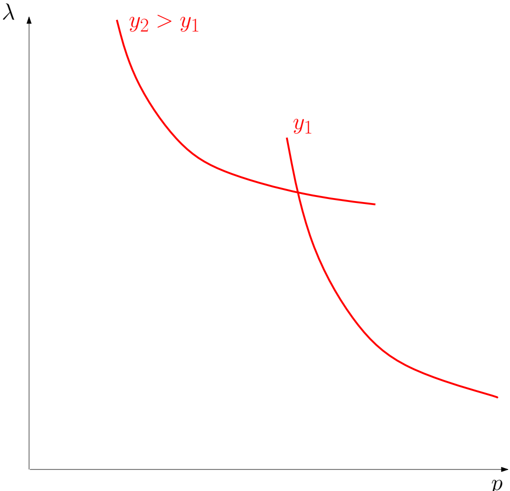
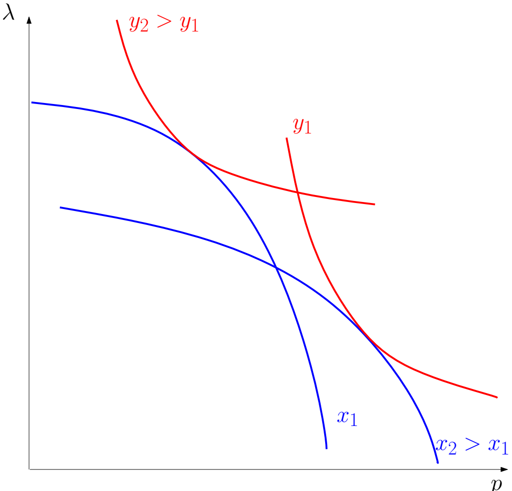


Illustration of -AM

Private Values

- With private values: single crossing
- –AM: High y_2 matches with low x_1



Assortative Matching

Main Theorems

$\exists$ numbers $\bar{n}$ and $\underline{n}$ larger than one $(n = 1/(1 - \max\{e_M\}))$

Theorem (+AM under $\bar{n}$-Root-Supermodularity)

$+$AM for all type distr. iff $f(x, y)$ is $\bar{n}$- root-supermodular.
-AM for all type distr. iff $f(x, y)$ is nowhere $\underline{n}$-root-supermod.

Corollary: *-AM for all distr. if $f(x, y)$ is weakly submod.*

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Proposition: If matching function is not CES
+AM for some distr. even if $f(x, y)$ not $\bar{n}$-root-supermod.

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The assortative assignment is constrained efficient.

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Theorem (Efficiency)

The assortative assignment is constrained efficient.

Proposition: *q^{-1} convex and bounds on derivatives:*
+AM for all distr. iff $f(x, y)$ is square-root-supermodular.

Proposition: If matching function is not CES
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Positive Assortative Matching

Proof: +AM iff $f(x, y)$ $\bar{n}$-root-supermodular

Seller y: $max_{p \in \mathcal{P}} m(\lambda(p, y))p$

Recall:

$$\lambda_{GH}(y, p) = \sup \{\lambda \in \mathbb{R}_+ : \exists x; q(\lambda) [f(x, y) - P] \geq U(x, G, H)\},$$

where $U(x, G, H) \equiv \max_{(y', p') \in \mathrm{supp} G} u(x, y', p', G, H)$.

$$\Rightarrow \max_{\lambda, p} \left\{ m(\lambda)p : \lambda = \sup \left\{ \lambda' : \exists x; q(\lambda') [f(x, y) - p] \geq U(x, G, H) \right\} \right.$$

$$\Rightarrow \max_{x, \lambda, p} \{ m(\lambda)p : q(\lambda) [f(x, y) - p] = U(x, G, H) \}.$$

(connection to competing mechanism design; McAfee 1993, Peters 1997, 1999)

Positive Assortative Matching

Proof: +AM iff $f(x,y)$ $\bar{n}$-root-supermodular

After substituting the constraint:

$$\max_{x\in\mathcal{X},\lambda\geq 0} m(\lambda)f(x,y)-\lambda U(x).$$

First Order Conditions:

$$\begin{aligned} m'(\lambda)f(x,y)-U(x) &= 0 && \text{(similar to Hosios '90)} \\ m(\lambda)f_x(x,y)-\lambda U'(x) &= 0 && \text{(similar to Becker '73)} \end{aligned}$$

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Hessian for SOC:

$$\begin{pmatrix} m''(\lambda)f(\mu,y) & m'(\lambda)f_x(\mu,y)-U'(x) \\ m'(\lambda)f_x(\mu,y)-U'(x) & m(\lambda)f_{xx}(\mu,y)-\lambda U''(x) \end{pmatrix}.$$

Along Equilibrium Allocation: Question: $a(\lambda)$? Magnitude?

$$\mu'(y)\left[f_{xy}-\underbrace{\frac{1-\lambda m'(\lambda)/m(\lambda)}{-\lambda m''(\lambda)/m'(\lambda)}}_{a(\lambda)}\frac{f_x(\mu,y)f_y(\mu,y)}{f(\mu,y)}\right]\geq 0,$$

Intuition and Explanation

What is $a(\lambda)$?

- It is the *elasticity of substitution ES_M* between buyers and sellers in the matching function $M(b,s) = sm(b/s)$.

$$a(\lambda) = \frac{M_b(\lambda,1)M_s(\lambda,1)}{M_{bs}(\lambda,1)M(\lambda,1)}$$

Why is it important?

- The Hosios condition: entry of sellers into one (x,y) based on *derivative of matches with respect to sellers (M_s)*.
- Our condition connects different (x,y) combinations via the *elasticity of substitution between buyers and sellers (ES_M)*.

Interpretation in terms of "match value" and "trading security":

$$\underbrace{\frac{f_{xy}\ f}{f_x\ f_y}}_{\substack{\text{match value improvement} \\ (CRTS:\ ES_f^{-1})}} > \underbrace{\frac{M_b\ M_s}{M_{bs}\ M}}_{\substack{\text{loss due to no trade} \\ (ES_M)}}$$

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- Our condition connects different (x,y) combinations via the *elasticity of substitution between buyers and sellers* (ES_M).

Interpretation in terms of "match value" and "trading security":

$$\underbrace{\frac{f_{xy}\ f}{f_x\ f_y}}_{\substack{\text{match value improvement} \\ (CRTS:\ ES_f^{-1})}} > \underbrace{\frac{M_b\ M_s}{M_{bs}\ M}}_{\substack{\text{loss due to no trade} \\ (ES_M)}} \Leftrightarrow \frac{f_{xy} f}{f_x f_y} \frac{M_{bs} M}{M_b M_s} > 1$$

Positive Assortative Matching

Sufficiency and Necessity

$\bar{a} = \sup a(\lambda)$, $\underline{a} = \inf a(\lambda)$. Recall:

$$\mu'(y)\left[f_{xy} - \underbrace{\frac{1-\lambda m'(\lambda)/m(\lambda)}{-\lambda m''(\lambda)/m'(\lambda)}}_{a(\lambda)} \frac{f_x(x,\mu) f_y(x,\mu)}{f(x,\mu)}\right] \geq 0, \tag{5}$$

Proposition: PAM $\forall\, B, S \;\Leftarrow\; f$ is strictly n-root-sm $(n = (1-\bar{a})^{-1})$.
Proposition: NAM $\forall\, B, S \;\Leftarrow\; f$ is nowhere n-root-sm $(n = (1-\underline{a})^{-1})$.
(includes weak submodularity, sometimes more)

Proof: Non-differential version of (5).

Proposition: PAM $\forall\, B, S \;\Rightarrow\; f$ is n-root-sm $(n = (1-\bar{a})^{-1})$.
Proposition: NAM $\forall\, B, S \;\Rightarrow\; f$ never str. n-root-sm $(n = (1-\underline{a})^{-1})$.

Proof: By contradiction: find distributions where (5) cannot hold.

Special Case 1

Square-Root-Supermodularity

Assume q^{-1} convex; bounds on derivatives ($|q'(0)|, |q''(0)| \in (0,\infty)$).

Proposition: $+\text{AM}\ \forall\ B, S \ \Leftrightarrow\ f$ is square -root-sm.

- $a(0) = 1/2$ (binding when some sellers cannot trade)
- $a(\lambda) \leq 1/2$ (if and only if $1/q(\lambda)$ is convex in λ)
- therefore $\bar{a} = 1/2$.

First Bullet Point:

$q(\lambda) = m(\lambda)/\lambda$

$\Rightarrow q'(\lambda) = (m'(\lambda) - q(\lambda))/\lambda$ bounded $\Rightarrow m'(0) = q(0)$

$\Rightarrow q''(\lambda) = (m'' - 2q')/\lambda$ bounded $\Rightarrow q'(0) = m''(0)/2$

$\Rightarrow a(0) = m'(0)q'(0)/[m''(0)q(0)] = 1/2$

Special Case 2

The Class of CES Matching Functions

Consider CES: $m(\lambda) = (1 + k\lambda^{-r})^{-1/r}$ $[M(\beta, \sigma) = (\beta^r + k\sigma^r)^{-1/r}\beta\sigma]$

$r > 0$, $k > 1$, $a(\lambda) = (1 + r)^{-1}$ constant

Proposition: Fix the type distributions. There is

- $+AM$ if f is n-root-supermodular; $(n = \frac{1+r}{r})$
- $-AM$ if f is nowhere n-root-supermodular; $(n = \frac{1+r}{r})$

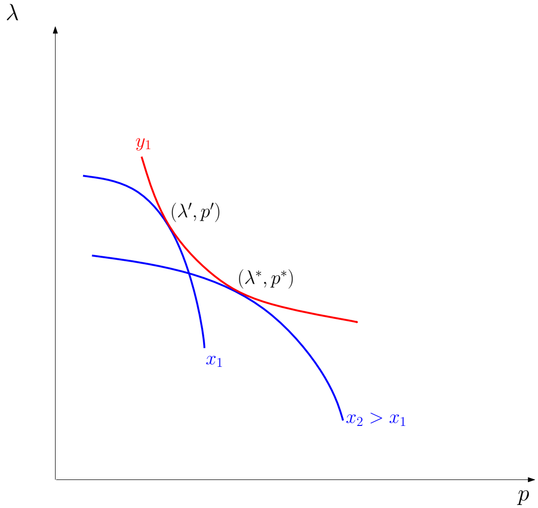
Corollary: CES with elasticity e, then PAM under:

1. Supermodularity if $e = 0$ (Leontief);
2. Square-Root-Supermodularity if $e = \frac{1}{2}$ (Telegraph Line);
3. Log-Supermodularity if $e = 1$ (Cobb-Douglas).

Assortative Matching

Graphical Interpretation

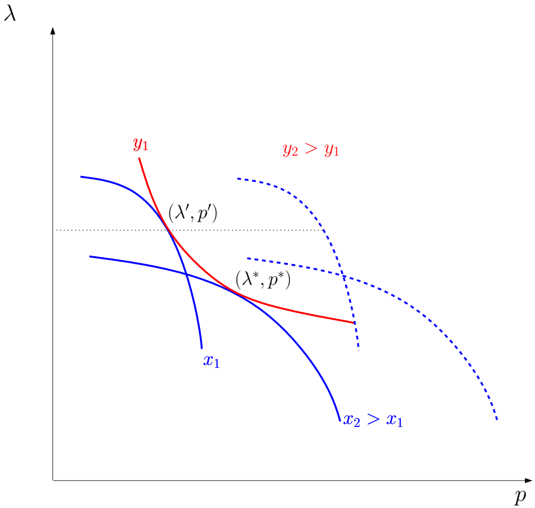
- IC in (λ, p, y), project in (λ, p) and vary y



Assortative Matching

Graphical Interpretation

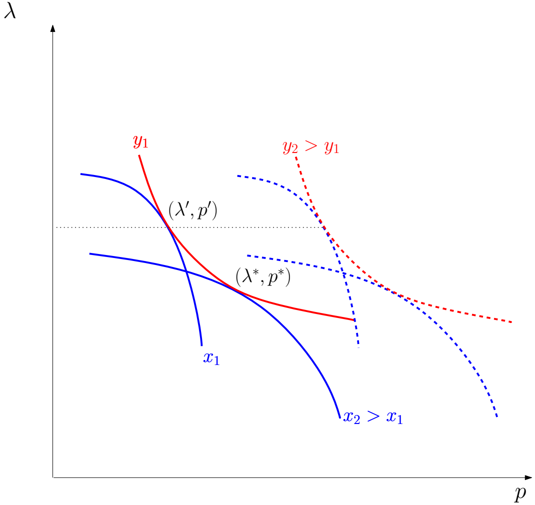
- Parallel shifts, identical distance when $f = x + y$



Assortative Matching

Graphical Interpretation

- Slope of iso-profit curve is flatter



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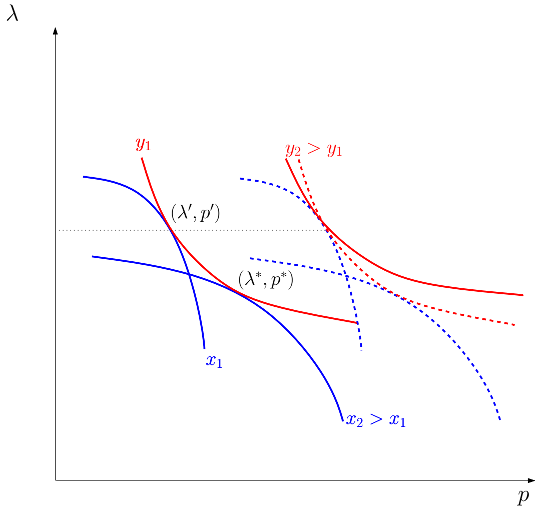
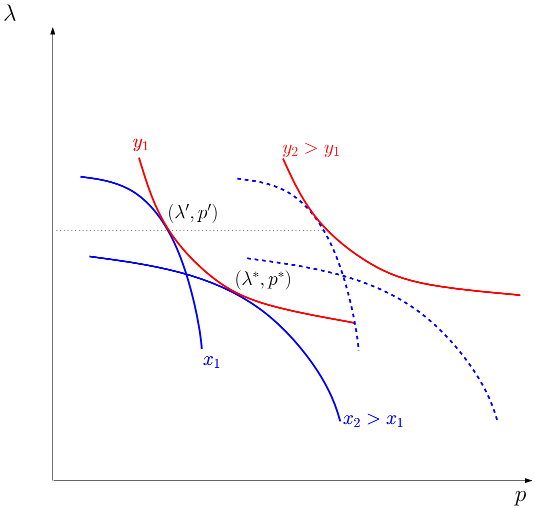


Assortative Matching

Graphical Interpretation

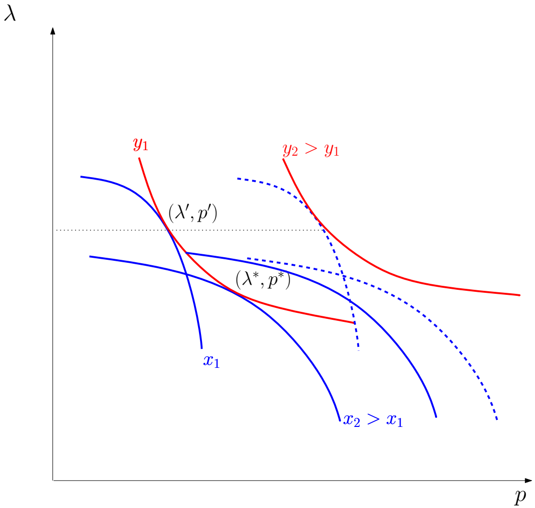
- High y_2 will match with low x_1



Assortative Matching

Graphical Interpretation

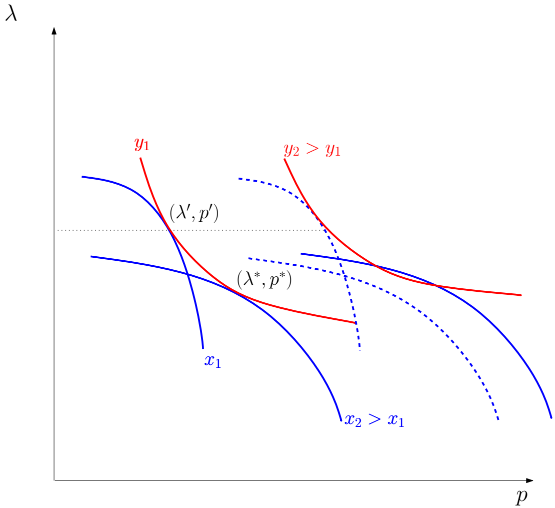
- High x IC moves less when submodularity



Assortative Matching

Graphical Interpretation

- Need root-supermodularity for IC to move "far enough"



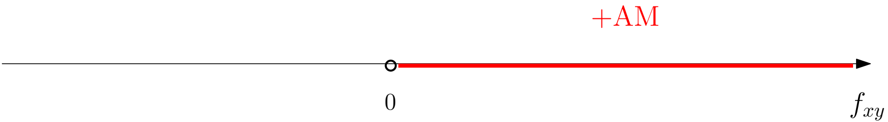
Assortative Matching

Comparing Logs and Roots

Competition
supermodularity
⇒ +AM
submodularity
⇒ –AM

Dec. Price Comp
root-supermodularity
⇒ +AM
sub- and modularity
⇒ –AM

Random Search
log-supermodularity
⇒ +AM
log-submodularity
⇒ –AM



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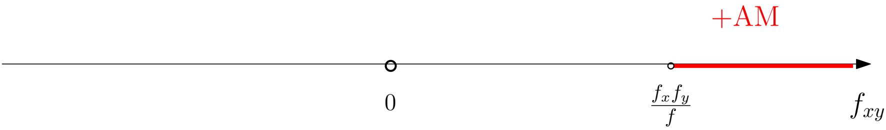

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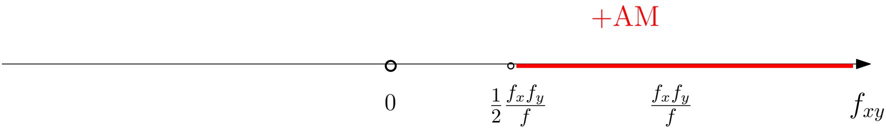

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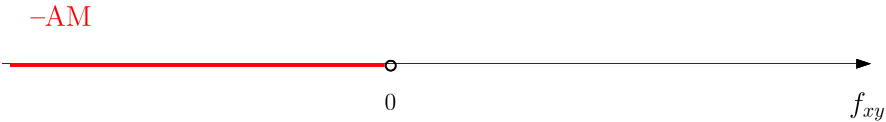

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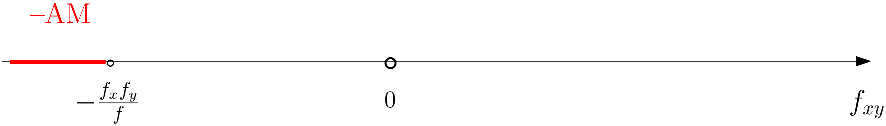

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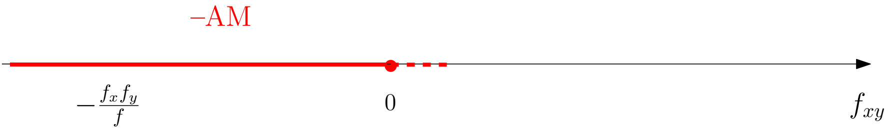


Existence

Proposition

If $f(x, y)$ is $\bar{n}$-root-supermodular (or nowhere $\underline{n}$-rs), then there exists an equilibrium for all type distributions.

Proof.

- construct equilibrium, monotonically increasing (+AM)
- solution to FOCs satisfies system of 2 differential equations in λ and μ with the appropriate boundary conditions
- verify SOCs along equilibrium allocation μ^*
- establish this is a global maximum by considering different solutions to the FOCs and showing that none other exist

$\square$

Efficiency

+AM constrained efficient under root-supermodularity

Planner chooses (G, H) to maximize total surplus

$$\max_{G,H} \int m(\lambda_{GH}(y,p)) f(x,y) dG$$

$$\text{s.t.} \qquad H_{\mathcal{X}} = B; \quad G_{\mathcal{Y}} = S; \quad \lambda_{GH} = dH_{\mathcal{X}\mathcal{Y}}/dG$$

Under our root-supermodularity conditions for PAM and NAM:

- solution coincides with decentralized equilibrium
- Hosios per (x,y) market, Root-SM to connect them

Prices

The equilibrium price schedule under PAM satisfies

$$p'(y) = \underbrace{f_y}_{Becker(1973)} + \underbrace{(\eta_q f_x - \lambda \eta_m f_y)\, a(\lambda)}_{Compensation\ \ through\ \ trading\ \ probabilities}$$

η_m elasticity of m

Insights:

1. Prices increasing in quality under PAM
2. Prices as in Becker under symmetry
3. Prices can be decreasing under NAM

Dynamic Framework

Dynamic Framework:

$$\max_{\lambda \in \overline{\mathbb{R}}_+} m(\lambda)\left[1-\delta\left(1-m(\lambda)\right)\right]^{-1} p$$

$$\text{s.t.} \qquad q(\lambda)\left[1-\delta\left(1-q(\lambda)\right)\right]^{-1}\left(f(x,y)-p\right) = U(x)$$

Necessary and sufficient condition for $+$AM:

$$f_{xy}(x,y) \geq A(\lambda,\delta)a(\lambda)\frac{f_x(x,y)f_y(x,y)}{f(x,y)}$$

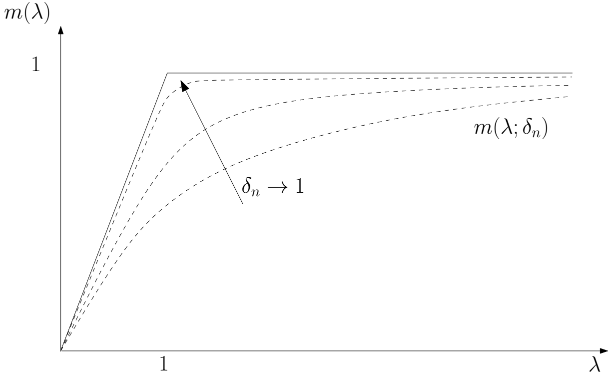
where

1. $A(\lambda,\delta) \in [0,1]$
2. $\lim_{\lambda\to 0} A(\lambda,\delta) = 1$ for all $\delta \in [0,1)$,
3. $\lim_{\delta\to 1} A(\lambda,\delta) = 0$ for all $\lambda > 0$.

Vanishing Frictions

- Two approaches to vanishing frictions:

  1. over time $\delta \to 1$; or 2. change in matching function
     - root-supermodularity necessary for +AM for any frictions
     - but necessary only at vanishing set of types
- Illustration: changing matching function



Wrap Up

- Complementarities are a source of high productivity in many environments (goods, labor, neighborhood,...)
- Imperfections in trade, but prices play allocative role
- Role of prices: ex-ante sorting, reduces frictions
- Highlights the interplay between frictions and match value:
  1. Match Value: tendency for +AM (if supermodular)
  2. Frictions: tendency for –AM (a-modular $\Rightarrow$ –AM)
- simple trade-off: Becker vs Elasticity in Matching
- root-supermodular: point where effect (1) outweighs (2)

Assortative Matching

Comparing Logs and Roots

Assortative Matching

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Comparing Logs and Roots

III. Directed Search

Large Firms

Large Firms: Directed Search and Sorting

- Existing literature on search and firm size: identical workers
(Smith 99, Acemoglu-Hawkins 06, Mortensen 09, Kaas-Kircher 10, Helpman-Itskhoki-Redding 10, Menzio-Moen 10,...).
- Vacancy filling prob $m(q)$. Job finding prob $m(q)/q$. Post (x, v_x, ω_x)

$$\max_{r_x, l_x, \omega_x, v_x} \int [F(x, y, l_x, r_x) - l_x\omega_x - v_x c]\, dx$$
$$\text{s.t. } l_x = v_x m(q_x); \quad \text{and} \quad \omega_x m(q_x)/q_x = w(x).$$

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$$\text{s.t. } l_x = v_x m(q_x); \quad \text{and} \quad \omega_x m(q_x)/q_x = w(x).$$

- Two equivalent formulations:
  1. $\max_{s_x, r_x} \int [G(x, y, s_x, r_x) - w(x)s_x]dx$, where
  $$G(x, y, s_x, r_x) = \max_{v_x} [F(x, y, v_x m(s_x/v_x), r_x) - v_x c].$$
  2. $\max_{r_x, l_x, v_x} \int [F(x, y, l_x, r_x) - C(x, l_x)]dx$, where
  $$C(x, l_x) = \min_{v_x, q_x} c v_x + q_x v_x w(x) \text{ s.t. } l_x = v_x m(q_x).$$

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- Determine unemployment. FOC

$$w(x) q_x = \frac{\eta(q)}{1 - \eta(q)} c$$

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- Check sorting, compute $w(x)$ as in previous part.
- Determine unemployment. FOC (simple closed form with const. elasticity α)

$$w(x)q_x = \frac{\eta(q)}{1-\eta(q)}c \quad = \quad \frac{1-\alpha}{\alpha}c$$

Proposition

The unemployment rate is falling in worker skills.

- $\eta(q)$ weakly decreasing $\Rightarrow q$ decreasing in x

Large Firms: Directed Search and Sorting

Proposition

The unemployment rate is falling in worker skills.

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Proposition

The vacancy rate is ambiguous in firm size.

- Consider PAM (likewise for NAM)
- Vacancies $(1/q)$ increasing in x
- Firm size ambiguous in y : $F_{23} \gtrless F_{14}$

III. Directed Search

Risk Aversion – Distribution of Assets

Question

The broad purpose of this paper:

- How does the distribution of assets affect job search decisions?

    1. Do workers with different assets get different productivity jobs?

Question

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- How does the distribution of assets affect job search decisions?

  1. Do workers with different assets get different productivity jobs?

  2. What is optimal level of government-provided unemployment insurance (UI) as a function of asset ?

Motivation

Model Ingredients

- Unemployment risk as source of income uncertainty
- Two sources of market incompleteness:
  1. Uninsurable Unemployment Risk
  2. Job search
- Heterogeneous asset holdings
- Access to asset markets $\Rightarrow$ consumption smoothing

$\rightarrow$ role of precautionary savings

- How UI affects LM outcome?
  - Incentive effects: which jobs to apply for
- The needs to smooth consumption and job search behavior

The Mechanism

The Labor Market as an Insurance Mechanism

- Heterogeneous firms: high productivity firms
  - have higher opportunity cost of unfilled job
  - Post high wages
- Risk averse workers self-insure w/ wage-unemployment bundle
- Capture precautionary savings motive
- Different asset holdings affect job search decision
- Private assets: differential risk tolerance $\Rightarrow$ truth telling

Related Literature

- Partial Equilibrium
  - Danforth (1979)
  - Hopenhayn-Nicolini (1992): optimal UI, consumption $\downarrow$
  - Shimer-Werning (2007, 2008): UI $\uparrow$ (constant if CARA)
- General Equilibrium
  - Acemoglu-Shimer (1999): homogeneous assets; CARA; focus on firm investment and job creation
  - Golosov-Menzio-Maziero (2011): homogenous agents, private job search decision
- Quantitative
  - Hansen-Imrohoroglu (1992)
  - Alvarez-Veracierto (2001)
  - Krusell, Mukoyama, Sahin (2011)

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$\Rightarrow$ New:

1. asset distribution + two-sided heterogeneity $\Rightarrow$ sorting
2. both consumption-saving decision *and* choice job finding prob

The Model

- Timing:
  - Two periods (generalize to infinite horizon)
- Agents:
  - Workers $a \in \mathcal{A} = [\underline{a}, \overline{a}] \subset \mathbb{R}_+$, distributed $\sim F(a)$
  - Firms $y \in \mathcal{Y} = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}_+$, distributed $\sim G(y)$ (large)
- Preferences and Technology:
  - Concave worker pref.: $u(c)$, where u is C^2.
  - Output: $v(y)$. Firm risk neutral.
  - Common Discount factor $\beta < 1$. Risk Free bond $R > 1$
- Matching Technology:
  - Search is Directed
  - Worker-to-firm ratio: λ
  - Matching prob: $m(\lambda)$; $m' > 0, m'' < 0$; worker $q(\lambda) = \frac{m(\lambda)}{\lambda}$

The Model

Actions

- Firm y: announce $w \Rightarrow$ distribution of firm strategies $P(y, w)$
- Worker a (assume for now observable):
  - consumption-savings decision a':
    1. period 1: $c_1 = a - a'$
    2. period 2: $c_{2,e} = Ra' + w, c_{2,u} = Ra'$
  - jobs search decision: firm y, and therefore $w, q(\lambda)$

  $\Rightarrow$ distribution of worker strategies $Q(a, a', y, w)$
- Measure Preserving market clearing condition:

$$P_{\mathcal{Y}}(\cdot) = G(\cdot) \text{ and } Q_{\mathcal{A}} = F(\cdot)$$

The Model

Payoffs

- Firm sets wages to maximize expected profits:

$$\pi(y, w) = m(\lambda)\,(v(y) - w)$$

- Worker simultaneously chooses consumption and makes job-search decision to maximize expected payoff:

$$\begin{aligned} U(a, a', y, w) \quad &= \quad u(c_1) + \beta\,[q(\lambda)u(c_{2,e}) + (1 - q(\lambda))u(c_{2,u})] \\ \text{s.t.} \quad & \quad c_1 = a - a' \\ & \quad c_{2,e} = Ra' + w \\ & \quad c_{2,u} = Ra' \end{aligned}$$

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- Beliefs

$$\lambda_{PQ}(a, w) = \sup \left\{ \lambda \in \mathbb{R}_+ : \exists a, q(\lambda)[y - w \geq \max_{y, w \in \operatorname{supp} P} U(a, a', y, w; P, Q)] \right\}$$

Definition

An equilibrium is a pair of market clearing distributions (P, Q) such that:

1. Worker optimality: $(a, a', y, w) \in \text{supp } Q$ only if (y, p) maximizes $U(a, a', y, w)$;
2. Firm optimality: $(y, w) \in \text{supp } P$ only if w maximizes π;

- Monotone matching (positive) $\mu : \mathcal{A} \to \mathcal{Y}$. Market Clearing:

$$\int_a^{\overline{a}} f(a)da = \int_{\mu(a)}^{\overline{y}} \lambda(y)g(y)dy.$$

Solution

- First solve *as if* a observable
- Wages from expected profits: $w = v(y) - \frac{\pi}{m}$
- Rewrite worker maximization problem (denoted by $\Phi(a, y, \pi)$):

$$\max_{a', \lambda} u(a - a') + \beta \left[qu\left(Ra' + v(y) - \frac{\pi}{m} \right) + (1 - q)\, u(Ra') \right]$$

- This is an allocation problem $\Phi(a, y, \pi)$ with:
  1. Non-linear frontier (see Legros-Newman, 2007)
  2. Search Frictions: matching is probabilistic
  3. Many-to-one matching ex ante (one-to-one ex post)
  4. Maximization problem "inside" match value (wrt. a' and λ)
  5. Equilibrium prices

Solution

- The FOCs (a', λ) to the maximization problem satisfy:

$$\begin{aligned} -u'(a-a') + \beta R\left[qu'(c_e) + (1-q)\,u'(Ra')\right] &= 0 \\ \beta q'\left[u(c_e) - u(Ra')\right] + \beta u'(c_e)\frac{m'\pi}{\lambda m} &= 0 \end{aligned}$$

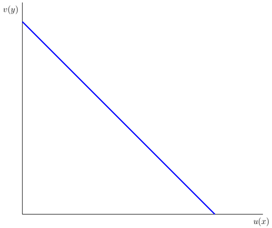
  i.e., consumption smoothing and optimal job search

- Optimal Allocation: $\max_y \Phi(a, y, \pi) \Rightarrow \Phi_y + \Phi_\pi \frac{\partial \pi}{\partial y} = 0$, implies:

$$\beta q u'(c_e)\left(v_y - \frac{\pi'}{m}\right) = 0$$

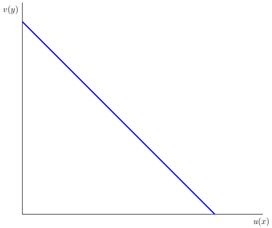
Solution

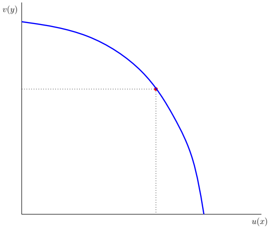
TU – NTU



Solution

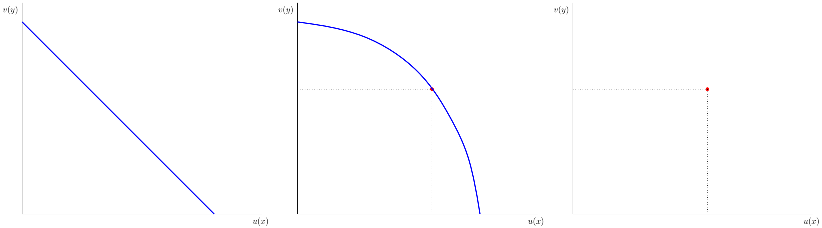
TU – NTU





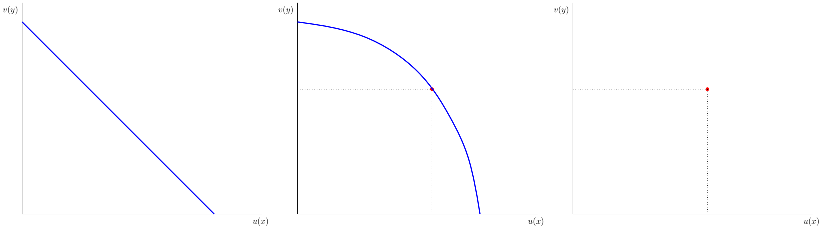
Solution

TU – NTU



Solution

TU – NTU

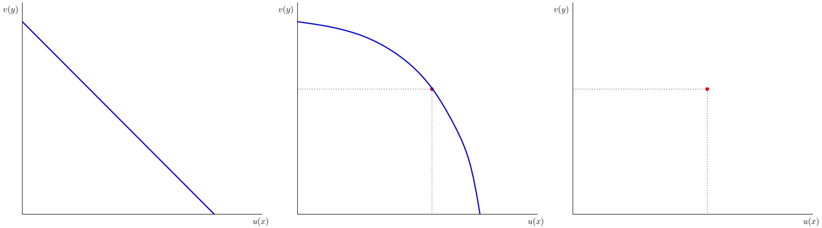


- TU:

$$\max_y f(a,y) - \pi(y) \quad \Rightarrow \quad f_y = \pi'(y)$$

Solution

TU – NTU



- TU:

$$\max_y f(a,y) - \pi(y) \quad \Rightarrow \quad f_y = \pi'(y)$$

- NTU:

$$\max_y \Phi(a,y,\pi) \quad \Rightarrow \quad \Phi_y + \Phi_\pi \frac{\partial \pi}{\partial y} = 0$$

Solution

Monotone Matching

- The allocation problem of a to y with frontier $\Phi(a, y, \pi)$
- Supermodularity of Φ:

$$\frac{d^2}{dady}\Phi = \Phi_{ay} + \Phi_{\pi y}\frac{\partial \pi}{\partial y} = \Phi_{ay} - \frac{\Phi_y}{\Phi_\pi}\Phi_{\pi a},$$

Solution

Monotone Matching

- The allocation problem of a to y with frontier $\Phi(a, y, \pi)$
- Supermodularity of Φ:

$$\frac{d^2}{dady}\Phi = \Phi_{ay} + \Phi_{\pi y}\frac{\partial \pi}{\partial y} = \Phi_{ay} - \frac{\Phi_y}{\Phi_\pi}\Phi_{\pi a},$$

- Higher a apply to higher $y \iff \Phi$ supermodular

$$\begin{aligned} \Phi_{ay} &> \frac{\Phi_y}{\Phi_\pi}\Phi_{a\pi} \\ -u''(a-a')a'_y &> \frac{\beta q u'(c_e) f_y}{\beta q u'(c_e)\frac{-1}{m}}(-u''(a-a')a'_\pi) \end{aligned}$$

Sorting

- From Implicit Function Thm (where ϕ is the maximand of Φ)

$$
\begin{array}{rcl}
a'_y & > & -mf_y a'_\pi \\
\left(\phi_{a'y} + mv_y \phi_{a'\pi}\right)\phi_{\lambda\lambda} & < & \left(\phi_{\lambda y} + mv_y \phi_{\lambda\pi}\right)\phi_{a'\lambda} \\
 & \cdots & \\
0 & < & \beta u'(c_e)\dfrac{m'}{\lambda} v_y \phi_{a'\lambda}
\end{array}
$$

- Positive sorting of a on $y \iff \phi_{a'\lambda} > 0$

$$
\beta R\left(q'[u'(c_e) - u'(Ra')] + u''(c_e)\frac{\pi m'}{\lambda m}\right) > 0,
$$

Assets – Productivity allocation

Using the FOC that $\phi_\lambda = 0$:

$$\frac{m'\pi}{\lambda m} = -q' \frac{u(c_e) - u(Ra')}{u'(c_e)}$$

Proposition

Workers with higher initial asset levels a will apply for more productive jobs provided

$$\frac{u'(c_e) - u'(Ra')}{u(c_e) - u(Ra')} < \frac{u''(c_e)}{u'(c_e)} \tag{U}$$

Assets – Productivity allocation

$$\frac{u'(c_e)-u'(Ra')}{u(c_e)-u(Ra')} < \frac{u''(c_e)}{u'(c_e)}$$

- Within HARA, condition (**U**) is equivalent to DARA:
  - $<$ CRRA – log
  - $=$ CARA – risk neutrality
  - $>$ quadratic
- DARA, $\frac{u''}{u'} < 0$ (or positive risk prudence $u''' > 0$):
  - sufficient for small w
  - not for large w; counter example: Taylor exp, arbitrary u''''

Assets – Productivity allocation

Under condition **U** $\approx$ DARA

- High asset workers ($a \uparrow$):
  1. apply for high productivity jobs ($y \uparrow$)
  2. *typically* earn higher wages ($w \uparrow$)
  3. have higher unemployment ($\lambda \uparrow \Rightarrow q(\lambda) \downarrow$)
  4. have higher expected consumption ($c \uparrow$)
  5. have higher expected utility ($U \uparrow$)
- High productivity firms ($y \uparrow$):
  1. *typically* post higher wages ($w \uparrow$)
  2. attract higher asset workers ($a \uparrow$)
  3. have higher expected profits ($\pi \uparrow$)
  4. fill vacancies faster ($\lambda \uparrow \Rightarrow m(\lambda) \uparrow$)

Equilibrium Properties

Under condition **U** ($\approx$ DARA)

- High asset holders have higher risk tolerance
- High productivity firms want to hire with high probability
  $\Rightarrow$ post high wage

$\Rightarrow$ natural complementarily between assets and productivity

Equilibrium Properties

Under condition **U** ($\approx$ DARA)

- High asset holders have higher risk tolerance
- High productivity firms want to hire with high probability
  $\Rightarrow$ post high wage

$\Rightarrow$ natural complementarily between assets and productivity

But, there is no technological complementarity (or single crossing condition)

Infinite Horizon

- The value to the unemployed $U(a)$; employed $E(a)$:

$$
\begin{aligned}
U(a) &= \max_{a',\lambda} \left\{ u(a-a') + \beta \left[qE(Ra') + (1-q)U(Ra') \right] \right\} \\
E(a) &= \max_{a'} \left\{ u(w+a-a') + \beta E(Ra') \right\}
\end{aligned}
$$

Infinite Horizon

- The value to the unemployed $U(a)$; employed $E(a)$:

$$\begin{aligned} U(a) &= \max_{a',\lambda}\left\{u(a-a')+\beta\left[qE(Ra')+(1-q)U(Ra')\right]\right\} \\ E(a) &= \max_{a'}\left\{u(w+a-a')+\beta E(Ra')\right\} \end{aligned}$$

- The FOC for employed is $u'(w+a-a') = \beta RE'(Ra')$. With $\beta R = 1 \Rightarrow a' = a/R = \beta a$:

$$E(a) = \frac{1}{1-\beta} u\left(w+(1-\beta)a\right)$$

Infinite Horizon

- The value to the unemployed $U(a)$; employed $E(a)$:

$$\begin{aligned} U(a) &= \max_{a',\lambda} \left\{ u(a-a') + \beta \left[qE(Ra') + (1-q)U(Ra') \right] \right\} \\ E(a) &= \max_{a'} \left\{ u(w+a-a') + \beta E(Ra') \right\} \end{aligned}$$

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$$E(a) = \frac{1}{1-\beta} u\left(w + (1-\beta)a\right)$$

- Firm problem (stationary + cloning assumption):

$$\begin{aligned} V(y) &= \max_{w} \left\{ m\left(v(y) - w\right) + \beta(1-m)V(y) \right\} \\ &= \max_{w} \left\{ \frac{m}{1-\beta(1-m)} \left[v(y) - w \right] \right\}. \end{aligned}$$

Infinite Horizon

- We can write the problem of the unemployed $U(a) = \Phi$ as:

$$\Phi(a, y, \pi) = \max_{a', \lambda} \left\{ u(a - a') + \beta \left[q \frac{1}{1-\beta} u\left(c_e\right) + (1-q)\Phi(Ra') \right] \right\}$$

where

$$c_e = (1-\beta)Ra' + v(y) - \pi \frac{1 - \beta(1-m)}{m}$$

Infinite Horizon

Proposition

Workers with higher initial asset levels a will apply for higher wage jobs provided

$$\frac{u'(c_e) - \Phi'(Ra')}{\frac{1}{1-\beta} u\left(c_e\right) - \Phi(Ra')} < \frac{u''(c_e)}{u'\left(c_e\right)} \qquad (\mathbf{U}_\infty)$$

Infinite Horizon

Proposition

Workers with higher initial asset levels a will apply for higher wage jobs provided

$$\frac{u'(c_e) - \Phi'(Ra')}{\frac{1}{1-\beta} u(c_e) - \Phi(Ra')} < \frac{u''(c_e)}{u'(c_e)} \qquad (\mathbf{U}_\infty)$$

Proposition

Under condition $(\mathbf{U}_\infty)$ and for a given worker with assets a, the job productivity y decreases in the duration of unemployment.

Calibration

- One period is set to be 6 weeks.
- $a \in \mathcal{A} = [0, 300]$ and $y \in \mathcal{Y} = [100, 200]$
- $u(c) = log(c)$, $f(y) = y$, $q(\theta) = \theta(1 + \theta^{\gamma})^{\frac{1}{\gamma}}$

Parameter	Definition	Value
β	discount factor	0.99
r	interest rate	0.005
b	unemployment benefit	60
k	cost of vacancy	50
λ	Probability of Separation	0.03
γ	elasticity of matching fn	1.2

Characterization of the Steady State

u(%)	$avg(\theta)$	$avg(w)$
4.7%	1.11	148.22

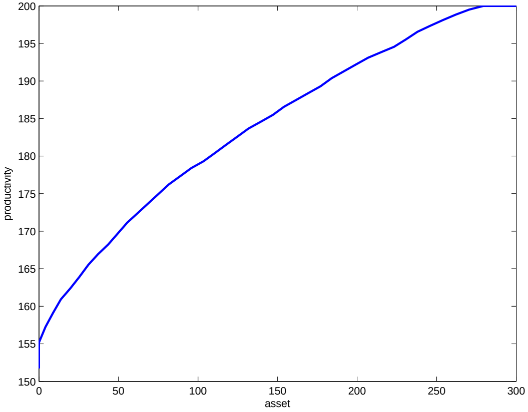


Figure: Allocation of firms and workers in labour market

Probability of job finding and Wage

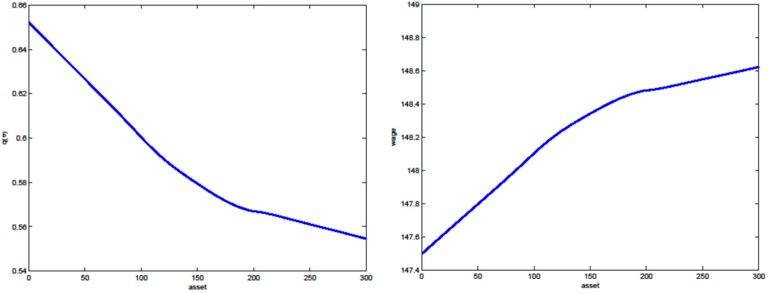


Figure: probability of job finding and wage as a function of asset

Value of workers and firms

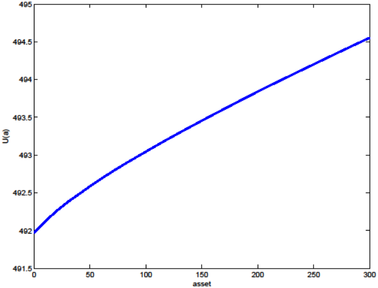


Figure: The value of unemployed workers as a function of asset and firms as a function of productivity

Distribution of asset and productivity

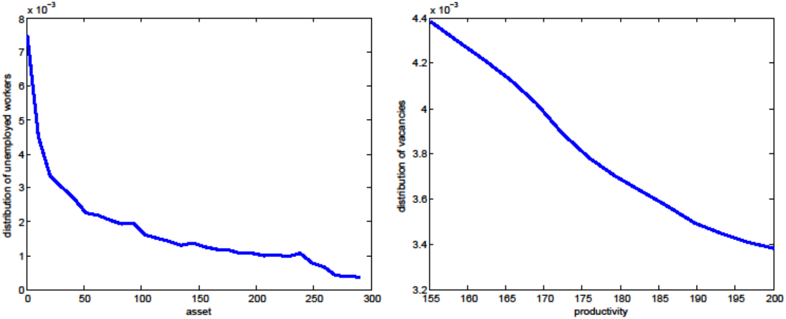
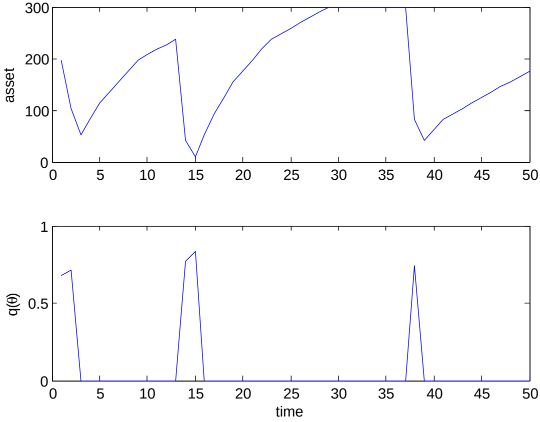


Figure: Distribution of workers and firms

Simulation



Welfare Effects of UI

Is UI welfare improving?

1. Consumption
2. Allocation and probability of job finding
3. Firms entry

Welfare Effects of UI

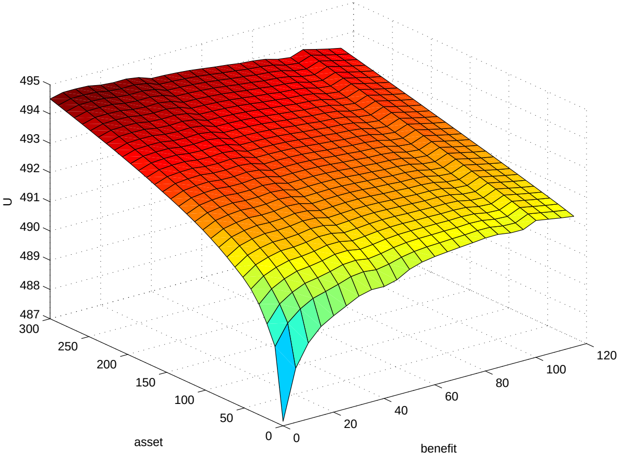
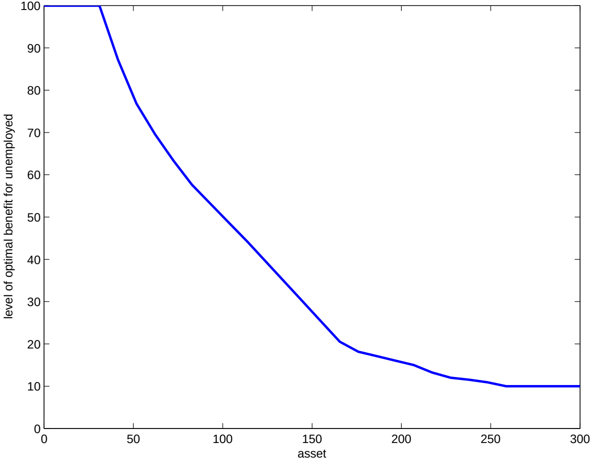


Figure: The value of unemployment

Optimal UI and asset holding



Consumption

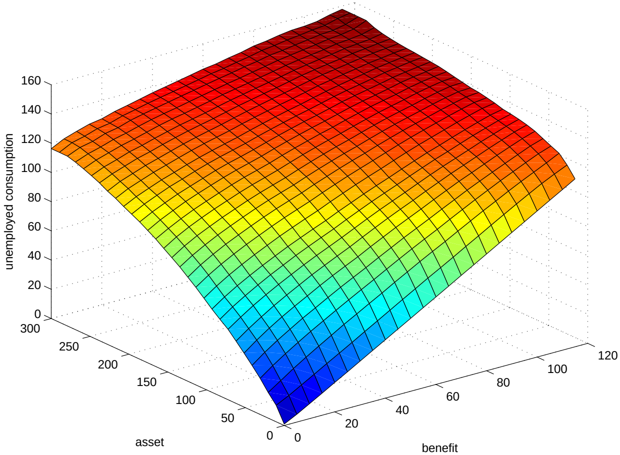


Figure: Consumption of unemployed workers

Allocation

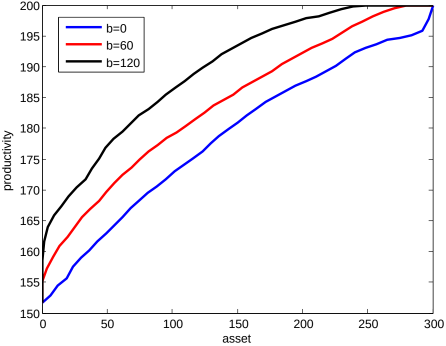


Figure: Change in allocation of asset holders to firms of different productivities

Probability of job finding

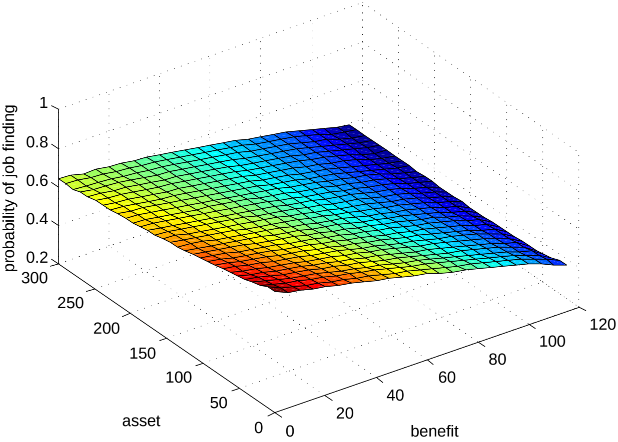


Figure: Probability of job finding as a function of asset and unemployment benefit

Unemployment and Firms entry

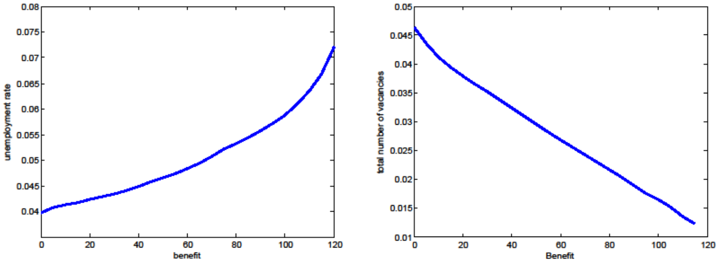
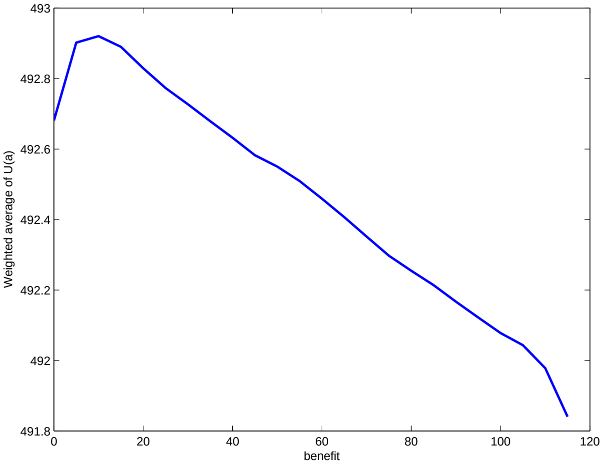


Figure: Unemployment rate and total vacancies as a function of unemployment benefit

Optimal level of UI

Average value of the unemployed



Comparison

Aiyagari(1994)

- The employment process is exogenously given
- UI and taxes are nondistortionary
- Welfare is monotonically increasing in benefit

Krusell et al(2010)

- Frictional labour market, Nash bargaining, homogenous firms
- Same probability of job finding for all workers
- Asset distribution does not play any role

Related empirical literature

- Silvio (AER-2006), Card, Chetty, and Weber (QJE-2007), and Lentz (RED-2009): document that higher asset holdings lead to prolonged job search
- Chetty (JPE-2008) shows that the elasticity of the job finding rate with respect to unemployment benefits decreases with liquid wealth
- Browning and Crossley (JPE-2001) show that unemployment insurance improves consumption smoothing for poor agents, but not for rich ones

Conclusion

- Interaction: search frictions, unemployment risk
- Wage/productivity increasing in assets

$\Rightarrow$ Assets affect wage inequality

- UI: interaction of consumption smoothing, distribution and firms entry
- Productivity and labour-market outcomes

III. Directed Search

Competing Mechanisms

Competing Mechanisms

- McAfee (1993), "Mechanism Design by Competing Sellers"
- Issue: heterogeneous agents: use mechanisms (auctions,...) instead of prices to extract rents from buyers
- Frictions in equilibrium: coordination problem
- But: number of participants in auction is determined competitively: reservation price $\downarrow$, # agents $\uparrow$
- Eeckhout-Coles (2003), extend the contract space (demand-contingent prices) $\rightarrow$ indeterminacy
- Peters and Severinov (1997), Peters (1998, 2000,...): foundations large market assumptions
- Shimer (2005): demand schedule for heterogeneous buyers/workers

Competing Mechanisms

- McAfee: finite # buyers and sellers, but ignore strategic impact on continuation value of other agents
- Shows $\exists$ equilibrium in second price auction + reservation price = seller's outside option (moreover, is weak best response to any other strategy profile)
- Peters and Severinov restrict attention to second price auctions, but finite # agents, solve for the Subgame Perfect equilibrium and take the limit as # buyers $\to \infty$
- Confirm result in McAfee: Bertrand type competition
- By undercutting other seller you can attract the market share (though not entirely due to the search frictions)

Other Information Issues

Matching and Moral Hazard

- Matching and the hold up problem: how is investment affected by matching of heterogeneous firms and workers?
- One solution to hold up: dynamics instead of matching
  - Folk Theorem: obvious
  - Che and Sakovics (Eca 2004). Holdup problem solved when continued investment is allowed (i.e., option to delay)
  - Logic: static, no investment if $\frac{1}{2}\phi_I - C < \frac{1}{2}\phi_N$
  - But if $\frac{1}{2}\phi_I - C > 0$, then in a dynamic setting, one shot deviation principle (will invest next period, so need to offer at least $\frac{1}{2}\phi_I$), and payoff:

$$\max\left\{\phi_N - \delta\frac{1}{2}\phi_I, \delta(\frac{1}{2}\phi_I - C)\right\}$$

    First payoff is accepted by other player; second rejected, in which case invest next period.
  - δ large, both deviations less than investing: solves hold up

Other Information Issues

Matching and Moral Hazard

- How does matching solve hold up problem?
- Complete information: Felli-Roberts (1999), Cole-Mailath-Postlewaite (1993, 2001)
- Bertrand competition (not trade in full contingent contracts) can solve hold up: return on investment is bounded by outside option of matching with next type
- Incomplete Information: Hoppe, Moldovanu, Sela (2009): signaling, but inefficiency does not disappear as $n \to \infty$
- HMS result: under equal bargaining shares, half of output is wasted on signaling. Random matching can be superior to signaling provided

$$\frac{\mathsf{Cov}(x, \mu(x))}{\mathbb{E}x \cdot \mathbb{E}\mu(x)} \leq 1$$

Topics in Labor Markets

Jan Eeckhout

2015–2016

IV. Further Topics:

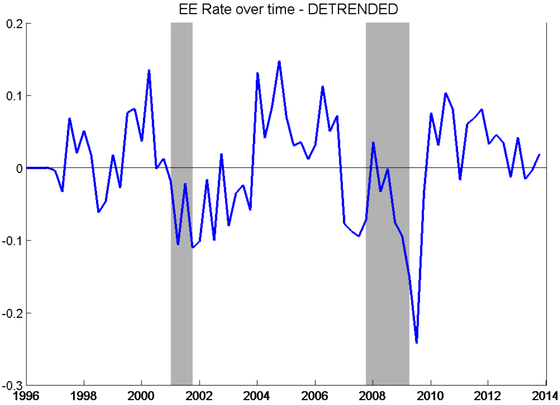
Matching and Uncertainty:

Unemployment Cycles

Motivation

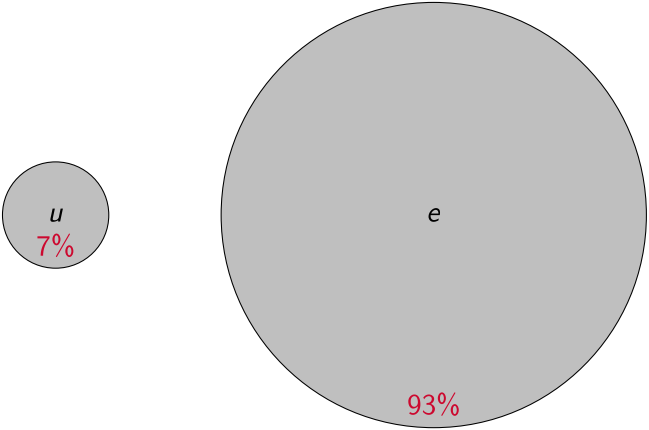
- Theory of cycles, solely driven by the labor market
- Labor market by itself can generate cyclical outcomes
    1. Mechanism: search behavior of the employed
    2. We illustrate theory with a Quantitative Exercise

Search Behavior of the Employed



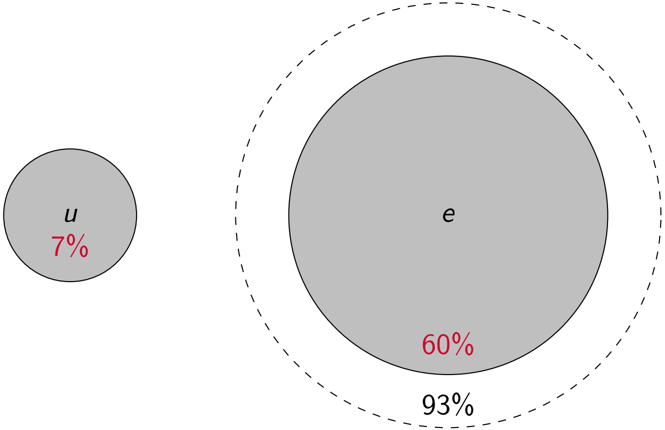
Composition Externality

Labor Force (on average)



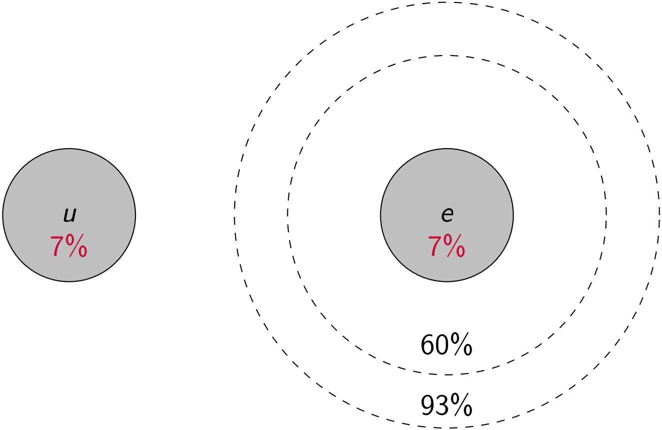
Composition Externality

Searchers



Composition Externality

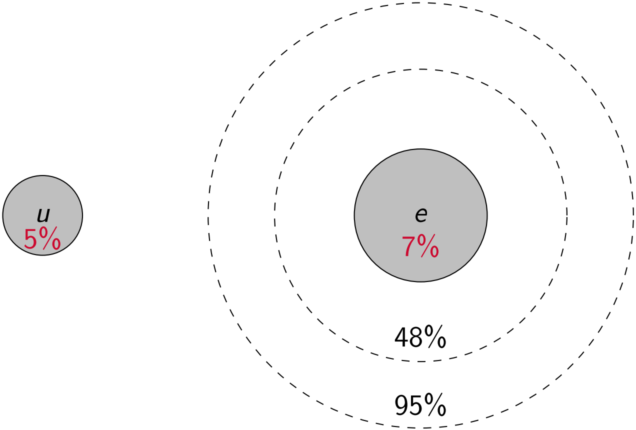
Effective Searchers



$\rightarrow$ *on average* 50% $\simeq \frac{7}{7+7}$ of jobs are filled by employed

Composition Externality

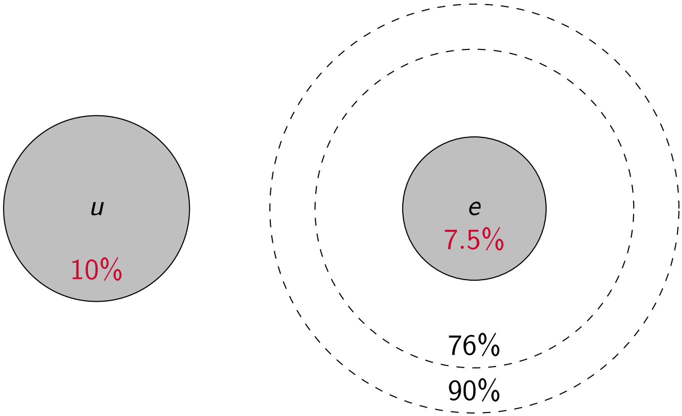
Boom



→ *Boom:* 62% $\simeq \frac{7}{7+5}$ of jobs are filled by employed

Composition Externality

Recession



→ *Recession:* 42% $\simeq \frac{7.5}{7.5+10}$ of jobs are filled by employed

The Mechanism

- Pro-cyclical on-the-job search (OJS) intensity of employed

  $\Rightarrow$ Multiple equilibria
- Strategic complementarity betw. search effort and vac. posting due to:
  1. Composition externality + job quality: newly created jobs by employed are more productive and more prevalent in Boom: 42% (R) $\rightarrow$ 62% (B)
  2. Duration: average job duration shorter in Boom

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Boom: OJS intensity $\uparrow \Rightarrow$ composition $\succ$ duration $\Rightarrow$ profits $\uparrow$ $\Rightarrow v \uparrow \Rightarrow$ matching prob $\succ$ search cost $\Rightarrow$ OJS intensity $\uparrow$

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  1. Composition externality + job quality: newly created jobs by employed are more productive and more prevalent in Boom: 42% (R) $\rightarrow$ 62% (B)
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Boom: OJS intensity $\uparrow \Rightarrow$ composition $\succ$ duration $\Rightarrow$ profits $\uparrow$
$\Rightarrow v \uparrow \Rightarrow$ matching prob $\succ$ search cost $\Rightarrow$ OJS intensity $\uparrow$

Recession: OJS intensity $\downarrow \Rightarrow$ composition $\prec$ duration $\Rightarrow$ profits $\downarrow$
$\Rightarrow v \downarrow \Rightarrow$ matching prob $\prec$ search cost $\Rightarrow$ OJS intensity $\downarrow$

Implications

1. Large fluctuations in u, v, EE without shifts in fundamentals
2. Jobless recovery: OJS crowds out unemployed searchers during recovery
3. Outward shift Beveridge curve in recovery (no change match efficiency)

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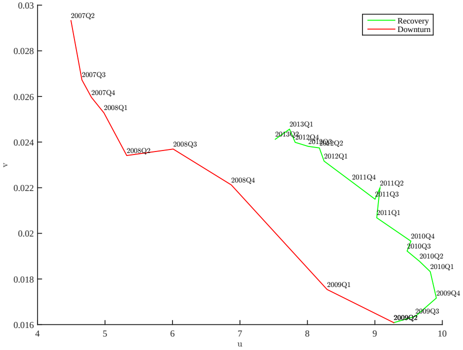


The Literature

- Multiple Equilibria in Search Markets:

  Increasing Returns: Diamond (1982)
  Selection: Burdett-Coles (1998)
  Demand External.: McAfee (1992), Kaplan-Menzio (2014), Schaal-Taschereau (2014)
  Decreasing Returns: Golosov-Menzio (2015)
  Marriage Market: Burdett-Imai-Wright (2004)
  Housing Market: Moen-Nenov (2014)

- Business Cycles and Search:

  Shimer (2005), Hall (2005), Hagedorn-Manovskii (2008)

The Model

The Model: Key Ingredients

1. On-the-job search
2. Job ladder (sorting)
3. Endogenous vacancy creation

The Model: Key Ingredients

1. On-the-job search
2. Job ladder (sorting)
3. Endogenous vacancy creation

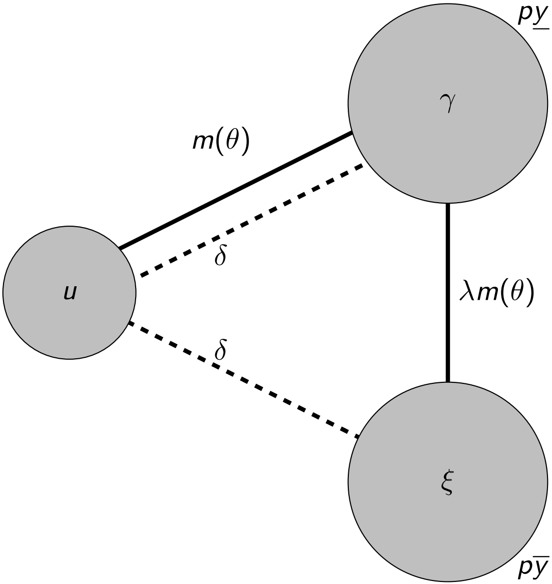
- Natural setup: random arrival diff. jobs $\Rightarrow$ selection + duration issue
- All action comes from OJS of those in low productivity job who transit to high productivity job

$\Rightarrow$ Focus on simple model: out of U, low prod. job; out of E high prod.

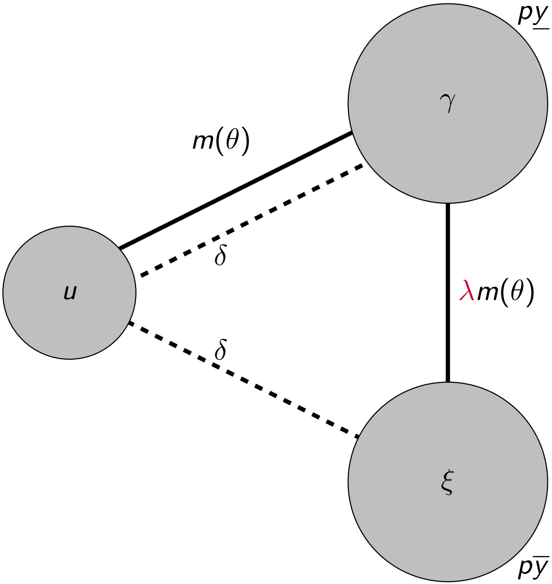
Agents, Actions, Payoffs + Wage Setting

- Workers: measure one; risk-neutral and homogenous
  - Employed (get w) or unemployed (get b)
  - Decision: Once on the job, *active* OJS at cost k?
  - Cost of search during unemployment (or passive OJS) normalized to zero
  - Objective: maximize discounted value of employment
- Firms: large number; ex-ante homogenous and risk-neutral
  - Decision: post a vacancy at cost c; free entry
  - Ex-post heterogeneity in their job productivity $y \in \{\underline{y}, \overline{y}\}$: $\underline{y}$ for UE match, $\overline{y}$ for EE match $\rightarrow$ Job ladder
  - Objective: maximize discounted sum of profits
- Wage setting: sequential auction; firms match outside offers

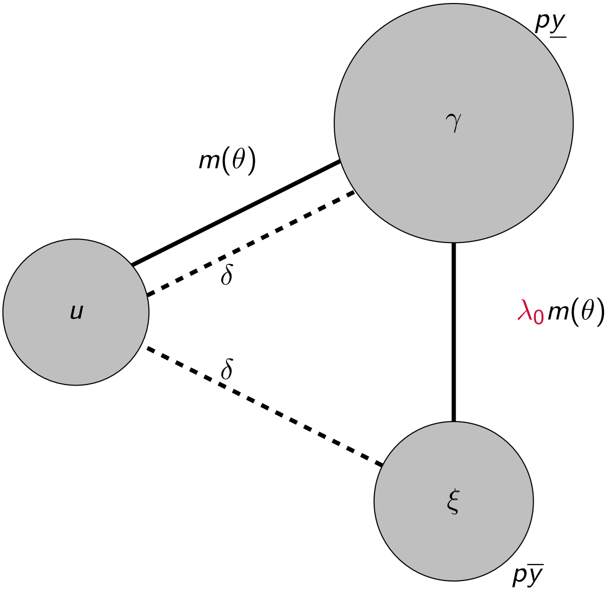
Labor Market



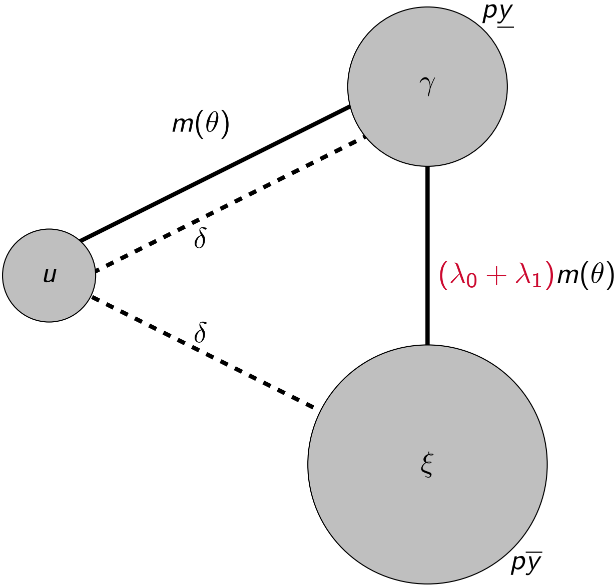
Labor Market



No Active OJS



Active OJS



$p\underline{y}$
γ
$m(\theta)$
δ
u
$(\lambda_0 + \lambda_1)m(\theta)$
δ
ξ
$p\overline{y}$

Firms

Bellman Equations

$$
\begin{aligned}
rV &= -c + q(\theta(\Omega)) \left[\frac{u}{s(\Omega)} \underline{J} + \frac{\lambda(\Omega)\gamma}{s(\Omega)} \overline{J} - V \right] + \dot{V} \\
r\underline{J} &= p\underline{y} - \underline{w}(\Omega) - [\lambda(\Omega) m(\theta(\Omega)) + \delta](\underline{J} - V) + \dot{\underline{J}} \\
r\overline{J} &= p\overline{y} - \overline{w}(\Omega) - \delta(\overline{J} - V) + \dot{\overline{J}}
\end{aligned}
$$

where

- $\Omega \in [0, 1]$ all workers' search decision
- we suppress time indices
- $\theta(\Omega) = \frac{v}{s(\Omega)} = \frac{v}{u + \lambda(\Omega)\gamma}$
- $\underline{w}(\Omega), \overline{w}(\Omega)$ set by PVR bargaining

Wages

Firms

Bellman Equations

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Wages

Workers

Bellman Equations

$$
\begin{aligned}
rU &= pb + m(\theta(\Omega))(\underline{E} - U) + \dot{U} \\
r\underline{E} &= \underline{w}(\Omega) - \omega pk + \lambda(\omega)m(\theta(\Omega))(\overline{E} - \underline{E}) - \delta(\underline{E} - U) + \dot{\underline{E}} \\
r\overline{E} &= \overline{w}(\Omega) - \delta(\overline{E} - U) + \dot{\overline{E}}
\end{aligned}
$$

where

- $\omega \in [0, 1]$ individual worker's search decision

Workers

Bellman Equations

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r\overline{E} &= \overline{w}(\Omega) - \delta(\overline{E} - U) + \dot{\overline{E}}
\end{aligned}
$$

where

- $\omega \in [0, 1]$ individual worker's search decision

Labor Market Dynamics

$$\begin{aligned}
\dot{\gamma} &= um(\theta(\Omega)) - \gamma[\delta + \lambda(\Omega)m(\theta(\Omega)] \\
\dot{\xi} &= \gamma\lambda(\Omega)m(\theta(\Omega)) - \xi\delta \\
1 &= u + \gamma + \xi
\end{aligned}$$

Equilibrium

Definition

An equilibrium is a path $\{U_t, \underline{E}_t, \overline{E}_t, V_t, \underline{J}_t, \overline{J}_t, \theta_t, \underline{w}_t, \overline{w}_t, u_t, \gamma_t, \xi_t, \omega_t, \Omega_t\}$ s.t. for all $t \geq 0$

1. $U_t, \underline{E}_t, \overline{E}_t, V_t, \underline{J}_t, \overline{J}_t$ satisfy the Bellman equations above;
2. Given Ω_t, $\omega_t = \Omega_t$ maximizes $\underline{E}_t$;
3. There is free entry: $V_t = 0$;
4. Wages: $\underline{w}_t$ such that $\underline{E}_t = U_t$ and $\overline{w}_t$ such that $\underline{J}_t = V_t$;
5. u_t, γ_t, ξ_t satisfy the laws of motion;
6. $\lim_{t\to\infty} \underline{J}_t$ is finite for initial conditions u_0, γ_0, ξ_0.

Multiple Steady State Equilibria: Existence

- Check one-shot deviations of workers in $\underline{y}$-jobs in interval dt
- Denote $\underline{E}(\omega|\Omega)$: value of $\underline{y}$ job when worker action is ω given Ω
  1. $\Omega = 1$: all workers active OJS $\Rightarrow$ profitable to stop active OJS $\omega = 0$?

     $$\underline{E}(1|1) > \underline{E}(0|1) \quad \Longleftrightarrow \quad m^{-1}\left(\frac{k(r+\delta)}{\lambda_1(\underline{y}-b))}\right) < \theta(1).$$

  2. $\Omega = 0$: all workers no active OJS $\Rightarrow$ profitable active OJS $\omega = 1$?

     $$\underline{E}(0|0) > \underline{E}(1|0) \quad \Longleftrightarrow \quad \theta(0) < m^{-1}\left(\frac{k(r+\delta)}{\lambda_1(\underline{y}-b))}\right).$$

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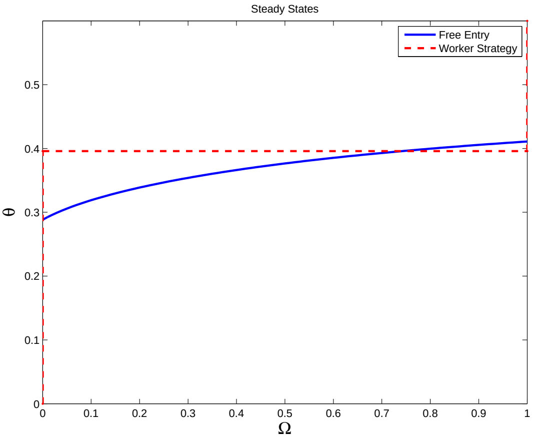
Lemma

There are multiple steady states if and only if

$$\theta(0) < m^{-1}\left(\frac{k(r+\delta)}{\lambda_1(\underline{y}-b)}\right) < \theta(1).$$

Proof

Steady State Equilibria

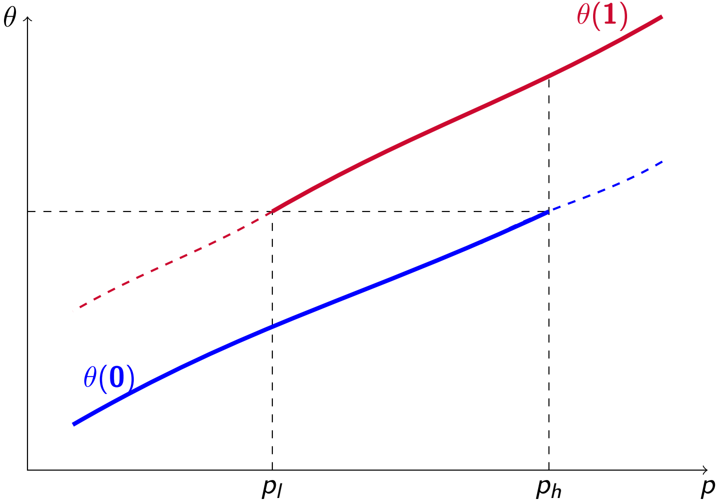


Multiple Steady States: Existence

Proposition

Let $m(\theta) = \phi \frac{\alpha\theta}{\alpha\theta+1}$. Then there are multiple steady state equilibria if and only if $p \in [p^l, p^u]$. The set $[p^l, p^u]$ is non-empty for an open set of parameters.

Multiplicity Bounds: p



Multiple Steady State Equilibria: Existence

Sufficient Sorting Needed for Active OJS

Proposition

Let $m(\theta) = \phi \frac{\alpha\theta}{\alpha\theta+1}$.

1. *If* $(\overline{y} - \underline{y} < \epsilon)$ *then there is a unique steady state with no active OJS;*
2. *If* $\overline{y}$ *is arbitrarily high (given* $\underline{y}$*), there is a unique steady state with active OJS;*
3. *For* $\overline{y} \in [\overline{y}^l, \overline{y}^u]$ *(given* $\underline{y}$*), there are multiple steady states.*

▸ Plot

Steady State Equilibria: Properties

Proposition

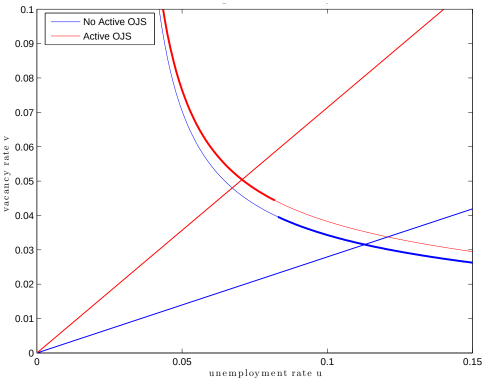
Assume there are multiple steady states. Then:

1. *unemployment is lower with active OJS:* $u(\mathbf{1}) < u(\mathbf{0})$;
2. *EE flows are higher with active OJS:* $EE(\mathbf{1}) > EE(\mathbf{0})$;

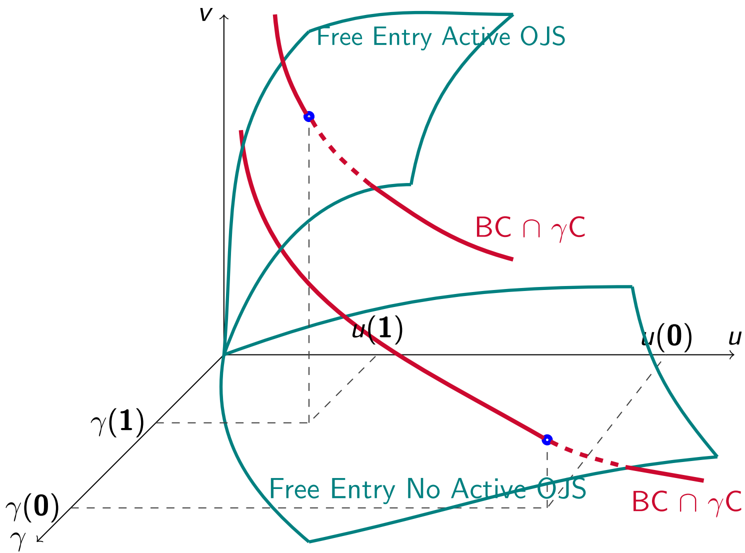
and under $m(\theta) = \phi\alpha\theta/(\alpha\theta + 1)$

3. *vacancies are higher with active OJS:* $v(\mathbf{1}) > v(\mathbf{0})$;
4. *conventional market tightness is higher with active OJS:* $\Theta(\mathbf{1}) > \Theta(\mathbf{0})$;
5. $BC(\mathbf{1})$ *is shifted outward relative to* $BC(\mathbf{0})$
6. $BC^s(\mathbf{1})$ *is shifted outward relative to* $BC^s(\mathbf{0})$
7. *Share of OJSearchers is higher with active OJS:* $\frac{\lambda(\mathbf{1})\gamma(\mathbf{1})}{s(\mathbf{1})} > \frac{\lambda(\mathbf{0})\gamma(\mathbf{0})}{s(\mathbf{0})}$.

Steady State Equilibria: Properties



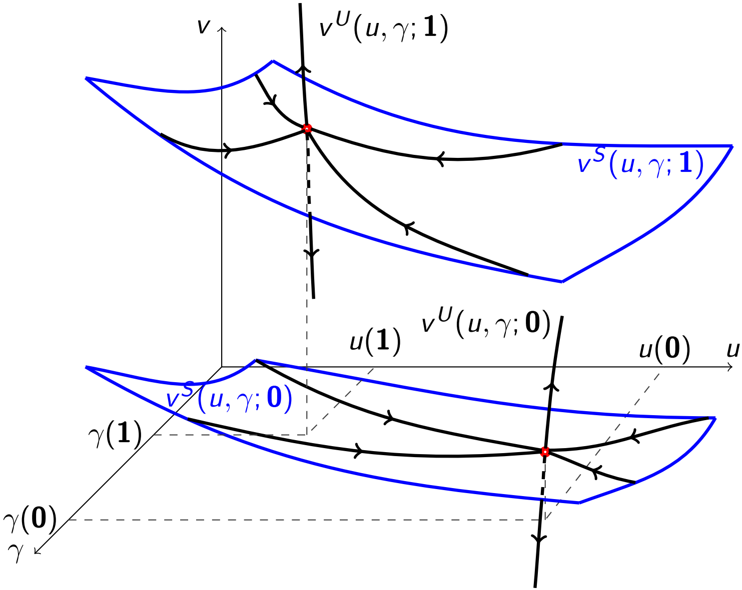
Steady State Equilibria: Properties



Dynamics

- Our model can be reduced to a dynamic system in $\mathbb{R}^3$: $\dot{u}(\Omega), \dot{\gamma}(\Omega), \dot{\theta}(\Omega)$ ▸ System
- Multiple SS equilibrium $\rightarrow$ multiple equil. paths in dynamic economy

Saddle-Path Stability



Validation and Quantitative Exercise

Validation and Quantitative Exercise

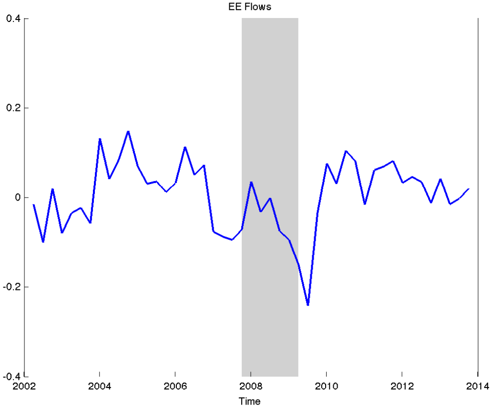
1. Direct evidence for mechanism: pro-cyclical search intensity
2. Quantitative exercise
   - Calibrate the model to US economy
   - Quantitative assessment:
     - Steady States: Labor Market Fluctuations and counterfactuals
     - Dynamics: Jobless recovery

The Data

- US quarterly data
- Main data source: Current Population Survey (CPS)
- Data on vacancies, unemployment, labor market transitions
- Vacancies: JOLTS (BLS) + online help-wanted ads
- Data spans 1996-2013 but main focus on Great Recession

1. Evidence on Pro-Cyclical Search Intensity

EE Flows (Detrended)



Decomposition of EE Flows: $EE = \lambda\gamma m(\theta)$

$$m(\theta) = \frac{UE}{u} \quad \text{and} \quad \lambda\gamma = \frac{EE \cdot u}{UE}$$

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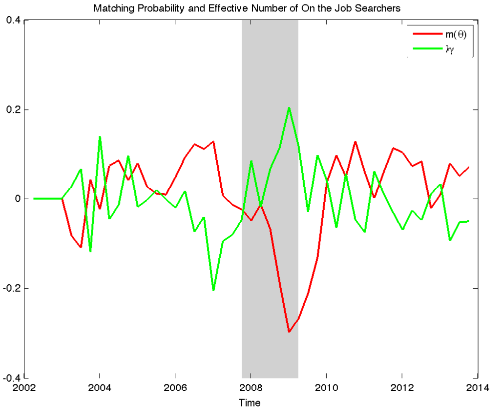
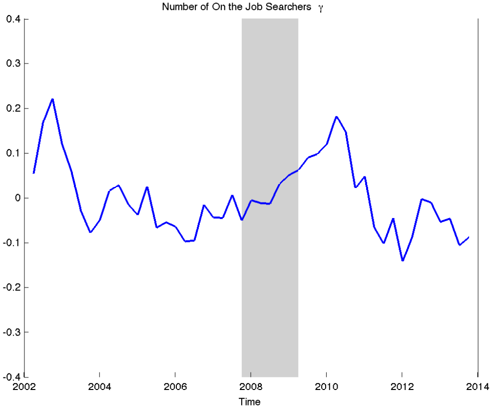


Decomposing $\lambda\gamma$

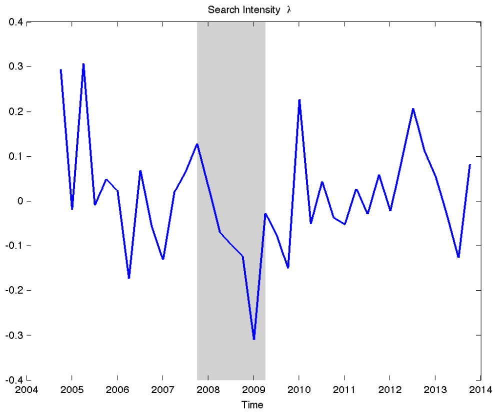
- Problem: No direct measure of search intensity λ
- Use CPS micro-data panel structure
- Check whether individuals was unemployed before current job or transited from another job
- Construct γ (employed after UE transition) and ξ (after EE transition)
- Then, search intensity is computed as: $\lambda = \frac{EE}{m(\theta)\gamma}$

Decomposition of EE Flows: γ



▸ ξ

Decomposition of EE Flows: $\lambda = \frac{EE}{m(\theta)\gamma}$



$\Rightarrow$ Pro-cyclical search intensity!

ATUS

2. Quantitative Exercise

Calibration

- Set parameters $(r, b, \delta, p, \underline{y})$ outside the model
- Calibrate $(\lambda_0, \lambda_1, \alpha, \phi, c, k, \overline{y})$ using GMM
- Target business cycle moments from the Great Recession
  - EE fluctuations (peak and trough)
  - $m(\theta)$-fluctuations (peak and trough)
  - wage differentials $\overline{w}/\underline{w}$ in *boom* (peak)
  - v, u*-levels* in *boom* (peak)
- Focus on 2 data points from last cycle with largest differences in EE

$\Rightarrow$ 2006Q3 *boom* $(\Omega = 1)$ and 2009Q3 *recession* $(\Omega = 0)$

Calibration

- We do *not* target unemployment and vacancy levels in the recession
- We do *not* restrict the estimates to fall into range of multiple SS (we get it)

Exogenously Set Parameters

Variable	Value		Notes
r	0.0113	discount factor	standard
$\underline{y}$	1	productivity first job	normalization
b	0.919	unemployment value	92% of $\underline{y}$; 58% of $\overline{y}$ (see belo
δ	0.05	job separation rate	average separation rate
p	1	productivity	normalization

Estimated Parameters

	Estimate	Parameter Description
λ_0	0.092	passive OJS intensity
λ_1	0.073	active OJS intensity
α	0.863	curvature matching function
ϕ	3.258	overall matching efficiency
c	9.404	vacancy posting cost
$\overline{y}$	1.577	high productivity
k	0.080	search cost

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$\overline{y}$	1.577	high productivity
k	0.080	search cost

$\Rightarrow$ Multiple Steady States Exist:

$p \in [p^l, p^u] = [0.994, 1.026]$

Moments

Targeted

- **Model 1**: Benchmark model, multiple steady st., fixed productivity p

	Data	**Model**
$EE(\mathbf{1})$	0.066	0.035
$EE(\mathbf{0})$	0.036	0.022
$u(\mathbf{1})$	0.047	0.055
$v(\mathbf{1})$	0.029	0.039
$m(\theta(\mathbf{1}))$	0.852	0.853
$m(\theta(\mathbf{0}))$	0.511	0.513
$\frac{\overline{w}(\mathbf{1})}{\underline{w}(\mathbf{1})}$	1.230	1.230

Moments

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$EE(\mathbf{1})$	0.066	0.035
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- Discrepancy between model and data: constant separation rate

Moments

Non-Targeted

	Data	Model
$u(\mathbf{0})$	0.096	0.089
$v(\mathbf{0})$	0.016	0.029
$\frac{\lambda(\mathbf{0})\gamma}{s(\mathbf{0})}$	0.423	0.327
$\frac{\lambda(\mathbf{1})\gamma}{s(\mathbf{1})}$	0.625	0.425

Labor Market Fluctuations

- Fluctuations between peak and trough of Great Recession
- $\Delta x = \frac{x(\mathbf{0}) - x(\mathbf{1})}{x(\mathbf{1})}$

	Data	**Model 1**	**Model 2**
ΔEE	-0.46	-0.37	
$\Delta m(\theta)$	-0.40	-0.40	
Δv	-0.47	-0.28	
Δu	1.06	0.60	
$\Delta \theta$	-0.61	-0.47	
$\Delta \Theta$	-0.74	-0.55	
$\Delta \lambda\gamma/s$	-0.32	-0.23	

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$\Delta m(\theta)$	-0.40	-0.40	-0.15
Δv	-0.47	-0.28	-0.08
Δu	1.06	0.60	0.17
$\Delta \theta$	-0.61	-0.47	-0.20
$\Delta \Theta$	-0.74	-0.55	-0.22
$\Delta \lambda\gamma/s$	-0.32	-0.23	-0.02

Model 1: Multiple equilibria, fixed productivity $\Delta p = 0$.
Model 2: Active OJS equil., Δp: $+2\%$ deviation from trend in boom, -3% in recession.

- Myopic agents: in recession ($\mathbf{\Omega} = 0$) change beliefs to boom ($\mathbf{\Omega} = 1$)
- Searchers:
  $s(\mathbf{0}) = u(\mathbf{0}) + \lambda_0\gamma(\mathbf{0}) \quad \rightarrow \quad s^R = u(\mathbf{0}) + (\lambda_0 + \lambda_1)\gamma(\mathbf{0})$
- Fraction κ of u-hires:

$$\kappa(\mathbf{0}) = \frac{u(\mathbf{0})}{u(\mathbf{0}) + \lambda_0\gamma(\mathbf{0})} = 0.67 \ \rightarrow \ \kappa^R = \frac{u(\mathbf{0})}{u(\mathbf{0}) + (\lambda_0 + \lambda_1)\gamma(\mathbf{0})} = 0$$

- Uncond. matching probability
  $\kappa(\mathbf{0})m(\theta(\mathbf{0})) = 0.34 \rightarrow \kappa^R m(\theta^R) = 0.30$

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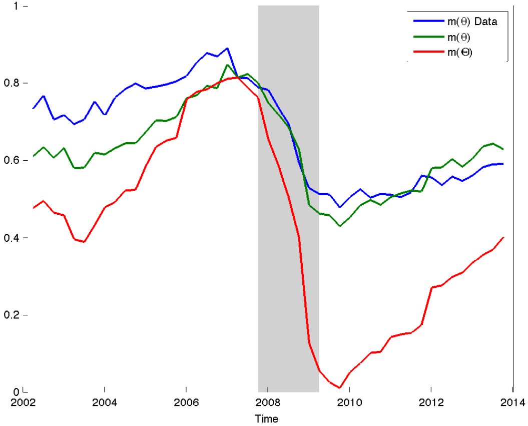
- Uncond. matching probability
  $\kappa(\mathbf{0}) m(\theta(\mathbf{0})) = 0.34 \rightarrow \kappa^R m(\theta^R) = 0.30$

$\Rightarrow$ Job-destructive Recovery

Jobless Recovery and Crowding Out

I. A Simple Exercise

- Effective matching probability $m(\theta)$ drops (but less so than $m(\Theta)$)



Jobless Recovery and Crowding Out

II. Productivity-Induced Dynamics

- Multiplicity selection criterion: history-dependent beliefs (Cooper 1994)
- Aggregate productivity p follows Markov process
- Agents are forward-looking
- Experiment: Economy has been in the recession for a while and positive shock $p \uparrow$ induces unique equilibrium with OJS

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- Aggregate productivity p follows Markov process
- Agents are forward-looking
- Experiment: Economy has been in the recession for a while and positive shock $p \uparrow$ induces unique equilibrium with OJS
- Limitations: saddle-path stability + linear approximation dynamic system

Jobless Recovery and Crowding Out

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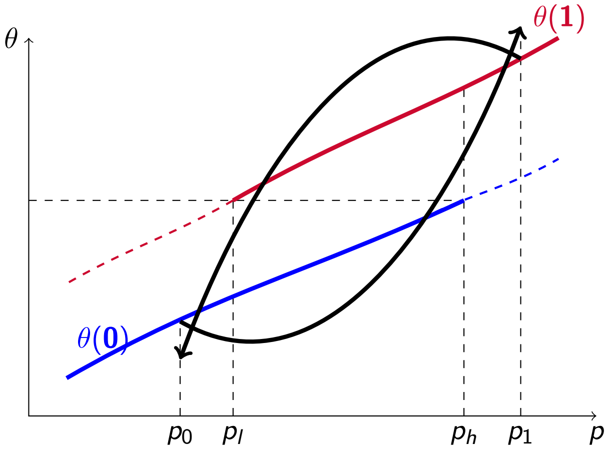
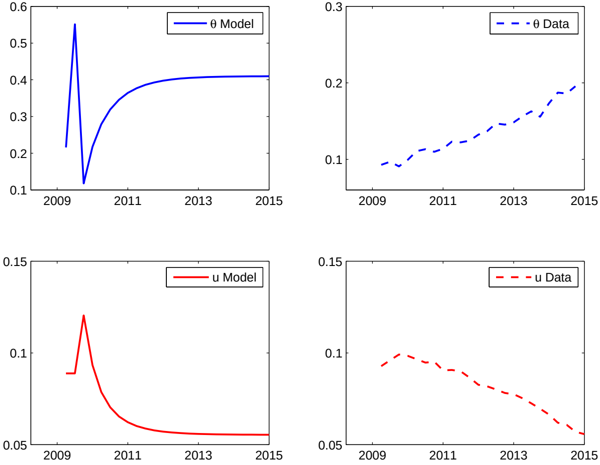


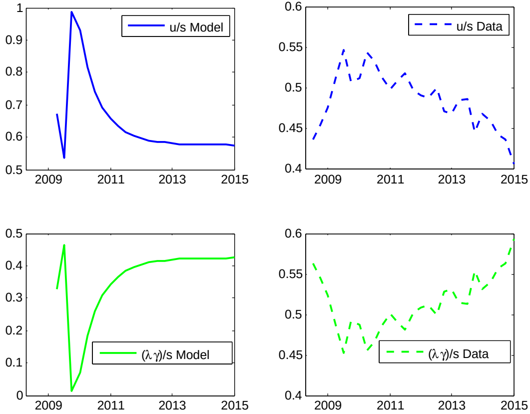
Jobless Recovery: Transition Paths

Market Tightness and Unemployment



Jobless Recovery: Transition Paths

Composition of New Jobs



Summary of Quantitative Results

- Fluctuations
  - Model generates sizable fluctuations v, u, EE without shift fundamentals
  - Small additional fluctuations from productivity change
- Jobless recovery
  - Unemployment initially grows during the recovery
  - Composition of u-jobs is initially higher in recovery

Conclusion

The labor market by itself can generate cycles

Topics in Labor Markets

Jan Eeckhout

2015–2016

Appendix

Wages

$$
\begin{aligned}
\underline{w}(\Omega) &= pb\left(\frac{r+\lambda(\Omega)m(\theta(\Omega))+\delta}{r+\delta}\right)-\frac{\lambda(\Omega)m(\theta(\Omega))}{r+\delta}p\underline{y}+\Omega pk \\
\overline{w}(\Omega) &= p\underline{y}
\end{aligned}
$$

Back

Proof of Lemma 1 Back

1. No deviation when no one searches: $\underline{E}(0|\mathbf{0}) > \underline{E}(1|\mathbf{0})$.

$$\underline{E}(1|\mathbf{0}) = \frac{1}{1+rdt}\left[dt(\underline{w}(\mathbf{0}) - pk) + (1-\delta dt)dt\lambda(1)m(\theta(\mathbf{0}))\overline{E} + (1-\delta dt)(1-dt\lambda(1)\right.$$

where $\overline{E} = \overline{E}(0|\mathbf{0})$.

$$\underline{E}(0|\mathbf{0})(1+rdt) > dt(\underline{w}(\mathbf{0}) - pk) + dt\lambda(1)(1-\delta dt)m(\theta(\mathbf{0}))\overline{E} + (1 - \delta dt - dt\lambda(1)m(\theta$$

Subtracting $\underline{E}(0|\mathbf{0})$ from both sides and dividing by dt and take the limit $dt \to 0$:

$$r\underline{E}(0|\mathbf{0}) > \underline{w}(\mathbf{0}) - pk + \lambda(1)m(\theta(\mathbf{0}))\overline{E} + (-\delta - \lambda(1)m(\theta(\mathbf{0})))\underline{E}(0|\mathbf{0}) + \delta U.$$

Substituting the equilibrium values for $\underline{E}(0|\mathbf{0}), \overline{E}, U$ and $\underline{w}(\mathbf{0})$ we get:

$$(\underline{y} - b)[\lambda(1) - \lambda(0)]m(\theta(\mathbf{0})) - k(r+\delta) < 0. \tag{6}$$

2. No deviation when all search: $\underline{E}(1|\mathbf{1}) > \underline{E}(0|\mathbf{1})$ (proceed similarly).

$$(\underline{y} - b)[\lambda(1) - \lambda(0)]m(\theta(\mathbf{1})) - k(r+\delta) > 0. \tag{7}$$

Putting (1) and (2) together gives the condition in the Lemma.

Multiple Equilibria: Dynamics

- Local stability around SS
- Our model can be reduced to a dynamic system in $\mathbb{R}^3$: $\dot{u}(\Omega), \dot{\gamma}(\Omega), \dot{\theta}(\Omega)$.

$$
\begin{aligned}
\dot{u}(\Omega) &= \delta(1-u) - u m(\theta(\Omega)) \\
\dot{\gamma}(\Omega) &= u m(\theta(\Omega)) - (\delta + \lambda(\Omega) m(\theta(\Omega)))\gamma \\
\dot{\theta}(\Omega) &= \frac{m(\theta(\Omega))u}{(1-\eta(\theta(\Omega)))(u+\lambda(\Omega)\gamma)} \quad \times \left[\frac{\lambda}{u} \left(-\frac{\theta(\Omega)c}{m(\theta(\Omega))} + \bar{J} \right) \left(-\dot{u}\frac{\lambda(\Omega)}{u} + \dot{\gamma} \right) \right. \\
&\quad -(p\underline{y} - \underline{w}(\Omega)) + \left(\frac{c}{q(\theta(\Omega))} \frac{u+\lambda(\Omega)\gamma}{u} - \frac{\lambda(\Omega)\gamma}{u}\bar{J} \right) (r + \delta + \lambda(\Omega) m(\theta(\Omega)
\end{aligned}
$$

Back

Condition for Multiple Equilibria

Back

Necessary and sufficient condition for existence of multiple steady states

$$-\frac{2(\phi\lambda_0+2r)}{4\alpha(\phi\lambda_0+r)}+\overline{y}-\alpha^2 p\phi b+\frac{\sqrt{\alpha^2(-8cr^2(\phi\lambda_0+r)(2cr-\alpha p\phi(\overline{y}-b))+(cr2(\phi\lambda_0+2r)+\alpha p\phi(-(\overline{y}-b)(\phi\lambda_0+}}{4\alpha^2 cr(\phi\lambda_0+r)}$$

$$<\frac{kr}{\alpha\left(\phi\lambda_1(\underline{y}-b)-kr\right)}<-\frac{2(\phi(\lambda_0+\lambda_1)+2r)+kr}{4\alpha(\phi(\lambda_0+\lambda_1)+r)}+\overline{y}- \quad \textbf{(ME)}$$

$$+\frac{\sqrt{\alpha^2(-8cr^2(\phi(\lambda_0+\lambda_1)+r)(2cr-\alpha p\phi(\overline{y}-b-k))+(cr2(\phi(\lambda_0+\lambda_1)+2r)+\alpha p\phi(kr-(\overline{y}-b)(\phi(\lambda_0+\lambda_1)+}}{4\alpha^2 cr(\phi(\lambda_0+\lambda_1)+r)}$$

Condition for Multiple Equilibria

Back

Necessary and sufficient condition for existence of multiple steady states

$$-\frac{2(\phi\lambda_0+2r)}{4\alpha(\phi\lambda_0+r)}+\overline{y}-\alpha^2 p\phi b+\frac{\sqrt{\alpha^2(-8cr^2(\phi\lambda_0+r)(2cr-\alpha p\phi(\overline{y}-b))+(cr2(\phi\lambda_0+2r)+\alpha p\phi(-(\overline{y}-b)(\phi\lambda_0+}}{4\alpha^2 cr(\phi\lambda_0+r)}$$

$$<\frac{kr}{\alpha\left(\phi\lambda_1(\underline{y}-b)-kr\right)}<-\frac{2(\phi(\lambda_0+\lambda_1)+2r)+kr}{4\alpha(\phi(\lambda_0+\lambda_1)+r)}+\overline{y}- \quad \textbf{(ME)}$$

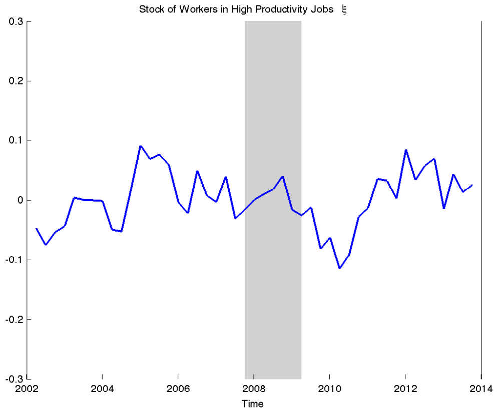
$$+\frac{\sqrt{\alpha^2(-8cr^2(\phi(\lambda_0+\lambda_1)+r)(2cr-\alpha p\phi(\overline{y}-b-k))+(cr2(\phi(\lambda_0+\lambda_1)+2r)+\alpha p\phi(kr-(\overline{y}-b)(\phi(\lambda_0+\lambda_1)+}}{4\alpha^2 cr(\phi(\lambda_0+\lambda_1)+r)}$$

Multiplicity bounds in terms of p:

$$p^l = \frac{2c\lambda_1 r(\underline{y}-b)[k(\lambda_0+\lambda_1)+\lambda_1(\underline{y}-b)]}{\alpha[\lambda_1\phi(\underline{y}-b)-kr][b^2\lambda_1+k(\lambda_0+\lambda_1)\overline{y}+\lambda_1(\overline{y}-k)\underline{y}-b(k\lambda_0+\lambda_1\overline{y}+\lambda_1\underline{y})]}$$

$$p^u = \frac{2c\lambda_1 r(\underline{y}-b)}{\alpha(\overline{y}-b)[\lambda_1\phi(\underline{y}-b)-kr]},$$

Decomposition of EE Flows: ξ



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Beveridge Curves

Steady state flow equations:

$$\begin{aligned} u &= \frac{\delta}{\delta + m(\theta(\Omega))} \\ \gamma &= \frac{\delta m(\theta(\Omega))}{[\delta + m(\theta(\Omega))][\delta + \lambda(\Omega) m(\theta(\Omega))]}. \end{aligned}$$

Beveridge Curves BC and BC^s:

$$\begin{aligned} v &= \frac{\delta u(1-u)[2\lambda(\Omega)(1-u)+u]}{\alpha[u(\delta+\phi)-\delta][\lambda(\Omega)(1-u)+u]} \\ v &= -\frac{(\delta s(2\delta(-1+s)+\phi(\lambda(-2+s)+s-\sqrt{\lambda^2(-2+s)^2+s^2-2\lambda s^2}))}{-2\alpha\delta(\delta+2\lambda\phi)+2\alpha(\delta+\phi)(\delta+\lambda\phi)s} \end{aligned}$$

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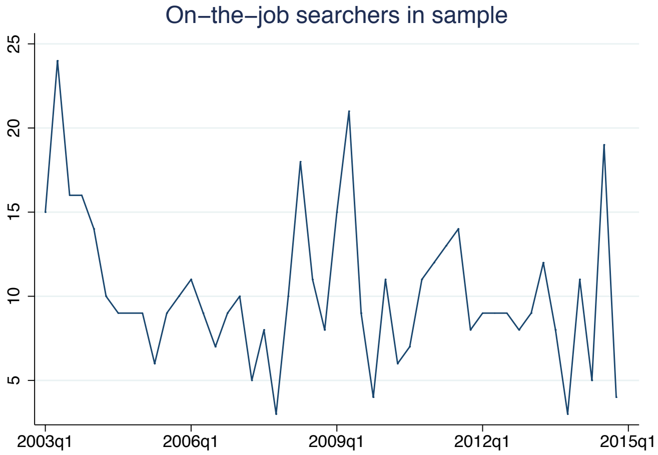
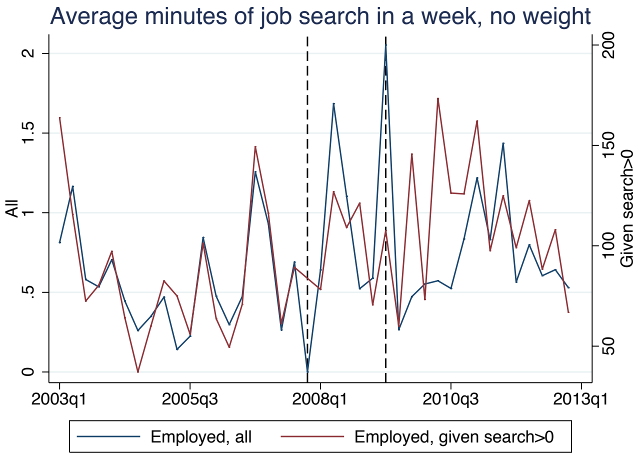
American Time Use Survey

Reporting non-zero search time



American Time Use Survey



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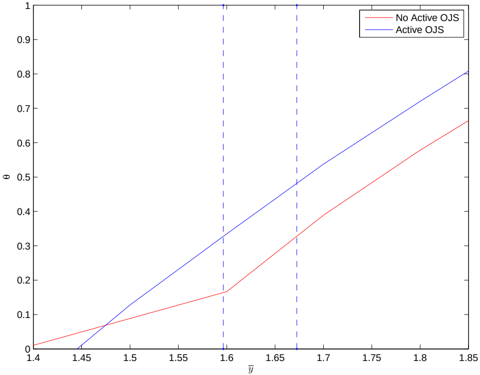
Multiplicity Bounds: $\overline{y}$ ($\underline{y} = 1$)



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Decomposition of EE Flows: $EE = \lambda\gamma m(\theta)$

$$m(\theta) = \frac{UE}{u} \quad \text{and} \quad \lambda\gamma = \frac{EE \cdot u}{UE}$$

Decomposition of EE Flows: $EE = \lambda\gamma m(\theta)$

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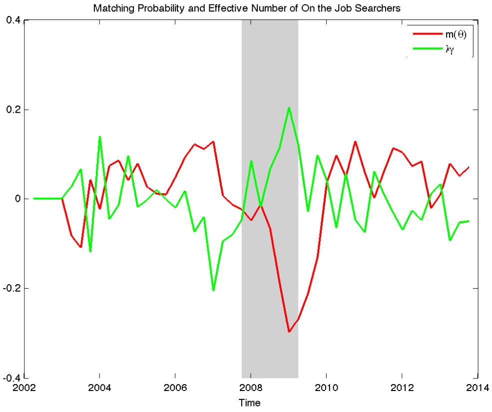
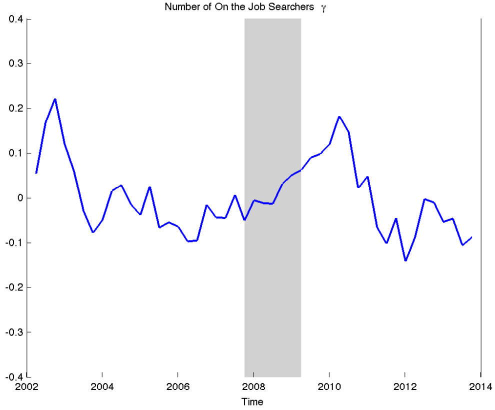


Decomposing $\lambda\gamma$

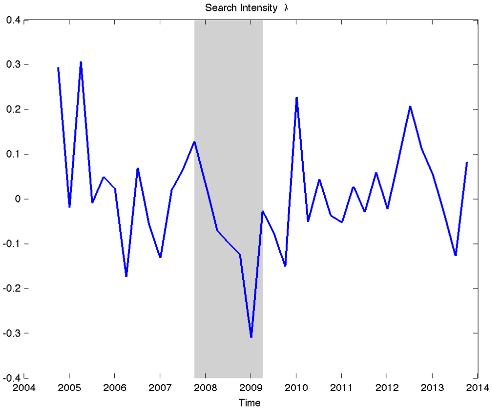
- Problem: No direct measure of search intensity λ
- Use CPS micro-data panel structure
- Check whether individuals was unemployed before current job or transited from another job
- Construct γ (employed after UE transition) and ξ (after EE transition)
- Then, search intensity is computed as: $\lambda = \frac{EE}{m(\theta)\gamma}$

Decomposition of EE Flows: γ



▸ ξ

DECOMPOSITION OF EE FLOWS: $\lambda = \frac{EE}{m(\theta)\gamma}$



$\Rightarrow$ Pro-cyclical search intensity!

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Topics in Labor Markets

Jan Eeckhout

2015–2016

IV. Further Topics:

Matching and Uncertainty:

Stochastic Sorting

Motivation

- Matching problem (Becker 1973) with stochastic types:
  1. match $\rightarrow$ ex ante characteristics x, y
  2. output $\rightarrow$ ex post realizations ω, σ
- Realistic $+$ can confront model with data:
  1. Attributes change
  2. Account for mismatch
  3. Noise is part of model

Examples

$$x \to \omega$$

$$y \to \sigma$$

	x, y	ω, σ
Marriage	x : man's education	ω : income
	y : woman's education	σ : income

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Marriage	x : man's education	ω : income
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Job Market	x : MBA degree	ω : worker productivity
	y : job level/position	σ : realized demand/technology

Examples

$$x \to \omega$$

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	x, y	ω, σ
Marriage	x : man's education	ω : income
	y : woman's education	σ : income
Job Market	x : MBA degree	ω : worker productivity
	y : job level/position	σ : realized demand/technology
Executives	x : past experience	ω : CEO performance
	y : initial market value	σ : stock price change

Application

Mismatched CEOs

- There is randomness CEO compensation + firm performance
- There is Sorting (Gabaix-Landier, Terviö)

$\rightarrow$ And... many CEOs are the wrong (wo)man for the job

Application

Mismatched CEOs

- There is randomness CEO compensation + firm performance
- There is Sorting (Gabaix-Landier, Terviö)

→ And... many CEOs are the wrong (wo)man for the job

- What is role of:
  1. Effort
  2. Sorting
  3. Mismatch

⇒ Estimate the technology and distributions

Related Work

1. Mismatch: confronting matching models with reality
   - Search Frictions: Shimer and Smith (2000)
   - Learning: Anderson-Smith (2011)
   - Matching under uncertainty (Het. pref.): Chiappori-Reny (2005), Legros-Newman (2007) (no mismatch); Chade (2006)
   - Unobserved heterogeneity + multidimensional types: Choo-Siow (2006) Galichon-Salanié (2011), Lindenlaub (2012)
2. CEO compensation
   - "Luck" (noise uncorrelated with effort): Bertrand-Mullainathan
   - Sorting: Gabaix-Landier, Terviö
   - ...

Motivation

Overview of Model Features

- Attributes (types) are stochastic $\rightarrow$ change over time
- Hetereogeneity in endowments
- Traits of matched partners are uncertain when match forms
- Who matches with whom?
  - Matching based on ex ante attributes
  - $\Rightarrow$ Ex ante: no mismatch (Becker)
  - Match value and payoff depend on ex post realization of types
  - $\Rightarrow$ Ex post: mismatch
- No rematching

The Model Setup

General Framework

- Agents

$$\begin{aligned} \text{Workers:} \quad & x \to \omega \sim F(\omega|x) \\ \text{Firms:} \quad & y \to \sigma \sim G(\sigma|y) \end{aligned}$$

$\rightarrow$ joint distribution $K(\omega, \sigma | x, y)$

- Output:

$$q(\omega, \sigma)$$

- Competitive equilibrium/stability/efficient matching $\mu(x)$
- Remark:
  - Special Case: Independence $K(\omega, \sigma | x, y) = F(\omega|x)G(\sigma|y)$
  - Assume continuous variables with K, F, G, q smooth

The Model Setup

General Framework

- Agents

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  - Assume continuous variables with K, F, G, q smooth

Transferable Utility (TU)

- The expected surplus of a match between a type x and y:

$$V(x, y) = \int_{\underline{\omega}}^{\overline{\omega}} \int_{\underline{\sigma}}^{\overline{\sigma}} q(\omega, \sigma) k(\omega, \sigma | x, y) d\omega d\sigma$$

  where k is the density of K
- Determinants of equilibrium allocation:
  1. Complementarity of match output $q(\omega, \sigma)$
  2. Distributions $K(\omega, \sigma | x, y) \rightarrow$ stochastic dominance

Transferable Utility (TU)

Theorem

(i) If K is supermodular (submodular) in (x, y), then

PAM (NAM) if q is supermodular (submodular) in (ω, σ)

(ii) If $\int_{\underline{\omega}}^{\omega} \int_{\underline{\sigma}}^{\sigma} K$ is supermodular (submodular) in (x, y), then

PAM (NAM) if $q_{\omega\sigma}$ is supermodular (submodular) in (ω, σ)

→ Condition on q is necessary if result to hold *for all K*

→ Proof: applying integration by parts iteratively sketch proof

Transferable Utility (TU)

- Special case: cond. independence: $K = F(\omega|x)G(\sigma|y)$ and FOSD ($F_x < 0$, $G_y < 0$)
  - If F and G degenerate, then we recover Becker
  - $cov\{\omega, \sigma\}$ positive under PAM and negative under NAM
- If $K(\omega, \sigma|x, y)$ log-supermodular in (ω, σ, x, y) then
  - $K(\omega, \sigma)$ log-supermodular
  - $K(\omega|\sigma)$ and $K(\sigma|\omega)$ FOSD
  - $\Rightarrow$ Stochastic notion of PAM in (ω, σ): higher $\omega \rightarrow$ higher σ

Transferable Utility (TU)

Some Observations

- TU: simple and tractable
- But: ex post payoffs not pinned down ($\exists$ continuum of splits)
- Most applications: information on ex post payoffs
- Non-linear preferences: pins down ex post payoffs
    1. Risk Sharing
    2. Contracting under moral hazard

Non Transferable Utility (NTU)

Risk Sharing

- Stochastic characteristics $\Rightarrow$ Uncertainty $\Rightarrow$ Risk sharing

$$
\begin{aligned}
\Phi(x, y, v) \quad &= \quad \max_{c_x, c_y} \int_{\underline{\omega}}^{\overline{\omega}} \int_{\underline{\sigma}}^{\overline{\sigma}} u(c_y(\omega, \sigma)) k(\omega, \sigma | x, y) d\omega d\sigma \\
\text{s.t.} \quad & \quad c_x(\omega, \sigma) + c_y(\omega, \sigma) = q(\omega, \sigma) \;\; \forall \;\; (\omega, \sigma) \\
& \quad \int_{\underline{\omega}}^{\overline{\omega}} \int_{\underline{\sigma}}^{\overline{\sigma}} u(c_x(\omega, \sigma)) k(\omega, \sigma | x, y) d\omega d\sigma \geq v
\end{aligned}
$$

- Pins down consumption and thus ex post payoffs

Non Transferable Utility (NTU)

Risk Sharing

- NTU matching problem $\Rightarrow$ Legros and Newman (2007)
- PAM (NAM) $\Leftrightarrow$ Generalized Increasing (Decr.) Differences
- Differential version of their condition (Spence-Mirrlees):
  - PAM if and only if

$$\Phi_{xy} > \frac{\Phi_x}{\Phi_v} \Phi_{vy}$$

- Focus on $K(\omega, \sigma | x, y) = F(\omega | x) G(\sigma | y)$ and FOSD

Non Transferable Utility (NTU)

Overview of Main Results

- Sorting pattern only depends on q, not on distributions
  - if one side is risk neutral (e.g. firm); or
  - both sides have CARA preferences

  $\rightarrow$ PAM if $q_{\omega\sigma} > 0$

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- If u is HARA (CRRA, log, CARA, quadratic,...)

  $\rightarrow$ PAM if $\hat{q}_{\omega\sigma} > 0$ where $\hat{q}$ is a transformation of q
  e.g. CRRA: $u = \frac{c^\alpha}{\alpha}$ then $\hat{q} = \frac{q^\alpha}{\alpha(1-\alpha)^{1-\alpha}}$

Non Transferable Utility (NTU)

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  $\rightarrow$ PAM if $\hat{q}_{\omega\sigma} > 0$ where $\hat{q}$ is a transformation of q
  e.g. CRRA: $u = \frac{c^\alpha}{\alpha}$ then $\hat{q} = \frac{q^\alpha}{\alpha(1-\alpha)^{1-\alpha}}$
- If u is HARA (with DARA), $h(\omega, \sigma) = \omega + \sigma$, and $F(\omega|x) = F\left(\frac{\omega-x}{s}\right)$ and $G(\sigma|y) = G\left(\frac{\sigma-y}{t}\right)$

  $\rightarrow$ optimal sorting is NAM (riskiness constant $\Rightarrow$ insurance driven by DARA and income effect: match high with low)

Application: Mismatched CEOs

- NTU $\Rightarrow$ pins down ex post payoffs
- Executives match with firms
- Key assumptions: only new hires
  1. Frictionless matching
  2. Moral hazard within a match
  3. No rematching, no separation
- Variation Holmström-Milgrom linear contracting model

Mismatched CEOs

Holmström-Milgrom with Matching

- Large number of risk averse CEOs, risk neutral firms
- Linear contracting model
- CEO-firm pair (x, y) match. Timing:
  1. Firm offers output-contingent (q) contract
  2. CEO type ω realized (public); CEO chooses effort e
  3. Firm type σ realized (not observed) $\rightarrow$ output q observed
  4. Payments as specified in the contract
- Output: $q = m(\omega, y)(e + \sigma)$ where $\omega, \sigma \sim \mathcal{N}$, $m_\omega > 0, m_y > 0$
- Linear contracts (α, β): $w(q, \omega) = \beta(\omega) + \alpha(\omega)q$
- CEO: CARA preferences $-e^{-r\left(w - \frac{e^2}{2}\right)}$; Reservation wage $a(x)$

Mismatched CEOs

Optimal Contracting Problem

- Principal's problem is (where β, α, e depend on ω):

$$\max_{\beta,\alpha,e} \quad \int \left(\mathbb{E}[q|e] - (\beta + \alpha\mathbb{E}[q|e])\right) dF(\omega|x)$$

$$\text{s.t.} \quad \int \left(\mathbb{E}\left[-e^{-r\left(\beta+\alpha q-\frac{e^2}{2}\right)}\right]\right) dF(\omega|x) \geq -e^{-ra} \quad \textbf{(PC)}$$

$$e \in \arg\max_{\hat{e}} \int -e^{-r\left(\beta+\alpha q-\frac{\hat{e}^2}{2}\right)} dG(\sigma|y), \forall\omega \quad \textbf{(IC)}$$

  where $q = q(\omega, \sigma, y), \alpha(\omega), \beta(\omega), e(\omega)$

- **Remark:** (**PC**) is ex ante, before ω is revealed, while (**IC**) must hold for each realization of ω

Mismatched CEOs

Sketch Derivation and Optimal Contract

- (**IC**) $\Rightarrow \alpha(\omega) = e(\omega)/m(\omega, y)$ for all ω
- Insert into objective function and (**PC**)
- Optimal Contract $(\alpha(\cdot), \beta(\cdot), e(\cdot))$ is

$$\alpha(\omega) = \frac{1}{1 + rs^2(y)}$$

$$\beta(\omega) = a(x) - \frac{m(\omega, y)t(y)}{1 + rs^2(y)} + \frac{m^2(\omega, y)}{2\left(1 + rs^2(y)\right)^2}\left(rs^2(y) - 1\right)$$

$$e(\omega) = \frac{m(\omega, y)}{1 + rs^2(y)}$$

Mismatched CEOs

- Equilibrium:

$$
\begin{aligned}
w &= a + \frac{m^2}{2\left(1+rs^2\right)} + \frac{m}{1+rs^2}\sigma \\
\pi &= mt - a + \frac{m^2}{2\left(1+rs^2\right)} + \frac{rs^2}{1+rs^2}m\sigma \\
q &= \frac{m^2}{1+rs^2} + m(t+\sigma)
\end{aligned}
$$

- Ex ante Match Surplus:

$$
\begin{aligned}
V(x,y) &= \int\int q(\omega,\sigma,x,y)dF(\omega|x)dG(\sigma|y) \\
&= mkt + \frac{m^2(k^2+u^2)}{1+rs^2}
\end{aligned}
$$

Mismatched CEOs

Endogenous Outside Option $a(x)$

- Ex post wages w are pinned down by the optimal incentive contract (above)
- Ex ante compensation determines $a(x)$
- From FOC:

$$\max_x V(x,y) - a(x) \quad \Rightarrow \quad a'(x) = V_x(x,x)$$

and therefore $a(x) = a(\underline{x}) + \int_{\underline{x}}^{x} V_x(\tau,\tau)d\tau$ or:

$$a(\underline{x}) + \int_{\underline{x}}^{x} \left(m(\tau)k'(\tau)t(\tau) + \frac{m(\tau)^2(2k(\tau)k'(\tau) + 2u(\tau)u'(\tau))}{1 + rs(\tau)^2} \right) d\tau$$

where $a(\underline{x}) \in [0, V(\underline{x},\underline{x})]$.

Mismatched CEOs

- Match Value is separable $\Rightarrow$ PAM $\iff V_{xy} > 0$

$$V(x, y, \overline{v}) = \int \int \left(\frac{m^2}{1 + rs^2} + m(t + \sigma) \right) dFdG - \frac{1}{r} \log(-\overline{v}(x))$$

$\rightarrow$ from CARA, quadratic cost, normal distribution

Mismatched CEOs

Empirical Exercise

- Work in progress!!
- What do we want to do with the model?

1. Use US data CEO compensation and firm profits to estimate:
   - Match value function
   - CEO and firm type distributions
2. Quantify mismatch in market for CEOs
3. Decompose value loss due to mismatch
   - Forgone complementarities
   - Changes in effort (incentives)

Mismatched CEOs

Data

- Data sources:
  - Wages: Execucomp (Compustat) – total compensation: TDC1
  - Profits: Compustat: change in MkVal
- Constructing the variables:
  1. Newly hired 2010 (4 separations, 53 missing obs.)
  2. Rank firms by 2010 market value: $y \sim U[0, 1]$
  3. Rank workers: $x = y$
  4. w: TDC1(2011)+TDC1(2012)
  5. π: MkVal(2012)-MkVal(2010)

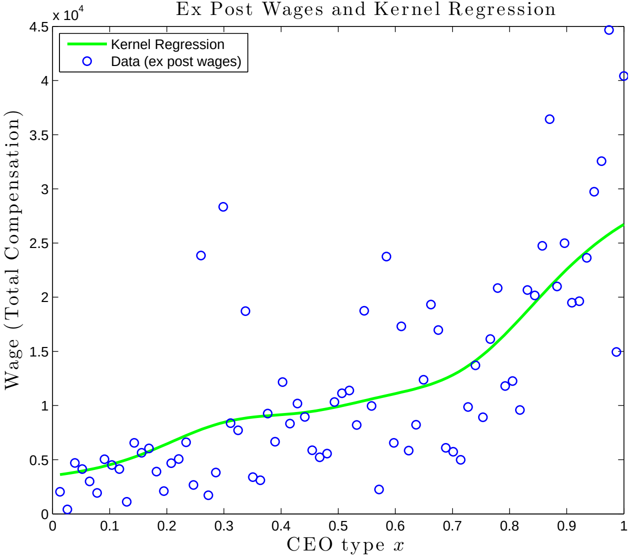
Mismatched CEOs

Top 10 companies in sample

	Company name	Market Cap. 2010 (billions)
1	Chevron	183
2	Bank of America	134
3	United Technologies	72
4	Caterpillar	59
5	Bristol-Myers Squibb	45
6	Morgan Stanley	41
7	Mastercard	29
8	Celgene	27
9	State Street	23
10	Transocean	22

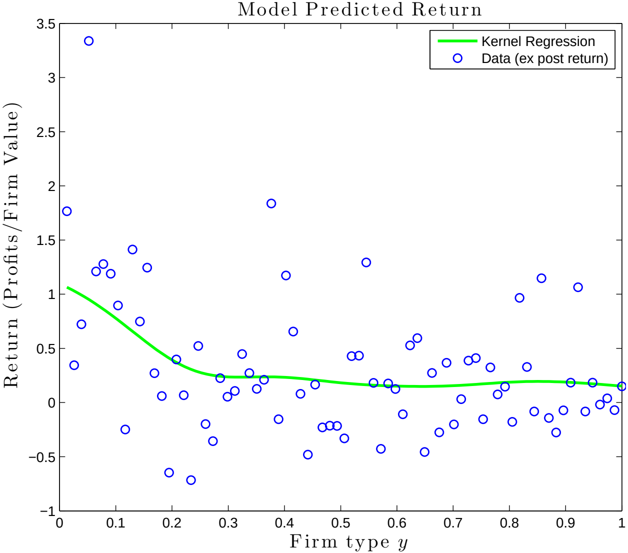
Mismatched CEOs

Wages



Mismatched CEOs

Profits (Return)



- Let

$$\begin{aligned} F(\omega|x) &= \mathcal{N}\left(k(x), u(x)^2\right) \\ G(\sigma|x) &= \mathcal{N}\left(0, s(x)^2\right). \end{aligned}$$

and $m = y\omega$

Mismatched CEOs

Implementing the Model

$$
\begin{aligned}
\mathbb{E}w(x) &= a(x) + \frac{x^2(k^2+u^2)}{2(1+rs^2)} \\
\mathbb{E}\pi(x) &= -a(x) + xkt + \frac{x^2(k^2+u^2)}{2(1+rs^2)} \\
\text{Var } w(x) &= \frac{x^2}{2(1+rs^2)^2}\left[x^2(u^4+2u^2k^2) + 2s^2(k^2+u^2)\right] \\
\text{Var } \pi(y) &= x^2t^2u^2 + \frac{x^2}{4(1+rs^2)^2}\left[2x^2(2k^2u^2+u^4)\right. \\
&\quad \left. +4r^2s^6(k^2+u^2) + 8xtku^2(1+rs^2)\right]
\end{aligned}
$$

- Solve explicitly for k, u, t, s for each x from the theory
- $a(x)$ is obtained recursively starting from exogenous $a(\underline{x})$
- Only one observation for each $x \Rightarrow$ use kernel(s) to obtain $\mathbb{E}w(x), \mathbb{E}\pi(x), \text{Var } w(x), \text{Var } \pi(y)$

Mismatched CEOs

Estimation: In Progress

1. Obtain estimated values for $k(x), u(x), t(y), s(y)$
2. Calculate $V(x, y)$ and verify $V_{xy}(x, y) > 0$
3. Properties of $q(\omega, \sigma)$ and $F(\omega|x), G(\sigma|y)$
   - Ex post complementarities $q_{\omega\sigma}$?
   - Stochastic Order on the distributions F_x, G_y?
4. How big is the mismatch? How much due to CEOs, how much due to firm noise?
5. Use ex post types ω to conduct counterfactual experiment by reassigning CEOs to ex post optimal match
   - What is output loss due to mismatch?
   - Decompose mismatch
     1. due to inefficient effort provision
     2. due to misallocation

Concluding Remarks

- Stochastic Sorting: Becker with realistic types
- Appealing:
  1. Characteristics change
  2. Mismatch in data
  3. “Noise” is integral part

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- Application: Holmström-Milgrom optimal contr. + matching
  Preliminary Results:
  1. CEOs are mismatched
     - Types are not very predictable
     - Strong ex post complementarity
  2. Huge Loss as a share of Market Value

  $\rightarrow$ Driven by mismatch, not by changes in effort provision

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  → Driven by mismatch, not by changes in effort provision

∴ **Focus on selection, rather than incentives**

Topics in Labor Markets

Jan Eeckhout

2015–2016

IV. Further Topics: Matching with Externalities: Competing Teams

The Problem

- We analyze assortative matching with externalities
  - In standard model $\longrightarrow$ match output depends only on the characteristics of the pair that matches
  - In our setup $\longrightarrow$ match output depends also on matching
- Natural extension of Becker (1973) $\longrightarrow$ Many applications
  - R&D competition
  - Oligopoly
  - Auctions
- Competing teams
  - Optimal and equilibrium matching
  - Inefficiency
  - Policy
- Related literature:
  - Small (to the best of our knowledge): Koopmans and Beckmann (1957); Sasaki and Toda (1996)

The Setup

Overview of the model:

- Large number of heterogeneous workers (and firms)
- Two stages:
  - Matching stage: Workers form teams of size two (or firms hire them) in a competitive labor market
  - Competition stage: Teams compete pairwise in output market
- Second stage induces matching with externalities in first stage
  - Match payoff of a team depends on composition of other teams
- Analysis of sorting patterns:
  - Planner v. Competitive Market
  - Wedge between them due to externalities

The Setup

- Continuum of agents
- Each has a characteristic ('type') $x \in \{\underline{x}, \overline{x}\}$, $\overline{x} > \underline{x}$
- Workers form teams of size 2
    - $\overline{X}$: team with two $\overline{x}$-type agents
    - $\underline{X}$: team with two $\underline{x}$-type agents
    - $\hat{X}$: team with one $\underline{x}$ and one $\overline{x}$-type agents
    - $\underline{X} < \hat{X} < \overline{X}$
- Transferable utility
- Matching μ partitions population in pairs:
    - PAM μ_+: half of the teams are $\overline{X}$ and half $\underline{X}$
    - NAM μ_-: all the teams are $\hat{X}$

The Setup

- Teams compete pairwise in downstream interaction (e.g., output market) against a randomly drawn team
  - $V(X_i|X_j)$: match output of team X_i when competing with X_j
  - V symmetric in components of X_i, and similarly in components of X_j
  - $\mathcal{V}(X_i|\mu_+) = \mathbb{E}_{\mu_+}[V(X_i|\tilde{X}_j)] = \frac{1}{2}V(X_i||\overline{X}) + \frac{1}{2}V(X_i|\underline{X})$
  - $\mathcal{V}(X_i|\mu_-) = \mathbb{E}_{\mu_-}[V(X_i|\tilde{X}_j)] = V(X_i|\hat{X})$

The Setup

An example of $V(X_i|X_j)$:

- Research: uncertainty about the exact outcome v_i
  1. Form R&D teams
  2. Draw uncertain research output v_i:
     - $v_i \in \{0, v\}$
     - probability to get v given team composition X_i: $p_i = p(X_i)$ (with $\overline{p} > \hat{p} > \underline{p}$)
  3. Winner takes all: $\max\{v_i, v_j\}$ (half in case of a tie)
- Expected payoff:

$$V(X_i|X_j) = p_i p_j \frac{v}{2} + p_i(1 - p_j)v = vp_i - \frac{v}{2} p_i p_j$$

$$\Rightarrow \quad \text{e.g. } V(\overline{X}|\underline{X}) = v\overline{p} - \frac{v}{2}\overline{p}\underline{p} \quad \text{and} \quad V(\overline{X}|\overline{X}) = v\overline{p} - \frac{v}{2}\overline{p}\overline{p}$$

$$\Rightarrow \quad \mathcal{V}(\overline{X}|\mu_+) = +\frac{1}{2}\left(v\overline{p} - \frac{v}{2}\overline{p}\underline{p}\right) + \frac{1}{2}\left(v\overline{p} - \frac{v}{2}\overline{p}^2\right)$$

The Setup

- **Planner:** Takes as given output market competition and chooses μ that maximizes sum of teams' outputs
  - PAM optimal if
  $$\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) \geq 2\mathcal{V}(\hat{X}|\mu_-)$$
  - NAM optimal if
  $$\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) \leq 2\mathcal{V}(\hat{X}|\mu_-)$$
  - Reduce to super or submodularity without externalities
  $$\mathcal{V}(\overline{X}) + \mathcal{V}(\underline{X}) \quad \text{v.} \quad 2\mathcal{V}(\hat{X})$$

The Setup

- **Competitive Equilibrium:** Agents take market wages and matching as given when they choose partners
  - Textbook notion; large market assumption justifies belief that they do not affect the allocation
  - $(\underline{w}, \overline{w}, \mu)$ such that (i) each type maximizes his payoff given wages; and (ii) choices are consistent with μ (market clearing)
  - PAM if

$$\begin{aligned} \mathcal{V}(\overline{X}|\mu_+) - \overline{w} &\geq \mathcal{V}(\hat{X}|\mu_+) - \underline{w} \\ \mathcal{V}(\underline{X}|\mu_+) - \underline{w} &\geq \mathcal{V}(\hat{X}|\mu_+) - \overline{w} \end{aligned}$$

  - This implies $\mathcal{V}(\cdot|\mu_+)$ supermodular, or

$$\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) \geq 2\mathcal{V}(\hat{X}|\mu_+)$$

  - Wages given by $\overline{w} = 0.5\mathcal{V}(\overline{X}|\mu_+)$ and $\underline{w} = 0.5\mathcal{V}(\underline{X}|\mu_+)$
  - Analogous construction for NAM
  - Reduces to super or submodularity without externalities
  - Two interpretations: partnerships, firms hiring teams

Sorting and Inefficiency

Proposition

There is an equilibrium with PAM allocation while there is NAM in the planner's solution if and only if

(i) $\mathcal{V}(X|\mu_+)$ *supermodular in* X*;*

(ii) $\mathcal{V}(\overline{X}|\mu_+) + \mathcal{V}(\underline{X}|\mu_+) - 2\mathcal{V}(\hat{X}|\mu_+) \leq 2[\mathcal{V}(\hat{X}|\mu_-) - \mathcal{V}(\hat{X}|\mu_+)]$

- Intuition:
  - "Supermodularity" (modified)
  - *Differential* externality NAM outweighs "supermodularity"
- Conditions for uniqueness
- Similar conditions for NAM equilibrium, PAM planner
  - Replace (i) by submodular $\mathcal{V}(X|\mu_-)$; reverse inequality in (ii)

Sorting and Inefficiency

- Additively Separable Payoffs
  - $\mathcal{V}(X_i|\mu) = g(X_i) + h(\mu)$
  - $h(\mu_+) = \frac{1}{2}h(\overline{X}) + \frac{1}{2}h(\underline{X})$ and $h(\mu_-) = h(\hat{X})$
  - PAM (NAM) equilibrium and NAM (PAM) planner iff

$$g \;\; \text{supermodular (submodular)}$$

$$g(\overline{X}) + g(\underline{X}) - 2g(\hat{X}) \leq (\geq) 2[h(\mu_-) - h(\mu_+)]$$

- Multiplicatively Separable Payoffs
  - $\mathcal{V}(X_i|\mu) = g(X_i)h(\mu)$
  - PAM (NAM) equilibrium and NAM (PAM) planner iff

$$g \;\; \text{supermodular (submodular)}$$

$$g(\overline{X}) + g(\underline{X}) - 2g(\hat{X}) \leq (\geq) 2g(\hat{X})\frac{h(\mu_-) - h(\mu_+)}{h(\mu_+)}$$

  - Need h 'sufficiently submodular' in X

Sorting and Inefficiency

We can also provide sufficient conditions in terms of V:

- PAM equilibrium and NAM planner if
  - $V(X|\overline{X}) + V(X|\underline{X})$ supermodular in X
  - $V(X_i|X_j)$ supermodular in (X_i, X_j)
  - $V(X|X)$ concave in X
- NAM equilibrium and PAM planner if
  - $V(X|\hat{X})$ submodular in X
  - $V(X_i|X_j)$ submodular in (X_i, X_j)
  - $V(X|X)$ convex in X
- Interpretation of NAM equilibrium and PAM planner:
  - Competition 'strategic substitutes' $\Rightarrow$ V submodular in (X_i, X_j)
  - PAM planner (with convexity condition)
  - Submodular in X_i $\Rightarrow$ NAM equilibrium (firms do not internalize externalitities)

Uncertainty

- Many economic environments involve uncertainty
- Patent race between research teams; Knowledge spillovers; Auctions between competing teams; Sports competitions;...
- Important for estimation
- Set up:
  1. Team composition X_i: labor market competition
  2. Team generates stochastic product v_i, from $F(v_i|X_i)$
  3. Output market competition $z(v_i, v_j)$
- Expected output of team X_i:

$$V(X_i|X_j) = \int\int z(v_i, v_j) dF(v_i|X_i) dF(v_j|X_j)$$

Uncertainty

- The value is 'additively separable' as follows:

$$V(X_i|X_j) = g(X_i) + h(X_j) + k(X_i, X_j).$$

Proposition

Let $S_i = S(v|X_i) = 1 - F(v|X_i)$ denote the survival function. The expected value $V(X_i|X_j)$ can be written as

$$\underbrace{z(\underline{v}, \underline{v}) + \int \frac{\partial z(v_i, \underline{v})}{\partial i} S_i dv_i + \int 2 \frac{\partial z(\underline{v}, v_j)}{\partial j} S_i dv_j}_{g(X_i)} + \underbrace{\int \frac{\partial z(\underline{v}, v_j)}{\partial j} S_j dv_j}_{h(X_j)} + \underbrace{\int \int \frac{\partial^2 z}{\partial i \partial j} S_i S_j dv_i dv_j}_{k(X_i, X_j)}$$

- The expressions for $\mathcal{V}(\cdot|\mu_+)$ and $\mathcal{V}(\cdot|\mu_-)$ easily follow from V

Corollary

Let $z(v_i, v_j) = av_i + bv_j + cv_iv_j$ *where* a, b, c *are constants and* $\underline{v} = 0$*. Then the value of the firm can be written as*

$$V_i = (a + 2b)m(X_i) + bm(X_j) + cm(X_i)m(X_j),$$

where $m(X) = \mathbb{E}[v|X]$*.*

Uncertainty

Corollary

Let $z(v_i, v_j) = av_i + bv_j + cv_iv_j$ *where* a, b, c *are constants and* $\underline{v} = 0$*. Then the value of the firm can be written as*

$$V_i = (a + 2b)m(X_i) + bm(X_j) + cm(X_i)m(X_j),$$

where $m(X) = \mathbb{E}[v|X]$*.*

- From $\int S_i dvi = \int [1 - F(v|X_i)]dv = \mathbb{E}[\tilde{v}|X_i]$
- Value only depends only on mean
- It easily follows that

$$\begin{aligned} \mathcal{V}(X_i|\mu_+) &= (a + 2b)m(X_i) + \frac{1}{2}(b + cm(X_i))\left(m(\overline{X}) + m(\underline{X})\right) \\ \mathcal{V}(X_i|\mu_-) &= (a + 2b)m(X_i) + (b + cm(X_i))m(\hat{X}) \end{aligned}$$

Economic Applications

I Spillovers
II Patent Race
III Auctions between Teams
IV Oligopolistic Competition

I. Spillovers

- Spillovers can be positive or negative
  - Positive: Development of a product by a firm helps another firm when developing a competing product
  - Negative: Development of a product by a firm adversely affects prospects of the other firm
- Assume $z(v_i, v_j) = v_0 + av_i + bv_j$, $a \geq 0$, $v_0 > 0$ large
- Assume $m(X) \geq 0$ for all X
- Then $V(X_i|X_j)$ is given by

$$V(X_i|X_j) = v_0 + (a + 2b)m(X_i) + bm(X_j)$$

I. Spillovers

Proposition

Let $z = v_0 + av_i + bv_j$, with $a \geq 0$.

1. *If $b \notin \left(-\frac{a}{3}, -\frac{a}{2}\right)$, the equilibrium allocation is efficient;*
2. *If $b \in \left(-\frac{a}{3}, -\frac{a}{2}\right)$, the equilibrium is inefficient: if m is supermodular (submodular), the equilibrium exhibits PAM (NAM), while the planner's solution exhibits NAM (PAM).*

- Positive spillovers always yield efficiency
  - Positive externality cannot offset private benefits
- Inefficiency can arise with negative spillovers
  - It occurs when b is in a range where private benefit parameter a is not large enough
  - Hence externality can dominate private benefit effect

I. Spillovers

- 'Romer-Lucas-like' setup
- Output: $A(\mu)g(X)$ where $A(\mu) = A(\sum g)$
- Inefficiency:
  - PAM equilibrium: $A(\overline{g} + \underline{g})(\overline{g} + \underline{g} - 2\hat{g}) > 0$
  - NAM planner: $A(\overline{g} + \underline{g})(\overline{g} + \underline{g}) < A(2\hat{g})2\hat{g}$

$\Rightarrow$ whenever g supermodular and $A(x)x$ is decreasing, or $A'(x) < -\frac{A(x)}{x}$

- Analogous conditions for PAM planner, NAM equilibrium

II. Patent Race

- Interesting application of negative spillovers
- Research: uncertainty about the exact outcome v_i
- A simple stochastic setting:
  1. Form teams X_i and X_j
  2. Draw uncertain research output v_i:
     - $v_i \in \{0, v\}$
     - probability to get v given X_i: $p_i = p(X_i)$ (with $\overline{p} > \hat{p} > \underline{p}$)
  3. Winner takes all: $\max\{v_i, v_j\}$
- Expected payoff:

$$V(X_i|X_j) = vp_i - \frac{v}{2}p_i p_j$$

- Planner maximizes $[1 - (1 - p_i)(1 - p_j)]v$

II. Patent Race

Proposition

Equilibrium is efficient. The allocation has PAM if p is supermodular, NAM if p is submodular.

- Depends on large market assumption
  - Random matching with opponents in a large market
  - External effect of meeting a high type team is negative
  - External effect of meeting a low type team is positive
  - These effects cancel out
- Inefficiency can arise in small markets (known opponent)

III. Auctions between Teams

- Team composition matters in auction: better estimates of value/cost of timber; make efficient use of bandwidth;...
- Uncertainty about outcomes: team-dependent
- Consider independent private values second price auction
- Order of events
  1. Teams are formed in a competitive labor market
  2. Valuation v_i from distribution of valuations $F(v_i|X_i)$
  3. Random pairwise matching of teams
  4. The two teams simultaneously submit their bids
- As usual, it is a dominant strategy for each bidder to submit a bid equal to the true valuation
- Large market with anonymous participants: e.g., eBay, telephone auctions, etc.

III. Auctions between Teams

- The value of an auction to team X_i when facing X_j is

$$V(X_i|X_j) = \int_{\underline{v}}^{\overline{v}} F(v|X_j)(1 - F(v|X_i)dv$$

III. Auctions between Teams

- The value of an auction to team X_i when facing X_j is

$$V(X_i|X_j) = \int_{\underline{v}}^{\overline{v}} F(v|X_j)(1 - F(v|X_i)dv$$

- Follows from

$$\begin{aligned}
V_i &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \max\{v_i - v_j, 0\} dF(v_i|X_i) dF(v_j|X_j) \\
&= \int_{\underline{v}}^{\overline{v}} \left(1 - v_j F_i(v_j) - \int_{v_j}^{\overline{v}} F_i dv_i - v_j(1 - F_i(v_j) \right) dF_j \\
&= \int_{\underline{v}}^{\overline{v}} \left(\int_{v_j}^{\overline{v}} (1 - F_i) dv_i \right) dF_j = \int_{\underline{v}}^{\overline{v}} n(v_j|X_i) dF_j \\
&= n(v_j|X_i) F_j(v_j)|_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} F_j n'(v_j|X_i) dv_j = \int_{\underline{v}}^{\overline{v}} F_j(1 - F_i) dv_j
\end{aligned}$$

where $n(v_j|X_i) = \int_{v_j}^{\overline{v}} (1 - F_i) dv_i$

III. Auctions between Teams

- It easily follows from V that

PAM $$\mathcal{V}(X_i|\mu_+) = \int_{\underline{v}}^{\overline{v}} \frac{F(v|\overline{X}) + F(v|\underline{X})}{2}(1 - F(v|X_i))dv$$

NAM $$\mathcal{V}(X_i|\mu_-) = \int_{\underline{v}}^{\overline{v}} F(v|\hat{X})(1 - F(v|X_i))dv$$

III. Auctions between Teams

Proposition

The equilibrium allocation is PAM while planner's solution is NAM if F is submodular in X for each v and

$$\int_{\underline{v}}^{\overline{v}} \mathcal{F}(1-\mathcal{F}) \leq \int_{\underline{v}}^{\overline{v}} \hat{F}(1-\hat{F})$$

where $\mathcal{F} = \frac{\overline{F}+\underline{F}}{2}$.

- F submodular: PAM equilibrium
- The expected value of $F(1-F)$ under NAM dominates PAM
- $\int_{\underline{v}}^{\overline{v}} F(1-F)dv = \mathbb{E}_{F^2}[v|X] - \mathbb{E}[v|X]$ larger under NAM than PAM. For example: same mean but $\hat{F}$ has higher variance

IV. Oligopolistic Competition

- Cournot duopoly with linear demand $P = a - bQ$.

$$q_i = \frac{a - 2c_i + c_j}{3b} \quad \text{and} \quad V_i = \frac{(a - 2c_i + c_j)^2}{9b}$$

- Costs depend on team composition $c_i = c(X_i)$ with $\overline{c} < \hat{c} < \underline{c}$

Proposition

If c is supermodular, there is an interval of a, $\underline{x}$, and $\overline{x}$, such that the equilibrium is NAM while the planner is PAM. Equilibrium is efficient if c is submodular or the planner's allocation is NAM.

- Only inefficiency: planner PAM, equilibrium NAM.
  - This occurs when c is supermodular
  - Set of $\underline{x}$ and $\overline{x}$ limits extent of complementarities
  - Intermediate levels of a: if very low enough, externality not strong enough to overturn the NAM equilibrium; if very high profits and the planner's objective are aligned
- We have results for Bertrand and consumer surplus

Policy Implications

- Sports competitions: US vs. Europe
  - US: intervention for balanced competition: PAM $\rightarrow$ NAM
  - Europe: laissez-faire: PAM
- We use the model with negative spillovers $z_i = v_0 + av_i + bv_j$
- Need to calculate wages
- Effects of policies:
  1. Taxes
     - Suitable taxes for hiring same type changes PAM to NAM
  2. Salary Cap
     - Bound on wage of high type cannot change PAM to NAM
  3. Rookie Draft
     - Senior and rookie high and low types
     - Sequential hiring at set type dependent wages
     - Low type seniors choose first
     - Equilibrium with NAM
     - Both senior types prefer it to PAM

Variations

We check the robustness of the results along three dimensions:

- Continuum of types
  - Example with uniformly distributed types on the unit interval and supermodular V
  - Derive conditions for NAM planner/PAM equilibrium
- 'Mixed matching'
  - With externalities, planner may want to match a fraction α as PAM and $1-\alpha$ as NAM
  - Not true without externalities
  - $\alpha = 1$ or 0 if planner's objective function is convex in α
  - We provide sufficient conditions, met in all of our applications
- Small markets
  - Analogous results for small number of agents
  - They take as given the allocation in a competitive equilibrium
  - Planner has similar conditions for PAM/NAM as well

Conclusion

- Assortative matching with externalities
  - Difficult problem in general (Koopmans and Beckmann (1957))
  - We analyze a tractable framework
- Competing Teams
  - Allocation problems with externalities/strategic interaction
  - If inefficient: discontinuous reallocation
- Complementarities in allocation problems:
  - Without externalities: correctly priced

  $\rightarrow$ no efficiency grounds for intervention
  - With externalities

  $\rightarrow$ role for intervention
- Extensions:
  - More than two types: Interesting mathematical problem
  - Stability and core

Topics in Labor Markets

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