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INEQUALITY

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ABSTRACT

In a growth model, rent-grabbing and free riding can give rise to inequality in productivity and firm size. Inequality among firms affects a firm's incentive to free ride or to grab rents, and, hence, the incentive to invest in research and training

We follow Lucas and Prescott (1971) and Hayashi (1982) and assume constant returns in production and in adjustment costs for investment, and perfect capital markets. Our conclusion, however, differs starkly from theirs: Average Tobin's q generally *exceeds* marginal q. That is, the unit value of capital is lower in big firms, and evidence dating back to Fazzari, Hubbard, and Petersen (1988) supports this claim quite decisively. Such evidence is usually taken to imply that small firms invest at a rate lower than its perfect capital market rate. In our model, however, it arises because small firms rely more on copying than big firms do: The marginal product of capital is equal across firms, but its average product is higher than that because small firms get a disproportionately high external benefit.

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Abstract

In a growth model we show that "rent-grabbing" and "free riding" aspects of the investment process can give rise to substantial inequality in productivity and firm size. Moreover, small firms have larger Tobin's q's than big firms do even though returns to scale are constant, and capital markets are perfect.

1 Introduction

We study a one-sector growth model with convex capital-adjustment costs. The nonstandard feature is that the entire distribution of capital – and not just its average or its maximum – enters the production function "externally". Such external effects should arise among agents that invest in learning, but they also can arise, in a reduced form sense, in situations like patent-races that may not involve external effects among the investors.

We get two main results. First, when the incentive to grab rents or to free ride is strong enough, inequality is the only long-run outcome. We analyze long-run growth and the distribution of incomes in a series of examples in which markets for knowledge do not exist and in which one cannot exclude others from using one's own knowledge. Inequality affects the incentive to invest, and the distribution of capital that induces everyone invests at the same rate is the long run equilibrium distribution.

We follow Lucas and Prescott (1971) and Hayashi (1982) and assume constant returns in production and in adjustment costs for investment, and perfect capital markets. The conclusion, however, is starkly different: Average Tobin's q generally *exceeds* marginal q. That is, the unit value of capital is lower in big firms, and evidence dating back to Fazzari, Hubbard, and Petersen (1988) supports this claim

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quite decisively. Such evidence is usually taken to imply that small firms invest at a rate lower than its perfect capital market rate. In our model, however, it arises because small firms rely more on copying than big firms do: The marginal product of capital is equal across firms, but its average product is higher than that because small firms get a disproportionately high external benefit.

The results change if one introduces markets for knowledge that internalize the externalities. We analyze the case in which pairs of firms can form research consortia and exclude others from using the knowledge that they generate. This exercise in the theory of clubs and assignments shows that the market outcome is efficient. We lose the theory of inequality when sorting among firms is positive, but retain it when assignments are negative.

2 Intuition

Let's start with a graphical exposition of the case in which the external benefit that a firm receives depends only on that firm's rank in the population distribution of capital stocks, k. The Figure 1 illustrates the effect that inequality then has on a firm's incentive to invest.

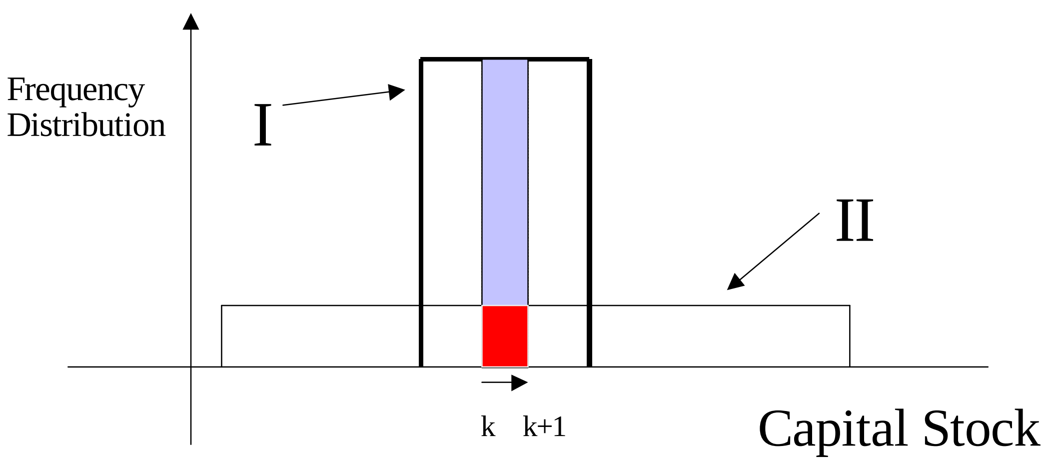

Figure 1: Inequality and the incentive to invest

The figure displays two hypothetical distributions of k in the population of firms. Distribution II is more spread out than distribution I. A firm that raises its stock of capital by one unit from k to $k+1$ will experience a gain in rank equalling the darker rectangle if Distribution II is the relevant one, whereas under Distribution I its gain in rank would be larger by the lighter rectangle.

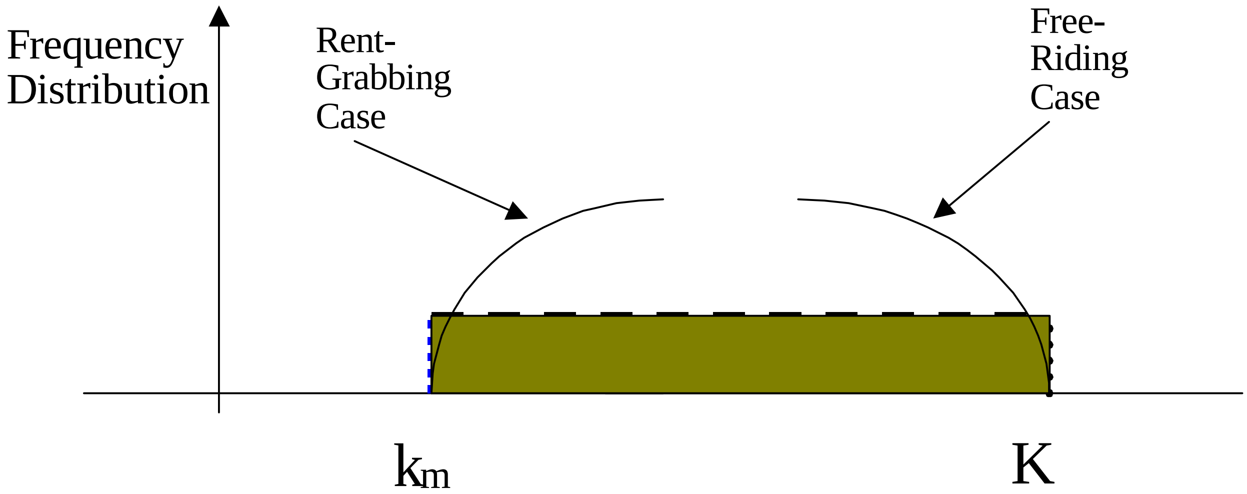

Figure 2: The density must be zero at one endpoint or the other

Whether inequality is a stimulus or a deterrent to investment and growth depends on whether we are in the rent-grabbing case in which rank raises access, or in the free riding case in which rank reduces access to usable knowledge of others. In the rent-grabbing case, leadership is a blessing, the prospect of a gain in rank is a stimulus to investment, and the incentive is higher under Distribution I. In other words, under rent-grabbing, inequality is *bad* for investment-incentives. On the other hand, if free-riding dominates, leadership is a curse, the prospect of a gain in rank is a deterrent to investment, and the incentive to invest is higher under Distribution II. In other words, under free riding , inequality is *good* for investment-incentives.

Figure 2 shows why the distribution of capital must, in general, have at least one tail in the sense that its density must be zero either at the minimum or maximum capital-level, or both.

For the free riding case, consider the example of a flock of birds flying south. There is a continuum of birds in the flock. The leader breaks the wind. The farther back a bird is, the wider the wind-tunnel it enjoys, and the easier it flies. If it flew faster, the bird would gain rank, and this would be bad. But the leader, bird K, wouldn't gain rank if *it* flew faster. For birds K and $K - \varepsilon$ to fly at the same speed, the density must be zero at K. For if, instead, it were strictly positive (as, say, the shaded density is), the marginal payoff to effort would jump up at K. Bird K would then pull away from bird $K - \varepsilon$ because it would face no disincentive from a gain in rank. A positive density at K would thus lead to a convexity and a kink in the payoff that go away only if the density at K is zero.

In the rent-grabbing case, the density must be zero at k_m, the *minimal* level of capital. Suppose that it wasn't and that, instead, it was positive, as shown by the shaded density. Bird k_m could fall back to $k_m - \varepsilon$ and not lose rank. Bird $k_m + \varepsilon$, on the other hand, faces a loss of rank if *it* were to fall back. There is, in other words, again an upward jump of the marginal payoff, this time at $k = k_m$, and the last bird would fly more slowly than the rest. This time the convexity and the kink are at k_m, and they go away only if the density at k_m is zero.

3 Model

Preferences: There is one final good. Preferences over consumption streams are $\sum_0^\infty \left(\frac{1}{1+\rho}\right)^t \frac{c_t^{1-\gamma}}{1-\gamma}$, where $\gamma \geq 0$. The capital market is perfect and the interest rate, r, is assumed to be constant. Maximization of this subject to the constraint that $\sum_0^\infty \left(\frac{1}{1+r}\right)^t c_t$ not exceed wealth leads to a growth rate of consumption x given by

$$x = \left(\frac{1+r}{1+\rho}\right)^{1/\gamma} - 1. \tag{1}$$

Production capacity: The agent runs his own firm. Let μ denote the economy-wide distribution function of firms' capital, k. Agent k's production capacity, or "potential output" is $f(k, \mu)$. While k is a scalar, μ is a function, and so f is a *functional.*

Accumulation of k : Starting from k, the cost of getting k' units of capital next period is $kC(\frac{k'}{k})$, where $C' > 0$. Also assume there are costs of rapid adjustment, so that $C'' > 0$. Output is

$$f(k, \mu) - kC\left(\frac{k'}{k}\right).$$

If there is no investment, capital depreciates by δ percent. That is, $C(1-\delta) = 0$.[1]

Firm's maximization problem: Let $v(k, \mu)$ be the value function. It satisfies

$$v(k, \mu) = \max_{k'} \left\{ f(k, \mu) - kC\left(\frac{k'}{k}\right) + \frac{1}{1+r} v(k', \mu') \right\}. \tag{2}$$

If v is differentiable, the first order condition of the problem in (2) is

$$\frac{1}{1+r} v_1(k', \mu') - C'\left(\frac{k'}{k}\right) = 0. \tag{3}$$

[1]Section 7.1 studies the effect of having μ enter C instead of f.

The envelope theorem says that

$$v_1(k,\mu) = f_1(k,\mu) - C\left(\frac{k'}{k}\right) + \frac{k'}{k}C'\left(\frac{k'}{k}\right). \tag{4}$$

Using (4) to eliminate $v_1(.)$ from (3), yields the second-order difference equation for k:

$$f_1(k',\mu') = (1+r)\,C'\left(\frac{k'}{k}\right) - \frac{k''}{k'}C'\left(\frac{k''}{k'}\right) + C\left(\frac{k''}{k'}\right). \tag{5}$$

Relative capital, z: Let K_t denote the largest k in the support of μ_t that, we assume, is bounded. Let

$$z \equiv k/K_t$$

denote capital relative to the current maximum. For any t, its cumulative density function, H_t is

$$H_t(z) = \mu_t\,(zK_t) \tag{6}$$

for $z \in [0,1]$.

A linear homogeneous production function: We now restrict f so that returns in the aggregate are constant. From (6) a sufficient statistic for μ is the pair (K,H). We can therefore write $f(k,\mu)$ in terms of the triple (k,K,H). We shall restrict the process by which external effects take place and assume that capacity can be written as

$$f(k,\mu) = F(k,k_a), \tag{7}$$

where k_a, a scalar, represents "accessed" knowledge, and that F is linear homogeneous.[2] At this point, one would normally assume that k_a is average knowledge, or frontier knowledge, and this is where we depart from the norm.

3.1 Type-dependent access: $k_a = KA(z \mid H)$

The main obstacle to the flow of information these days surely isn't geographical distance, but distance in human-capital space. In this space, a firm is located at k and the locations of other firms are described by (K,H). We assume that accessing the knowledge of others does not use up resources and that:

$$k_a = KA(z \mid H).$$

We emphasize the following features of the "access function" A:

[2]We relax this assumption in section 7.2.

(*i*) *A search-theoretic interpretation*: If we write

$$A(z \mid H) = \int_0^1 \alpha(z,s)dH(s), \tag{8}$$

then one may think of $\alpha(z,s)$ as the knowledge that firm z can get from firm s, and of $dH(s)$ as the probability of contacting such a firm. Moreover, a firm has no market power because it has no effect on H and, hence, on other firms' k_a's.[3]

(*ii*) *A "usability" interpretation*: Patents temporarily prevent a firm from making money with knowledge that it has accessed given that someone else has accessed it first. This would imply that $A(z \mid H)$ is zero for all $z < 1$.

(*iii*) *A may be increasing or decreasing in z*: It is increasing if, as Nelson and Phelps (1966) have emphasized, the more you know, the easier you can learn from others. Or it could be decreasing if, as Jovanovic and Rob (1989) and Parente and Prescott (1994) have emphasized, the more you know, the less there is left for you to learn – the "fishing out" phenomenon. Jovanovic and Nyarko (1996, sec. 3.3) provide an information-theoretic example in which A is inversely proportional to the distance between z and s.

(*iv*) *Under equality it is an "aK" model*: Let $H = \mathbf{1}$ denote the distribution that assigns all mass to $z = 1$. By Euler's theorem, each firm's capacity then is

$$F\left[K, KA(1 \mid \mathbf{1})\right] = \left\{F_1(1, A(1 \mid \mathbf{1})) + F_2(1, A(1 \mid \mathbf{1}))A(1 \mid \mathbf{1})\right\} K = aK \tag{9}$$

4 Equilibrium

The linear homogeneity of F and the functional form of A imply that f_1 depends only on the pair (z, H):

$$f_1 = \frac{dF}{dk} = F_1 + F_2 A' \equiv \phi(z \mid H). \tag{10}$$

If we rule out "leapfrogging", the only way that H can be constant is if every firm grows at the same rate x. We assume that x is constant we have, for any $k > 0$, $k' = (1+x)\,k$ and $k'' = (1+x)k'$, and (5) reads

$$\phi(z \mid H) = C\,(1+x) + (r-x)\,C'(1+x). \tag{11}$$

If every firm's k grows at the same, constant rate, each firm's z is constant over time, and so is H.

[3]One may think of $\alpha(z,s)$ as also reflecting firm z's effort to direct its search towards firms of type s, as in Jovanovic and MacDonald (1994).

Solve the optimum saving condition (1) for r to get $r(x) = (1+\rho)(1+x)^{\gamma} - 1$, and substitute it into the right-hand side of (11) to get

$$\psi(x) \equiv C(1+x) + [r(x) - x] C'(1+x). \tag{12}$$

Therefore (11) boils down to,

$$\phi(z \mid H) = \psi(x). \tag{13}$$

This eliminates r from the system, and reduces the objects of steady state equilibrium to just two: x and H.

Definition 1 *A constant growth equilibrium is a growth-rate $x > -\delta$, and a distribution function H on a subset of $(0,1]$ so that (13) holds for all $z \in \text{supp}H(z)$.*

The definition is incomplete for two reasons. First, condition (13) is necessary for a maximum, but, as we illustrated with Figure 2, it is not sufficient for firms at one of the corners – $z \in \{z_m, 1\}$. We shall, in each example need to check that each firm is indeed at a global maximum. Second, if $\gamma \leq 1$, consumers' lifetime utility becomes unbounded when r gets large. This limits how large x can be before $r(x)$ ceases to have meaning. The growth factor of utility is $\frac{1}{1+\rho}(1+x)^{1-\gamma}$, which means that when $\gamma \leq 1$ admissable x must not exceed an upper bound of

$$x^M \equiv (1+\rho)^{1/(1-\gamma)} - 1 \tag{14}$$

Equilibria will be denoted by x^* and H^*.

The definition asks that firms all grow at the same rate. If a subset of firms were to grow at a rate that was slower than the rest, relative inequality would be widening for ever, and H^* would then have a mass-point at zero. We do not wish to consider such cases, and we therefore do not allow any firms to be at $z = 0$. Indeed, if a firm ever reached $z = 0$ it would be stuck there forever. The investment technology implicit in the cost $kC(k'/k)$ is defined only for $k > 0$.

Because costs of adjusting k are convex, and because $r(.)$ is increasing, the effective marginal cost of investment, $\psi(.)$, is increasing:

Lemma 1 *ψ is strictly increasing.*

Proof. $\psi'(x) = r'(x)C'(1+x) + [r(x) - x] C''(1+x) > 0$, because $\gamma > 0$, $r'(x) > 0$, and $r(x) > x$, and because C' and C'' are both positive. ■

4.1 Inequality

By "inequality" we mean the variance or the range of the following variables, each of which has been normalized by division by K:

- Capital stocks $z \equiv k/K$. Investments in k and, hence, stocks of k, are likely to be unmeasured, but one can infer k by measuring a firm's "total factor productivity" that measures a firm's efficiency.
- Outputs, $F(z, A(z \mid H^*))$,
- Profits, $F(z, A(z \mid H^*)) - zC(1 + x^*)$,
- Firm values
$$\frac{F(z, A(z \mid H^*)) - zC(1 + x^*)}{r(x^*) - g} \equiv v(z), \tag{15}$$
- Average Tobin's q's, $v(z)/z$.[4]

Equality in k implies that all z's are unity. Equality is an equilibrium if the equation

$$\phi(1 \mid \mathbf{1}) = \psi(x) \tag{16}$$

has a solution for x that exceeds $-\delta$, and if investment is less than output. When this is not the case, equality cannot arise in steady state. In fact, if $A(\cdot)$ is sufficiently sensitive to z then equality is ruled out:

Proposition 2 *For equality to exist it is necessary that*

$$\frac{-F_1[1, A(1 \mid \mathbf{1})]}{F_2[1, A(1 \mid \mathbf{1})]} \leq A'(1 \mid \mathbf{1}) \leq \frac{(\psi(x^m) - F_1[1, A(1 \mid \mathbf{1}))}{F_2[1, A(1 \mid \mathbf{1})]}, \tag{17}$$

where

$$x^m \equiv \min \left\{ x^M, -1 + C^{-1} \left(F\left[1, A(1 \mid \mathbf{1})\right] \right) \right\}.$$

[4] "Tobin's q" is the market value of the firm divided by the replacement cost of its capital. There is no market for k in the model – the only cost of accumulation is the foregone output cost $kC(k'/k)$. But it is straightforward to interpret the empirical measure in terms of our model. Suppose that each unit of human capital, k, requires ς machines to work with before it could be productive. If machines and goods trade one for one, the purchase of machines requires that the investment cost be augmented by an amount $\varsigma(k' - k)$, then ς is be the theoretical counterpart to the per-unit replacement cost of capital. The internal adjustment cost then represents the output foregone because of the accumulation of the requisite human capital. But for the presence of μ in f, this would make our firms identical to the firms in Lucas and Prescott (1971) and Hayashi (1982).

Proof: Under equality, (13) reads $F_1[1, A(1 \mid \mathbf{1})] + F_2[1, A(1 \mid \mathbf{1})]A'(1 \mid \mathbf{1}) = \psi(x)$, or

$$A'(1 \mid \mathbf{1}) = \frac{(\psi(x) - F_1[1, A(1 \mid \mathbf{1})])}{F_2[1, A(1 \mid \mathbf{1})]}. \tag{18}$$

The first inequality in (17) follows because $\psi(x) \geq 0$. The second inequality follows from the following two observations:

(i) Firms have access to perfect capital markets. Off the steady state they may incur losses, but a firm cannot have negative profits for ever. It cannot grow faster than at a rate at which all output is invested, so that $F(1, A(1 \mid 1)) = C(1 + x)$. Denote this maximal growth-rate by $x^1 = -1 + C^{-1}(F[1, A(1 \mid \mathbf{1})])$. Then by Lemma 1, $\psi(x) \leq \psi(x^1)$.

(ii) The definition of equilibrium requires that consumers cannot feasibly attain infinite utility, and this, from (14), requires that $x < x^M$. ■

Corollary 3 *When A is representable by (8), in (18) we can make the following substitutions:* $A(1 \mid \mathbf{1}) = \alpha(1, 1)$ *and* $A'(1 \mid \mathbf{1}) = \alpha_1(1, 1)$.

4.2 Illustrations of economies in which equilibrium must involve inequality

The following three cases illustrate the failure of condition (17). Each assumes the production function $F(k, k_a) = k^\theta k_a^{1-\theta}$.

Illustration 1: $\alpha(z, s) = z^{\lambda_1} s^{\lambda_2}$. Now (17) reads

$$\frac{-\theta}{1-\theta} \leq \lambda_1 \leq \frac{\psi\left[C^{-1}(1) - 1\right] - \theta}{1-\theta}.$$

If λ_1 does not fall in this interval, constant growth cannot involve equality.

Illustration 2 (Students and teachers): $\alpha(z, s) = \omega_0 - (s - z - \omega)^2$ so that a type z student is best taught by a type $s = z + \omega$ teacher. The parameter ω_0 is set to ensure that $\alpha(z, s)$ is positive for all $(s, z) \in [0, 1]^2$. If $\omega = 0$, it is optimal for equal types to match. Matching is random, however, so that (8) still holds. As a function of z, then $\alpha(z, s)$ has an inverted-U shape, peaking at the point $s - \omega$. Using the same production function, since $\alpha_1(z, s) = 2(s - z - \omega)$, $\alpha_1(1, 1) = -2\omega$, and $\alpha(1, 1) = \omega_0 - \omega^2$. The first inequality in (17) now reads

$$\frac{F_1(1, \omega_0 - \omega^2)}{F_2(1, \omega_0 - \omega^2)} = \frac{\theta}{1-\theta}\left(\omega_0 - \omega^2\right) \geq 2\omega \tag{19}$$

If θ is small or if ω is large, the condition fails. Intuitively, if acquired knowledge matters enough and if having a smarter teacher also matters enough, there is an incentive to deviate to a lower z. As $\omega \to 0$ equality is always an equilibrium.

Illustration 3: $A(z \mid H) = \omega.[H(z)]^{\rho}$. Now $A(1 \mid \mathbf{1}) = \omega$. We also can imagine approaching equality with a sequence of differentiable functions H, in which case[5] $A'(1 \mid \mathbf{1}) = \lim_{H_n \to \mathbf{1}} \rho\omega H'(1) = +\infty$, and the second inequality in (17) fails for all values of the parameters. Equality therefore cannot occur. Instead, this example has a steady state with inequality that we shall characterize fully in Section 5.1.

It is not a rare event, therefore, for equality to fail to be a steady-state equilibrium. The sections that follow will solve several examples the equilibria of which have inequality. In each, the support of H^* will be an interval without holes in it. Yet, nothing in the definition of equilibrium ensures that this will be so generally.

Since we know little about the exact form of $A(\cdot)$ we shall analyze a set of examples. Equality is not an equilibrium in any of them. We begin with examples in which a larger z allows more access.

5 Leadership a blessing: A increasing in z

This section discusses two examples in which A is increasing. The first example is that of illustration 3 above, and it has the property that in the cross section of firms Tobin's q is a constant, independent of z. In the second example, q declines with z.

5.1 Example 1: $A(z \mid H) = \omega.[H(z)]^{\rho}$

The parameter $\rho \geq 0$ is an index of how fast access rises with rank. In Section 4.2, Illustration 3, we showed that equality is not an equilibrium. Instead, an equilibrium is the Pareto distribution

$$H^*(z) = z^{1/\rho} \qquad\qquad z \in [0, 1].$$

That is because $\omega[H^*(z)]^{\rho} = \omega z$, so that $A(z \mid H^*) = \omega z$, and so that

$$F(z, A(z \mid H^*)) = zF(1, \omega)$$

The value of the firm in (15) now simplifies to

$$\frac{F(1,\omega) - C(1+x)}{r-g} z. \tag{20}$$

Since all firms face the same purchase price of capital, firms' outputs, profits, and values are all proportional to z. This means that average and marginal Tobin's q's coincide. Inequality in z depends on ρ, As $\rho \to 0$, H^* converges to a mass point at $z = 1$, and as $\rho \to \infty$, H^* converges to a mass point at $z = 0$. So the density of z, of profits, and of Tobin's q's can, in general, be skewed to the left or to the right.

[5]Assuming we approach $\mathbf{1}$ by a sequence of differentiable functions such as, e.g., $H_n(z) = z^n$ for $n = 1, 2, \ldots$.

5.1.1 Other cases in which average q = marginal q.

The access function in example 1 had the property that there is at least one distribution function, call it H^*, that renders A proportional to z. Let the constant of proportionality be π, so that $A(z \mid H^*) = \pi z$. Firms' capacities are then proportional to their capital stocks because

$$f(k, \mu) = kF\left(1, \frac{KA(\frac{k}{K}, H^*)}{k}\right) = kF(1, \pi).$$

Since the cost function is also linear homogeneous, a firm's value is then proportional to its k, and its marginal and average Tobin's q coincide. Let where z_m denotes the smallest z in the support of H. We now state the generalized version of example 1:

Proposition 4 *Let $\pi > 0$ be given. If H^* solves, for H, the equation*

$$A(z \mid H) = \pi z \tag{21}$$

for all $z \in supp(H)$, and if

$$lim_{\varepsilon \to 0} A'(z_m - \varepsilon \mid H) \leq \pi, \tag{22}$$

then H^* is an equilibrium distribution, and the growth rate solves

$$F_1(1, \pi) + \pi F_2(1, \pi) = \psi(x). \tag{23}$$

Proof: Since $A' = \pi$, (13) and (10) imply that at H^* (23) holds for all values of z. Since (22) holds for z in a neighborhood of z_m, there is no kink at z_m and no convexity.■ [6]

Since a firm's output is proportional to k, this class of examples does fit the observation that larger firms have lower Tobin's q's (Fazzari, Hubbard and Petersen 1988). Since k is also a firm's productivity, the class of examples does not fit the observation that the elasticity of q with respect to a firm's TFP-level is less than unity (Dwyer 1997), the rejection from the proportionality in (20) is not overwhelming.

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[6] Another example that meets the restrictions of Proposition 4 is $\alpha(z, s) = z\hat{\alpha}(s)$, in which case

$$A(z \mid H) = z \int_0^1 \hat{\alpha}(s) \, dH(s).$$

It leads to multiple steady states – a continuum, in fact – because of the positive strategic complementarity in the access subgame. One steady state distribution is $H = \mathbf{1}$ – equality.

5.2 Example 2: A increasing in z, but $A(z_m \mid H) > 0$

Suppose that for *any* H, the least efficient firm can freely access one unit of k_a from a source other than its rivals. This implies that for $z \leq z_m$,

$$A(z \mid H) = A\,(z_m \mid H) = 1. \tag{24}$$

The function $\phi(\cdot) \equiv dF/dk = F_1 + F_2 A'$ is constant on the interval $[z_m, 1]$. For the firm at z_m to be at a maximum, ϕ must not exhibit a downward jump anywhere on an open neighborhood of z_m. This is the convexity at k_m that we saw in Figure 2. Let us now discuss it again with the help of Figure 3.

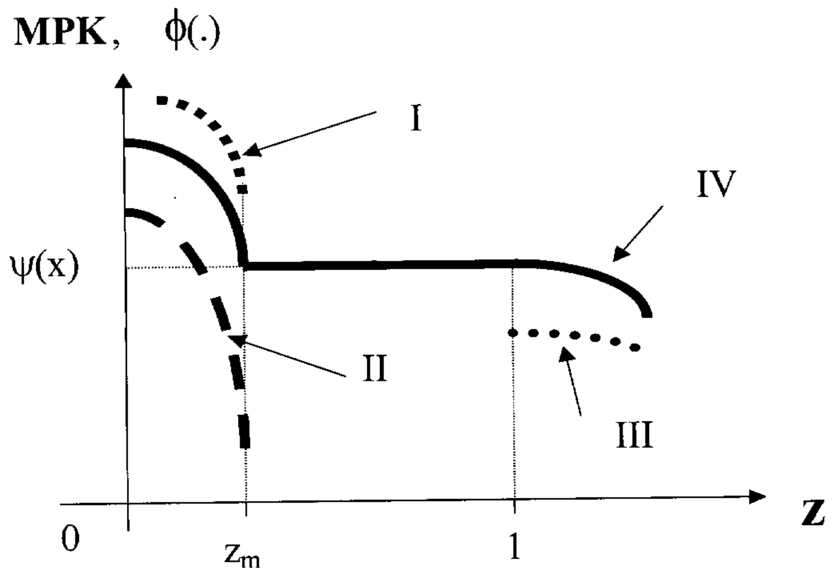

Figure 3: Marginal Tobin's q

Since F_2 and A' are both nonnegative, a downward jump at z_m, (such as the one that would occur if MPK behaved as in curve I) is ruled out. And an upward jump (such as the one that would occur if MPK behaved as in curve II) is ruled out by the second order conditions. Therefore, the second order conditions imply that

$$F_1\,[z_m, A(z_m \mid H^*)] = \psi(x), \tag{25}$$

which, if $F_2 > 0$, requires that

$$A'(z_m \mid H^*) = 0. \tag{26}$$

This condition will turn out to imply that Tobin's q decreases as z rises.

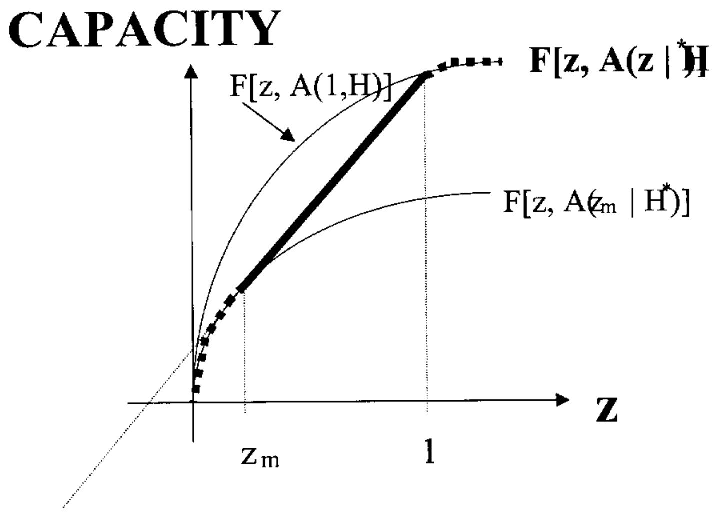

Figure 4: Potential output when $A' > 0$

No conditions corresponding to (25) and (26) are required at $z = 1$, because at that value of z, because neither an MPK like line IV in Figure 3 nor one like III would violate the second order conditions.

Figure 4 plots $F(z, A(z \mid H^*))$ – the heavy curve – which is concave in z when (25) holds. This means that average q will exceed marginal q.

Figure 4 shows that the decreasing Tobin's q result hinges on the second-order conditions: If A' were not zero at the point $z = z_m$, the envelope curve, $F\left[z, A(z \mid H^*)\right]$ would have a kink at z_m and the heavy straight line could then be equal in slope to, or even steeper than the 45° line. We can derive the equilibrium access function $A(z \mid H^*)$ when the production function is Cobb-Douglas. Let $F(k, k_a) = k^\theta k_a^{1-\theta}$, so that (25) reads

$$\psi(x) = \theta z_m^{-(1-\theta)}, \quad \text{or,} \quad z_m = \left(\frac{\theta}{\psi(x)}\right)^{1/(1-\theta)}. \tag{27}$$

Condition (13) reads

$$\theta \left(\frac{z}{A}\right)^{\theta-1} + (1-\theta)\left(\frac{z}{A}\right)^{\theta} \frac{dA}{dz} = \psi(x), \tag{28}$$

which leads to the differential equation

$$\frac{dA}{dz} = \frac{\theta}{1-\theta}\left\{\left(\frac{A}{z}\right)^{\theta} z_m^{-(1-\theta)} - \frac{A}{z}\right\} \tag{29}$$

with the boundary condition $A(z_m) = 1$.

The solution: As the appendix verifies, the general solution to (29) is

$$A = \left(z_m^{-(1-\theta)}\theta z^{1-\theta} + Cz^{-\theta}\right)^{1/(1-\theta)},$$

where C is arbitrary. The boundary condition $A(z_m) = 1$ implies that $C = z_m^{\theta}(1-\theta)$, which gives

$$A = \left(\theta\left(\frac{z}{z_m}\right)^{1-\theta} + (1-\theta)\left(\frac{z}{z_m}\right)^{-\theta}\right)^{1/(1-\theta)}$$

for $z \in [z_m, 1]$. This solution depends on a parameter, z_m, yet to be determined. But note that A is increasing and *convex*. To solve for it we need to know what $A(1)$ is, and to get this boundary condition we must specify a form for $A(z \mid H)$.

Assume that the access function is

$$A(z \mid H) = 1 + H(z),$$

which implies that $A(1) = 2$ for all H, a condition that will enable us to solve for z_m. Now $A(1) = 2$ implies that $\left(z_m^{-(1-\theta)}\theta + C\right)^{1/(1-\theta)} = 2$. And, since $C = z_m^{\theta}(1-\theta)$, we end up with a restriction on z_m alone:

$$\theta z_m^{\theta-1} + (1-\theta)\, z_m^{\theta} = 2^{1-\theta}. \tag{30}$$

The left-hand side decreases monotonically from infinity to one, whereas the right-hand side exceeds one. The unique solution for z_m is plotted in Figure 5.

Human capital differentials therefore increase with θ.

Output differentials: Let I_θ denote the ratio of the output of the leader to the laggard, conditional on θ:

$$I_\theta = \frac{F[1, A(1 \mid H)]}{F[z_m, A(z_m \mid H)]} = \frac{2^{1-\theta}}{z_m^{\theta}}$$

From (30) one can show that $\lim_{\theta\to 0} z_m^{\theta} = \lim_{\theta\to 1} z_m^{\theta} = 1/2$, so that as θ increases from zero to one, I_θ falls from 4 to 2. Output differentials therefore rise with the strength of the external effect, $1-\theta$.

Externalities and growth: The figure also shows that the ratio θ/z_m rises as θ does. A fortiori, so does $\theta z_m^{-(1-\theta)}$ and so by (27), so does $x = \psi^{-1}\left(\theta z_m^{-(1-\theta)}\right)$, the long-run growth rate. So, growth is higher when the share of own-capital in production is higher, and externalities reduce growth.[7]

[7] The optimal rate of growth is $\psi^{-1}(1)$. It is attained when $\theta = 1$ and there are no spillovers. As θ declines, so does the growth rate.

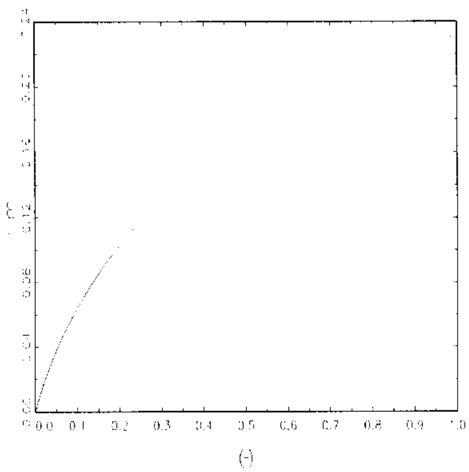

Figure 5: The dependence of z_m on θ

Skewness of H: Since A is convex, so is H. Since $A'(z_m) = H'(z_m) = 0$, the density, H', rises monotonically from zero up to a maximum at $z = 1$. It has a left-tail, and no right-tail. In contrast to Example 1 (which had no particular implications for skewness), Example 2 produces skewness in the wrong direction.

6 Leadership a curse: A decreasing in z

The ability to access usable knowledge may, in many kinds of activities, decline as one's knowledge increases. Goolsbee and Klenow (1998), for instance, find that spillovers in the decision to adopt a computer emanate entirely from a small group of elite computer owners – people who use their computers intensively and have owned several computers in their lifetimes. It seems reasonable to expect such spillovers to flow mainly towards the novices. If so, this is an example in which A decreases in z.

In this class of examples, the larger is z, the smaller is A, and so the ratio z/A increases with z. Tobin's q declines with z because (24) holds. We now assume that once a firm is at the head of the pack, it has reached the minimum k_a. That is,but that

$$A(z \mid H) = A(1 \mid H) \quad \text{for } z \geq 1.$$

Equality is not an equilibrium because a mass point would induce firms to pull back their investments and take advantage of the increased access to the knowledge of others that backwardness offers. In order to eliminate the convexity at $z = 1$ (discussed

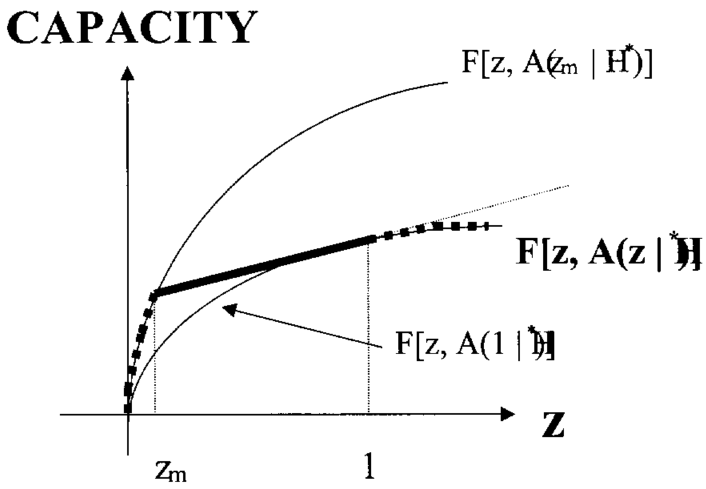

Figure 6: Potential output when $A' < 0$

graphically in Section 2) we require that

$$F_1\left[1, A(1 \mid H^*)\right] = \psi(x), \tag{31}$$

which, if $F_2 > 0$, now requires that

$$A'(1 \mid H^*) = 0. \tag{32}$$

Figure 6 shows that if (32) failed and A' were, instead, positive at $z = 1$, $F\left[z, A(z, H)\right]$ would have a kink at $z = 1$, and the firm's objective would not be concave.

6.1 Example 3: $A(z_m) > 0$, and A decreasing in z

As in Example 2, we shall normalize externalities to equal unity when they are at their lowest. Since A is now decreasing in z, this means that, instead of (24), we now have $A(1 \mid H) = 1$. Again, let $F(k, k_a) = k^\theta k_a^{1-\theta}$, so that (31) reads

$$\theta = \psi(x). \tag{33}$$

Condition (13) is again given by (28). Combine it with (33) to get the differential equation

$$\frac{dA}{dz} = \frac{\theta}{1-\theta}\left(\frac{A}{z}\right)^\theta \left[1 - \left(\frac{A}{z}\right)^{1-\theta}\right], \tag{34}$$

which is negative because A exceeds one, except at $z = 1$, where it equals zero. Note that (34) is a special case of (29). The solution now is

$$A = \left(\theta z^{1-\theta} + C z^{-\theta}\right)^{1/(1-\theta)},$$

where C is arbitrary. Now $A(1 \mid H) = 1$ implies that $C = (1 - \theta)$, and therefore

$$A = \left(\theta z^{1-\theta} + (1-\theta)\, z^{-\theta}\right)^{1/(1-\theta)}$$

for $z \in [z_m, 1]$. Differentiation shows that A is now decreasing, and still convex. This solution is not final yet, because z_m is yet to be determined, and for that we shall need to specify $A(z \mid H)$.

6.2 Restricting A further to $A(z \mid H) = 2 - H(z)$

We chose this example for its tractability. For some distributions H, however, it violates free disposal of k in production because $dF/dk = F_1 - F_2 H'$ is negative! This happens when H is concentrated so that, as a result, $H'(z)$ becomes large. This will not happen in equilibrium, though.

Since $A(z_m) = 2$,

$$\theta z_m^{1-\theta} + (1-\theta)\, z_m^{-\theta} = 2^{1-\theta}.$$

The solution for z_m is plotted in Figure 7.

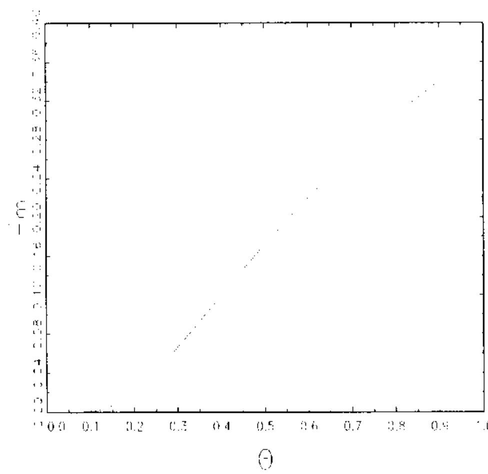

Figure 7: The dependence of z_m on θ

The human capital differential still increases in θ. The negative MPK problem leads to non-existence of a steady state when external effects matter enough. This occurs for $\theta \leq 0.10$, and so we rule out θ's in this region.

Output differentials: Let I_θ again denote the output differential conditional on θ:

$$I_\theta = \frac{F\left[1, A(1 \mid H)\right]}{F\left[z_m, A(z_m \mid H)\right]} = \frac{1}{z_m^\theta 2^{1-\theta}} = \frac{1}{\theta z_m + 1 - \theta} = \frac{1}{1 - \theta\left(1 - z_m\right)},$$

which rises from 1.1 when $\theta = 0.1$, to 2.7 when θ approaches 1.

External effects and growth: We can solve for the rate of growth from the condition $F_1(1 \mid A(1)) = \psi(x)$, so that now $x = \psi^{-1}(\theta)$ so that, once again, growth is higher when appropriability is higher.

7 Robustness

This section reports a diverse set of departures from the maintained assumptions. First, the conclusions seem to be survive if, instead of entering f, μ is assumed to enter C. They also largely survive if we relax the assumption that F is linear homogeneous. The first two subsections deal with these two departures from the model. The remaining three subsections will show that without type-dependent access, we get equality, and that if we remove external effects altogether, we get an indeterminacy in H. We end the section with a brief and partial analysis of welfare.

7.1 External effects in $C(.)$ instead of in $f(.)$

Instead of entering the production function f, suppose that μ affects learning possibilities. Starting from k, the cost of getting k' units of capital next period is $c(k, k', k_a)$. We shall suppose that c is homogeneous of degree one so that

$$c(k, k', k_a) = kC\left(\frac{k'}{k}, \frac{k_a}{k}\right),$$

where C is increasing in its first argument, and, since the externality reduces costs, decreasing in the second. Capital still depreciates by δ percent, so that $C\left(1 - \delta, \frac{k_a}{k}\right) = 0$. We need a linear homogeneous production function, so we assume that $f(k) = \eta k$.The firm's value still is $v(k, \mu)$, except that the Bellman equation now reads

$$v(k, \mu) = \max_{k'}\left\{\eta k - kC\left(\frac{k'}{k}, \frac{k_a}{k}\right) + \frac{1}{1+r}v(k', \mu')\right\}. \tag{35}$$

If v is differentiable, the first order condition now is

$$\frac{1}{1+r}v_1(k',\mu') - C_1\left(\frac{k'}{k},\frac{k_a}{k}\right) = 0. \tag{36}$$

The envelope theorem now says that

$$v_1(k,\mu) = \eta - C + \frac{k'}{k}C_1 + \left[\frac{k_a}{k} - \frac{dk_a}{dk}\right]C_2. \tag{37}$$

We shall continue to assume that $k_a = KA(z \mid H)$, so that

$$\frac{k_a}{k} = \frac{A(z \mid H)}{z} \quad \text{and} \quad \frac{dk_a}{dk} = \frac{dk_a}{dz}\frac{dz}{dk} = A'.$$

Using these facts and using (37) to eliminate $v_1(.)$ from (36), yields the second-order difference equation for k that, when evaluated in steady state, reads

$$\eta = (r(x) - x)\,C_1 + C + \left[\frac{A}{z} - A'\right]C_2. \tag{38}$$

The functions C, C_1, and C_3 are all evaluated at the point $\left(1+x, \frac{A(z|H)}{z}\right)$. Therefore (38) is of the form

$$\eta = M(x,z,H) \tag{39}$$

which, as before, must hold for all z in the support of H.

The analogy of Proposition 4 is the following result:

Proposition 5 *An H^* that solves, for H, the equation $A(z \mid H) = \pi z$ is an equilibrium, and under it*

$$\eta = (r(x) - x)\,C_1(1+x,\pi) + C(1+x,\pi)$$

Proof: An H^* that solves, for H, the equation $A(z \mid H) = \pi z$ for all $z \in (0,1]$ implies that $\frac{A(z|H)}{z} = A'(z \mid H) = \pi$, so that the last term on the right-hand side of (38) is zero. The claim then follows. ■

Condition (39) is not additively separable in x and z, and is therefore harder to analyze than (13) . The analogy of a $\psi(\cdot)$ that increases in x is an $M(.)$ that increases in x. When it is, there is at most one solution for x for any H. In the case covered by the above proposition, this is obviously true. But it holds more generally: As Lemma 1 shows, the term $(r(x) - x)\,C_1 + C$ is increases with x. And when $A' < 0$, and if $C_{12} > 0$, the last term is too, because then $\left[\frac{A(z|H)}{z} - A'(z \mid H)\right] > 0$. .

7.1.1 Cobb-Douglas example

Let

$$C(a,b) = a^{1+\theta}b^{-\theta},$$

so that, since in the steady state $a = 1+x$ and $b = \frac{A(z|H)}{z}$,

$$C_1 = (1+\theta)\left(\frac{(1+x)\,z}{A}\right)^{\theta} \quad \text{and} \quad C_2 = \theta\left(\frac{(1+x)\,z}{A}\right)^{1+\theta},$$

and (38) reads

$$\eta = (r(x)-x)\,(1+\theta)\left(\frac{(1+x)\,z}{A}\right)^{\theta} + (1+x)^{1+\theta}\left(\frac{A}{z}\right)^{-\theta} + \left[\frac{A}{z} - A'\right]\theta\left(\frac{(1+x)\,z}{A}\right)^{1+\theta}.$$

Rearranging,

$$\eta\left(\frac{A}{z}\right)^{\theta} = (r(x)-x)\,(1+\theta)\,(1+x)^{\theta} + (1+\theta)\,(1+x)^{1+\theta} - \frac{z}{A}A'\theta\,(1+x)^{1+\theta}$$

or, since

$$\frac{z}{A}A' = \frac{d\ln A}{d\ln z},$$

$$\frac{d\ln A}{d\ln z} = \frac{1}{\theta\,(1+x)^{1+\theta}}\left[m(x) - \eta\left(\frac{A}{z}\right)^{\theta},\right]$$

where

$$m(x) = (r(x)-x)\,(1+\theta)\,(1+x)^{\theta} + (1+\theta)\,(1+x)^{1+\theta}$$

Now let $y = \ln A$ and $u = \ln z$, in which case $\left(\frac{A}{z}\right)^{\theta} = \exp\{\theta\,(y-u)\}$, and the differential equation becomes

$$\frac{dy}{du} = \frac{1}{\theta\,(1+x)^{1+\theta}}\left[m(x) - \eta\exp\{\theta\,(y-u)\},\right]$$

This differential equation is of the form

$$\frac{dy}{du} = a - b\exp\{\theta\,(y-u)\},$$

Its exact solution (using Maple) is :

$$-\frac{\ln\left(-\exp\left(-\theta\,(y\,(u)-u)\right) + a\exp\left(-\theta\,(y\,(u)-u)\right) - b\right)}{(-1+a)\,\theta} - u = C_1$$

7.2 Increasing returns to scale

We have assumed throughout that F is linear homogeneous, and that its inputs are reproducible – production does not require labor services. Per capita output is $y = F(k, k_a) - kC\left(\frac{k'}{k}\right)$. In steady state, the output-capital ratio of each firm,

$$\frac{y}{k} = F(z, A(z, H)) - C(1 + x),$$

is constant through time (but not, necessarily, among firms at a point in time) as per capita output.

Let F be homogeneous of degree λ. Then $F(k, k_a) = K^{\lambda} F\left(\frac{k}{K}, A(z \mid H)\right)$, so that $F_1(k, k_a) = K^{\lambda-1} F_1(z, A(z \mid H))$ and $F_2(k, k_a) = K^{\lambda-1} F_2(z, A(z \mid H))$. If adjustment costs were of the form $k^{\lambda} C\left(\frac{k'}{k}\right)$, then marginal costs would equal $k^{\lambda-1} C\left(\frac{k'}{k}\right)$, which, on a growth path of x would equal $K^{\lambda-1} z^{\lambda-1} C(1 + x)$.

In a steady state (x, H), $v(k, \mu)$ now satisfies

$$v(k, \mu) = \max_{k'} \left\{ f(k, \mu) - k^{\lambda} C\left(\frac{k'}{k}\right) + \frac{1}{1+r} v(k', \mu') \right\}.$$

The first order condition is $\frac{1}{1+r} v_1(k', \mu') - k^{\lambda-1} C'\left(\frac{k'}{k}\right) = 0$. The envelope theorem now says that $v_1(k, \mu) = f_1(k, \mu) - k^{\lambda-1}\left[\lambda C\left(\frac{k'}{k}\right) - \frac{k'}{k} C'\left(\frac{k'}{k}\right)\right]$. Since

$$f_1(k, \mu) = K^{\lambda-1}\left[F_1(z, A(z \mid H)) + F_2(z, A(z \mid H)) A'(z \mid H)\right] = K^{\lambda-1} \phi(z, H),$$

the second-order difference equation for k now reads:

$$(1 + r) C'\left(\frac{k'}{k}\right) = z^{1-\lambda} \phi(z, H) - \left[\lambda C\left(\frac{k''}{k'}\right) - \frac{k''}{k'} C'\left(\frac{k''}{k'}\right)\right],$$

so that in steady state,

$$z^{1-\lambda} \phi(z, H) = \bar{\psi}(x, \lambda),$$

where $\bar{\psi}(x, \lambda) \equiv \lambda C(1 + x) + (r(x) - x) C'(1 + x)$. When $\lambda = 1$, this condition collapses to (13), as it must. Evidently, the essence of the linear homogeneous case extends to the increasing returns case.

7.3 A independent of z: Equality

Without type-dependent access, we get no theory of inequality. Indeed, if F is strictly concave in k, we still have long run growth, but equality is the only outcome because, as in many other models, the diminishing marginal product of capital is a force for convergence:

Proposition 6 *If k_a does not depend on z, and F is concave in k, equality is the only long run outcome, and the rate of growth satisfies the condition*

$$F_1(1, \alpha_1(1,1)) = \psi(x). \tag{40}$$

Proof: The first-order condition in steady state reads $F_1(k, k_a) = \psi(x)$. This equation cannot hold for two distinct k, except when F_1 is a constant (i.e. $F_{11} = 0$). This implies that $k = k_a$. Finally, the linear homogeneity implies that F_1 depends only on k/k_a, and this ratio is one. ■.

Predictably, when $A' = 0$, no rent-grabbing can arise, and growth is slower than optimal:

Proposition 7 *If k_a does not depend on z, the equality-constrained socially optimal rate of growth satisfies the condition*

$$F_1(1,1) + F_2(1, \alpha(1,1))\alpha(1,1) = \psi(x). \tag{41}$$

Proof: Assume that every agent has capital k_0 at $t = 0$. A growth rate x is optimal only if it the solution to the problem of maximizing the stream of consumption weighted by the discounted marginal utilities. That is, using (9), maximizes discounted output $\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[\tau k_t - k_t C\left(\frac{k_{t+1}}{k_t}\right)\right]$. The first order conditions to this problem, evaluated under constant growth, boil down to (41). ■

The situation is described by Figure 8 where x^* is equilibrium growth-rate, and x^o is the optimal rate.

7.4 F independent of k_a: "Gibrat's Law"

We say that "Gibrat's Law" holds if any initial distribution of firms replicates itself. Since we only consider steady states, this can take place in our equilibrium only if any H satisfies the definition of steady state equilibrium at some growth-rate x. This is stronger than the usual definition of the law, which, in our context, says that a firm's growth rate does not depend on its size.

If f did not depend on μ at all then a homogeneous f is of the form $f(k) = \eta k$; ϕ would not depend on H, and the long-run rate of growth would be unique and given by $\psi^{-1}(\eta)$. Then x would solve the following version of (11):

$$\eta = \psi(x), \tag{42}$$

The unique rate of growth would be $\psi^{-1}(\eta)$. Although x would be determined uniquely, *any* H would be an equilibrium, and the model would have no transitional dynamics. Any initial distribution of z, including the Dirac distribution, would replicate itself indefinitely, and "Gibrat's Law" would hold.

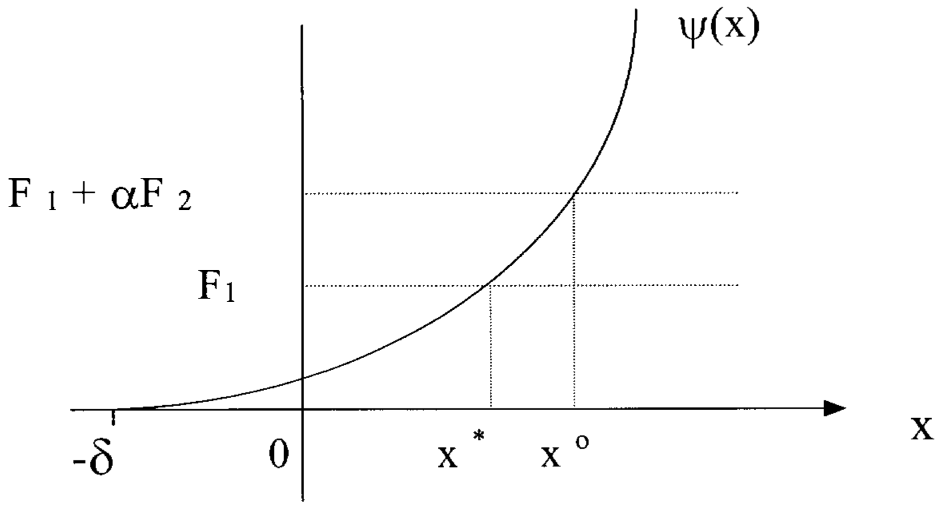

Figure 8: Equilibrium and equality-constrained optimum when k_a does not depend on k

Gibrat's Law" can, in fact, arise more generally. Even if $F_2 > 0$ and $A_z > 0$, it may still be the case that for all (z, H),

$$\frac{dF(k, k_a)}{dk} = F_1 + F_2 A' = \text{cons tan t}$$

Since F is linear homogeneous, $F_2 = 0$ implies that $\frac{dF}{dk} = F_1$ is a constant. However, consider the separable production function $F = Bk + Ck_a$ with $C \neq 0$. "Gibrat's Law" holds as long as $A''(z \mid H) = 0$ for all (z, H), simply because $\frac{dF}{dk} = B + A'C$ is then a constant. In general, any F and H that satisfy $F_{11} + 2A'F_{12} + F_{22}(A')^2 + F_2 A'' = 0$, $\forall(z, H)$ satisfies "Gibrat's Law"

In our framework, for a theory of inequality, $A' \neq 0$ is a necessary condition: Without an access function that varies with z, the model either predicts equality (the case where $F_2 > 0$), or (when $F_2 = 0$) that the distribution of income is arbitrary.

7.5 Equality?

When A does depend on z, equality can, in general, still be a long-run equilibrium. This is illustrated by example 1, where, when $\rho = 0$, equality is an equilibrium. (23) now reads $F_1(1, \omega) + \omega F_2(1, \omega) = \psi(x)$. The rate of growth depends on ω, and Figure 1 applies, with ω in place of α.

The next proposition provides sufficient conditions under which equality is the only long run outcome. These are more general than those of Proposition 6 because they cover the case in which $A' \neq 0$.

Proposition 8 *Let $F_{12} > 0$. If $A''(z,H) \leq 0$ and $A'(z,H) \leq 0$, $\forall (z,H)$ (with at least one inequality being strict) and if $\alpha_1(1,1)$ satisfies (17), a unique steady state equilibrium exists with equality and with growth rate*

$$x^* = \psi^{-1}\left(F_1(1,\alpha_1(1,1)) + F_2(1,\alpha_1(1,1))A'(1 \mid \mathbf{1})\right).$$

Proof. Suppose, contrary to the assertion, that in a steady state equilibrium there is inequality and that for $z \in [z_m, 1]$, $F_1 + A'F_2$ is constant and equal to $\psi(x)$. Then the second derivative, $F_{11} + 2A'F_{12} + F_{22}(A')^2 + F_2A'' = 0$. But linear homogeneity of F implies that $F_{11} \leq 0, F_{22} \leq 0$, and $F_{12} \geq 0$, and, given the assumptions of this proposition this expression is always negative. This rules out inequality. Finally, since (17) holds, equality is an equilibrium. Finally, by Lemma 1, ψ is strictly monotone, and therefore x^* is the only solution for x. ■

7.6 Equilibrium vs. optimum under equality

If equilibrium has equality as its long-run outcome, the equilibrium condition reads

$$F_1[1,\alpha(1,1)] + F_2[1,\alpha(1,1)]\alpha_1(1,1) = \psi(x), \tag{43}$$

which is solved by x^*. If, on the other hand, the planner were to start from a situation in which everyone was equal and if he were also constrained to treat everyone equally, the optimal rate of growth would be x^o, that solves (41). Since ψ is strictly increasing we therefore have the following result:

Proposition 9 *If x^* is an equality equilibrium growth rate and x^o the equality-constrained optimal rate,*

$$x^* \lesseqgtr x^o \qquad as \qquad \alpha_1(1,1) \lesseqgtr \alpha(1,1).$$

This result describes how private incentives in the access game relate to social incentives. Since $k_a = K\alpha(1,1)$, accessible knowledge grows by $\alpha(1,1)$ units for each unit increase in K, whereas the private return (in access) to capital accumulation is $\alpha_1(1,1)$.

Corollary 10 *In case (b), $x^* < x^o$, whereas in case (a) $x^* \lesseqgtr x^o$.*

Proof. In case (b) $\alpha_1 < 0$, whereas α itself is positive. ■

8 Markets for knowledge

Firms form, it is said, to internalize externalities. In our model, the externalities are, in general, economywide, and only a national monopoly would internalize them fully. To handle this meaningfully, we shall assume that externalities are not economywide but that, instead, one can learn from at most one person at a time. Two-person coalitions with side payments will then fully internalize the externality. The coworkers access one another's knowledge, but not anyone else's.

Production capacity and access to knowledge: If workers of type k and S are in the same firm, worker k can then access an amount of knowledge $k_a = K\alpha(\frac{k}{K}, \frac{S}{K})$, and worker S can access an amount of knowledge $k_a = K\alpha(\frac{S}{K}, \frac{k}{K})$. Since knowledge is confined within firms, k_a cannot depend on the knowledge of other workers in the economy, and this means that they cannot depend on K. But this can only be true if $\alpha(\cdot)$ itself is linear-homogeneous, so that

$$K\alpha\left(\frac{k}{K}, \frac{S}{K}\right) = \alpha(k, S) \tag{44}$$

This is a property that we shall assume for the remainder of this section.

Worker k's capacity to produce is[8]

$$\bar{f}(k, K\alpha) = F\left(k, K\alpha\left(\frac{k}{K}, \frac{S}{K}\right)\right) = KF(z, \alpha(z, s)),$$

where $s = S/K$. Worker S can produce $KF(s, \alpha(s, z))$.

Side payments: We assume complete markets with Walrasian prices. Each worker retains property rights over his net output, and pays to, or receives a transfer from the other worker. Worker k gets a gross payment

$$P(k) = Kp(\frac{k}{K})$$

from the agent he is paired with. If he is paired with S, he pays that agent

$$P(S) = Kp(\frac{S}{K})$$

The net payment by worker k to worker S is $P(S) - P(k)$. We restrict that equilibrium prices be linear homogeneous as in the two preceding equations, simply because this is necessary for the decisions problems to be stationary along a balanced growth path

Agent k's maximization problem: Each worker controls his own k and maximizes the net present value of his own output net of the costs of his own investment,

[8] Being a function of two scalars, $\bar{f}$ is a different function from f, which depends on the scalar k and the measure μ.

and net of transfers. His decision variables, at each date, are his investment in k, and the identity of his partner, S. Let $v(k)$ be the worker's value function. It satisfies

$$v(k) = P(k) + \max_{k',S}\left\{\bar{f}(k, K\alpha) - kC\left(\frac{k'}{k}\right) - P(S) + \frac{1}{1+r}v(k')\right\}. \qquad (45)$$

If v is differentiable, the first order conditions of the problem (45) are

$$\frac{1}{1+r}v'(k') - C'\left(\frac{k'}{k}\right) = 0, \qquad (46)$$

and

$$\bar{f}_2(k, K\alpha)\alpha_2 - P'(S) = 0. \qquad (47)$$

Since k' and S are "enveloped out",

$$v'(k) = P'(k) + \bar{f}_1(k, K\alpha) + \bar{f}_2(k, K\alpha)\alpha_1 - C\left(\frac{k'}{k}\right) + \frac{k'}{k}C'\left(\frac{k'}{k}\right). \qquad (48)$$

Using (48) to eliminate $v'(.)$ from (46), yields the second-order difference equation for k:

$$P'(k') + \bar{f}_1(k', K'\alpha') + \bar{f}_2(k', K'\alpha')\alpha_1' = (1+r)\,C'\left(\frac{k'}{k}\right) - \frac{k''}{k'}C'\left(\frac{k''}{k'}\right) + C\left(\frac{k''}{k'}\right). \qquad (49)$$

Steady state growth. As there now are no externalities among firms, the only link between the decisions of different firms is the interest rate r and the price function P. These prices will need to be such that every firm wishes to grow at the same rate. We would like to know if such an equilibrium places any restrictions on the rate of growth, x, and the distribution of capital over workers, k. From (47),

$$P'(S) = p'(s) = F_2\left(z, \alpha\left(z, s\right)\right)\right)\alpha_2\left(z, s\right)$$

Interchanging the arguments for a given match between s and z

$$P'(k) = p'(z) = F_2\left(s, \alpha\left(s, z\right)\right)\right)\alpha_2\left(s, z\right)$$

Let the equilibrium assignment be $S = K\xi\left(\frac{k}{K}\right)$so that in intensive form,

$$s = \xi\left(z\right).$$

This says that even though the workers can look for new partners at each instant, they will find it optimal to remain with the same type for ever.

Next, note that $\bar{f}_1(k, K\alpha) = F_1(z, \alpha(z, s))$, so that in equilibrium

$$\bar{f}_1(k, K\alpha) = F_1[z, \alpha(z, \xi(z))].$$

In steady state, $z = z'$, and hence $\xi(z) = \xi(z')$. Therefore (49) reads

$$p'(z) + F_1[z, \alpha(z, \xi(z))] + F_2[z, \alpha(z, \xi(z))]\,\alpha_1(z, \xi(z)) = \psi(x) \tag{50}$$

Evaluating $p'(z)$ at the equilibrium match $s = \xi(z)$, and substituting into (50) it follows that (50) becomes

$$F_2(\xi(z), \alpha(\xi(z), z))\,\alpha_2(\xi(z), z) + F_1[z, \alpha(z, \xi(z))] + F_2[z, \alpha(z, \xi(z))]\,\alpha_1(z, \xi(z)) = \psi(x). \tag{51}$$

Interpretation: The first term is the marginal contribution of agent z's capital to his partner s's output for which he gets rewarded $p'(z)$. The remaining terms are the marginal contributions to his own output.

When can equality survive? At $\xi(z) = z = 1$, (51) reads

$$F_1(1, \alpha(1,1)) + F_2(1, \alpha(1,1))\,(\alpha_1(1,1) + \alpha_2(1,1)) = \psi(x).$$

Proposition 2 applies to this case too, with the sole difference that in the inequalities displayed in (17), $\alpha_1(1,1)$ should be replaced by $\alpha_1(1,1) + \alpha_2(1,1)$, where α_2 corrects for the externality. The logic of the proof is exactly the same. Thus we have shown that

Proposition 11 *In a decentralized market equilibrium, a constant growth-path with equality does not exist unless*

$$\frac{-F_1[1, \alpha(1,1)]}{F_2[1, \alpha(1,1)]} - \alpha_2(1,1) \leq \alpha_1(1,1) \leq \frac{(\psi(x^m) - F_1[1, \alpha(1,1)])}{F_2[1, \alpha(1,1)]} - \alpha_2(1,1), \tag{52}$$

where x^m is defined in Proposition 2.

8.1 Equilibrium sorting

We shall show next that in the noncooperative assignment game sorting is positive if the two worker-types are strategic complements in the joint net surplus function $v(k) + v(S)$. That is, if

$$\frac{\partial^2 [v(k) + v(S)]}{\partial k \partial S} > 0.$$

Proposition 12 *k and S are strategic complements (substitutes) if for any pair (z, s)*

$$F_{12}(z, \alpha)\alpha_2(z, s) + F_2(z, \alpha)\alpha_{12}(z, s) + F_{22}(z, \alpha)\alpha_1(z, s)\alpha_2(z, s) > (<) 0 \tag{53}$$

Proof. The joint net surplus function is $v(k) + v(S)$. Then,

$$\frac{\partial^2 [v(k) + v(S)]}{\partial k \partial S} = \frac{\partial}{\partial k}\left(\frac{\partial v(k)}{\partial S}\right) + \frac{\partial}{\partial S}\left(\frac{\partial v(S)}{\partial k}\right) > 0.$$

and a similar expression holds for the second term. Now by the envelope theorem

$$\begin{aligned}\frac{\partial}{\partial k}\left(\frac{\partial v(k)}{\partial S}\right) &= \frac{\partial}{\partial k}\left(\bar{f}_2(k, K\alpha)\alpha_2 - P'(S)\right) = \bar{f}_{12}(k, K\alpha)\alpha_2 + \bar{f}_{22}(k, K\alpha)\alpha_1\alpha_2 + \bar{f}_2(k, K\alpha)\alpha_{12} \\ &= \frac{1}{K}\left\{F_{12}(z,\alpha)\alpha_2(z,s) + F_2(z,\alpha)\alpha_{12}(z,s) + F_{22}(z,\alpha)\alpha_1(z,s)\alpha_2(z,s)\right\}\end{aligned}$$

Therefore, if (53) condition holds everywhere, i.e. for both (k, S) and (S, k), the claim is valid. ■

Example – The standard assignment game: This arises if output depends only on accessed knowledge. That is, if $F(k, k_a) = k_a$. Since here $F_1 = 0$, $F_2 = 1$, and $F_{22} = 0$, condition (53) reduces to the familiar one:

$$\alpha_{12} > (<) 0.$$

8.2 Planner's optimum

This subsection will show that the equilibrium maximizes aggregate discounted output so that, in this sense, it is socially optimal. Even though matches are chosen in each period, in steady state growth equilibrium the types of the matched agents never change. We assume that the planner's steady state growth path has the same property. It follows that the optimal investment policy must maximize the discounted output of each infinitely-lived match.

Optimal investment in stable matches: The planner's rate of discount would equal the marginal rate of substitution of the consumers, and we shall retain the notation of $\left(\frac{1}{1+r}\right)^t$ for this variable. We assume that the planner cannot transfer knowledge among matches. We now use the assumption in (44). For the match between workers k and S, the planner solves the investment problem

$$\max_{(k_t,S_t)_{t=1}^{\infty}} \sum \left(\frac{1}{1+r}\right)^t \left\{F(k_t, \alpha(k_t, S_t)) - k_t C\left(\frac{k_{t+1}}{k_t}\right) + F(S_t, \alpha(S_t, k_t)) - S_t C\left(\frac{S_{t+1}}{S_t}\right)\right\} \tag{54}$$

subject to k_0, and S_0 given. Note that the planner is not solving an assignment problem here, he just decides on the sequence of capital levels that the two workers should have, conditional on remaining together for ever. We shall now show that the first-order conditions to this problem are the same as (51) as long as r is the same, and as long as the assignment is the same.

Proposition 13 *If $S_0 = K\xi(k_0/K)$, and if r is constant, the solution to (54) is*

$$k_t = (1+x)^t k_0, \text{ and } S_t = (1+x)^t S_0,$$

where x solves (51). That is, the planner chooses the same growth rate as the equilibrium growth rate.

Proof. The planner's value function is

$$W(k,S) = \max_{k',S'} \left\{ F(k, \alpha(k,S)) - kC\left(\frac{k'}{k}\right) + F(S, \alpha(S,k)) - SC\left(\frac{S'}{S}\right) + \frac{1}{1+r} W(k',S') \right\}$$

The first-order conditions are $\frac{1}{1+r} W_1(k',S') = C'\left(\frac{k'}{k}\right)$and $\frac{1}{1+r} W_2(k',S') = C'\left(\frac{S'}{S}\right)$. The envelope theorem says, for instance, that

$$W_1(k,S) = F_1(k, \alpha(k,S)) + F_2(k, \alpha(k,S))\,\alpha_1(k,S) + F_2(S, \alpha(S,k))\,\alpha_2(S,k) - C\left(\frac{k'}{k}\right) + \frac{k'}{k} C'\left(\frac{k'}{k}\right), \tag{55}$$

so that, using (48),

$$W_1(k,S) = v'(k) + F_2(S, \alpha(S,k))\,\alpha_2(S,k) - P'(k) = v'(k). \tag{56}$$

and similarly,

$$W_2(k,S) = v'(S) \tag{57}$$

So, if the planner maintains the same assignment this completes the proof. ■

Equilibrium growth is therefore optimal. This is because prices induce investment to be such that the marginal cost $\psi(x)$ equals the sum of a worker's contribution to his *own* output, α_1, and his marginal contribution to the output of the *other* worker, α_2.

Optimal matching: It remains to be shown that the pattern of assignment is the same for the planner as it is in equilibrium. The planner wants positive sorting if, for all (k,S), $W_{12} > 0$, and negative sorting if $W_{12} < 0$.

Proposition 14 *The planner's and the equilibrium sorting patterns are the same.*

Proof. (56) and (57) imply that $\frac{\partial^2[v(k)+v(S)]}{\partial k \partial S} = 2W_{12}(k,S)$. ■

8.3 Positive sorting

We show next that when k and S are strategic complements, Gibrat's Law holds, and we lose the theory of inequality. When $\frac{\partial^2[v(k)+v(S)]}{\partial k \partial S} > 0$, we have positive sorting. In this case, $\xi(z) = z$, and (51) reduces to

$$F_1[z, \alpha(z,z)] + F_2[z, \alpha(z,z)]\,[\alpha_1(z,z) + \alpha_2(z,z)] = \psi(x) \tag{58}$$

The simplified production function facilitates the characterization of the positive sorting equilibrium.

Proposition 15 *If k and S are strategic complements ($\frac{\partial^2[v(k)+v(S)]}{\partial k \partial S} > 0$), the resulting positive sorting equilibrium satisfies Gibrat's law.*

Proof. Since $\alpha(\cdot)$ is homogeneous, $\alpha(z,z)$ is proportional to z. Since F_1 is homogeneous of degree zero, it does not depend on z, because the ratio of the two arguments of F_1 is constant in z. Similarly for F_2. Finally, since $\alpha(\cdot)$ is homogeneous, α_1 and α_2 are homogeneous of degree zero as well. ■

Corollary 16 *In an economy with positive sorting and provided equilibrium exists, equality is always an equilibrium.*

Note that in the standard assignment game example where $F(k,k_a) = k_a$, (51) reduces to $\alpha_2(z,z) + \alpha_1(z,z) = \psi(x)$.

The Gibrat's Law outcome that the previous proposition proves would be of great interest if we believed that markets for knowledge were complete. Since markets for many types of knowledge go not exist, however, this result is merely a benchmark.

8.4 Negative sorting

We shall now show that when k and S are strategic substitutes $\left(\frac{\partial^2[v(k)+v(S)]}{\partial k \partial S} < 0\right)$ and when, as a result, sorting is negative, inequality will sometimes be the only equilibrium outcome, just as it was in the incomplete markets case. The allocation now satisfies $H(\xi(z)) = 1 - H(z)$ so that if H is invertible

$$\xi(z) = H^{-1}[1 - H(z)].$$

Equation (51) now contains H and we can now hope that we can, consequently, restrict H. If H has a density h, since $h(\xi(z))\xi'(z) = -h'(z)$,

$$\xi'(z) = \frac{-h(z)}{h[H^{-1}(1 - H(z))]} \tag{59}$$

If $F(k,k_a) = k_a$, equation (51) reads

$$\alpha_2(\xi(z), z) + \alpha_1(z, \xi(z)) = \psi(x).$$

For a steady state to exist, this condition must hold for all z, so that $\frac{d}{dz}\{\alpha_2(\xi(z), z) + \alpha_1(z, \xi(z))\} = 0$, and this implies, in turn, that

$$\xi'(z) = \frac{-[\alpha_{11}(z,\xi(z)) + \alpha_{22}(\xi(z), z)]}{\alpha_{12}(\xi(z), z) + \alpha_{12}(z, \xi(z))} = \frac{-h(z)}{h[H^{-1}(1 - H(z))]}, \tag{60}$$

where the second equality follows from (59).

8.4.1 Example 4: $F(k, k_a) = k_a$, and $\alpha(z, s) = z(\theta - s^4)$

Since $\alpha_1(z, s) = (\theta - s^4)$, and $\alpha_2(z, s) = -4zs^3$, (51) reads

$$\alpha_1(z, s) + \alpha_2(s, z) = \theta - s^4 - 4sz^3 = \psi(x) \tag{61}$$

Let $C \equiv \theta - \psi(x)$, so that (61) reads $s^4 + 4sz^3 = C$, and so that:

$$\xi(s) = \sqrt[3]{\frac{(C - s^4)}{4s}}. \tag{62}$$

It remains to solve for C. Since $\xi(1) = z_m$, it must be that $z_m = \sqrt[3]{\frac{(C-1)}{4}}$. Since $\xi(z_m) = 1$, this means that Substituting into (62) $s = z_m$,

$$\xi(z_m) = \sqrt[3]{\frac{C - \left(\sqrt[3]{\left(\frac{(C-1)}{4}\right)}\right)^4}{4\sqrt[3]{\left(\frac{(C-1)}{4}\right)}}} = 1,$$

which yields 3 solutions for C: $C = 1.08$, $C = 5$, and $C = 208.9$. If we choose a value for θ appropriately, we can rule out two of these solutions. In particular, if $\theta = 2$, say, then $C = 2 - \psi(x) = 1.08$ is the only solution at which $\psi(x)$ is positive. In this case $z_m = \sqrt[3]{\frac{.08}{4}} = 0.27$. A plot of $\xi(z)$ is in Figure 9:

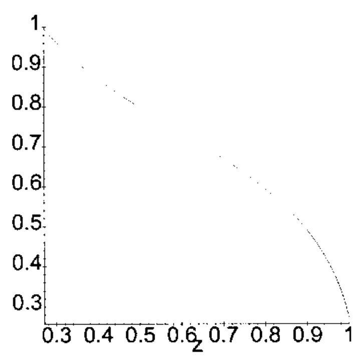

Figure 9: Plot of $\xi(z)$ in Example 4

We now derive the support of the equilibrium distribution. The mutual consent of matching partners requires that $\xi[\xi(z)] = z$, for all z in the support of H. Substitution into (62) yields

$$\xi\left[\xi(z)\right] = \sqrt[3]{\frac{1.08 - \left(\sqrt[3]{\left(\frac{(1.08-z^4)}{4z}\right)}\right)^4}{4\left(\sqrt[3]{\left(\frac{(1.08-z^4)}{4z}\right)}\right)}},$$

which we plot in Figure 10:

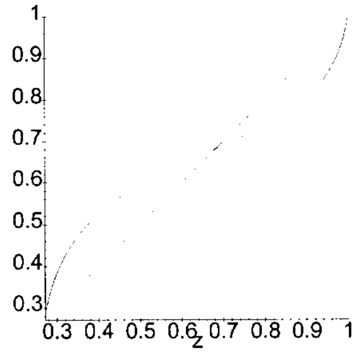

Figure 10: Plot of $\xi\left[\xi(z)\right]$ in Example 4

The curve evidently does not coincide with the 45° (dashed) line. The entire interval $z \in [0.27, 1]$ cannot therefore be in the support of the equilibrium distribution $H(z)$. Only intersections with the 45° line satisfy $\xi\left[\xi(z)\right] = z$, and these intersections occur at the points .27, .64, and 1. This, then, is the support of the equilibrium distribution. There exists a continuum of equilibrium distributions on this support, as long as they are symmetric around the point .64 and provided the solution is feasible:

$$H(z) = \left\{ \begin{array}{ll} \theta & \text{if } z = .27 \\ 1-\theta & \text{if } z = .64 \\ 1 & \text{if } z = 1 \end{array} \right\}$$

Such a distribution has mass θ at the points .27 and 1, and the remaining mass of $1-\theta$ is at the point .64.

On the one hand, this result is encouraging because any equilibrium distribution must have a range of [.27, 1]. Equality is not an equilibrium because (61) and the fact that $\psi(x) \geq 0$ imply that

$$s^4 + 4sz^3 = \theta - \psi(x) \leq 2,$$

but at $s = z = 1$, the left-hand side of this expression equals 5, a contradiction. On the other hand, however, almost all the agents could be at the midpoint – z^*, which is *de facto* equality. This seeming paradox arises because the function α is not homogeneous of degree 1, so that its first derivatives are not homogeneous of degree zero. This is not so reasonable in a model in which every other function is linear homogeneous, but it is the one tractable example we have so far for the negative sorting case, and it does show that inequality may be necessary for steady state growth.

The upshot of all this is that the introduction of even complete markets does not overturn entirely the conclusions of the incomplete markets model. In particular, if the access function has sufficient strategic substitutability between the team members, equality fails as an equilibrium and we, instead, get inequality.

9 Conclusion

Growth theorists have assumed that human capital enables one to use more efficiently the knowledge of others, to produce output as in Lucas (1988), or to accumulate knowledge as in Romer (1990). They assumed, however, that accessed knowledge depends on the mean or on the maximum of the population distribution of human capital. We generalized this by allowing access to depend on all the moments of the distribution, one of which is the variance.

This allows for theory of inequality based on external effects. In several examples we solved for the exact distribution of output as a function of the form of externalities. This mechanism for generating inequality is unexplored in the growth literature even though there are notable models of heterogeneous agents are linked by external effects in human capital. In his industry-level model for example, Nelson (1988) sketches a "nonsymmetric" equilibrium in which innovators coexist with imitators and in which both groups grow at the same rate, but he does not explain why inequality arises in the first place. Since then, Tamura (1991), Glomm and Ravikumar (1992), and Bénabou (1996) have studied how inequality evolves in a group of heterogeneous agents in models with externalities, and they have even carried out fully dynamic analyses whereas we look only at balanced growth. In those models, however, inequality either explodes, or it disappears unless fueled by repeated idiosyncratic shocks. Such models capture important reasons for inequality – shocks and initial conditions – but one that clearly differs from the mechanism that we highlight, namely, externalities.

Externalities also offer a new explanation for why Tobin's q is smaller for efficient (and, hence, big) firms. Small, inefficient firms benefit disproportionately from externalities and this causes them to have a higher average product of capital. The usual explanation for this observation is that small firms are financially constrained. We have shown, however, that one may expect to see such a relation even if capital markets are perfect.

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10 Appendix: Check of the solutions to (29) and (34)

Check that the solution to the ordinary differential equations (29) and (34) works. The solution is $A = \left(z_m^{-(1-\theta)}\theta z^{1-\theta} + Cz^{-\theta}\right)^{1/(1-\theta)}$ where C is an arbitrary constant. Differentiating,

$$\frac{dA}{dz} = \frac{(.)^{\frac{1}{1-\theta}-1}}{1-\theta}\left[z_m^{-(1-\theta)}\theta z^{1-\theta}\frac{1-\theta}{z} - Cz^{-\theta}\frac{\theta}{z}\right].$$

But

$$z_m^{-(1-\theta)}\theta z^{1-\theta}\frac{1-\theta}{z} - Cz^{-\theta}\frac{\theta}{z} = \frac{\theta}{z}\left[z_m^{-(1-\theta)}z^{1-\theta}(1-\theta) - Cz^{-\theta}\right]$$

But

$$\begin{aligned}\left[z_m^{-(1-\theta)}z^{1-\theta}(1-\theta) - Cz^{-\theta}\right] &= -\left[\theta z_m^{-(1-\theta)}z^{1-\theta} + Cz^{-\theta}\right] + z_m^{-(1-\theta)}z^{1-\theta} \\ &= -A^{1-\theta} + z_m^{-(1-\theta)}z^{1-\theta}\end{aligned}$$

Therefore

$$z_m^{-(1-\theta)}\theta z^{1-\theta}\frac{1-\theta}{z} - Cz^{-\theta}\frac{\theta}{z} = \frac{\theta}{z}\left[-A^{1-\theta} + z_m^{-(1-\theta)}z^{1-\theta}\right]$$

Therefore

$$\frac{dA}{dz} = \frac{(.)^{\frac{1}{1-\theta}-1}}{1-\theta}\frac{\theta}{z}\left[-A^{1-\theta} + z_m^{-(1-\theta)}z^{1-\theta}\right]$$

Now $\frac{1}{1-\theta} - 1 = \frac{\theta}{1-\theta}$. Therefore $(.)^{\frac{1}{1-\theta}-1} = A^{\theta}$

$$\begin{aligned}\frac{dA}{dz} &= \frac{A^{\theta}}{1-\theta}\frac{\theta}{z}\left[-A^{1-\theta} + z_m^{-(1-\theta)} z^{1-\theta}\right] \\ &= \frac{\theta}{1-\theta}\left[-\frac{A}{z} + z_m^{-(1-\theta)}\left(\frac{A}{z}\right)^{-\theta}\right]\end{aligned}$$